On a Tauberian theorem of O. Szász.

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O. SZÁSZ proved[1]) the following generalisation of Littlewood's Tauberian theorem on power series[2]):

Let $\sum_0^\infty a_k$ be summable by Abel's method. If $p > 1$ and

$$\frac{1}{n}\sum_1^n k^p |a_k|^p \qquad \text{is bounded,}$$

then $\sum_0^\infty a_k$ converges. This theorem does not hold for the limiting case $p = 1$, as it can be seen from Example 1, given below[3]). In the present paper it shall be proved that if

$$V_n = \frac{\sum_0^n k|a_k|}{n} \tag{1}$$

is not only bounded, but converges to a finite number V, then we can assure the convergence of the series $\sum_0^\infty a_k$. The following preliminary remark illustrates the difference between the qualitative and quantitative hypothesis concerning V_n: the boundedness of V_n does not imply $a_n \to 0$ (see e. g. Example 1) which holds evidently if V_n converges, since

$$|a_n| = V_n - V_{n-1} + \frac{V_{n-1}}{n}.$$

After having proved our theorem, we give three examples. From the

1) O. SZÁSZ, Verallgemeinerung eines Littlewoodschen Satzes über Potenzreihen, *Journal London Math. Soc.*, **3** (1928), pp. 254–262.

2) J. E. LITTLEWOOD, The converse of Abel's theorem on power series, *Proc. London Math. Soc.*, (2) **9** (1911), pp. 434–448.

3) Another example can be obtained from the example given by L. NEDER, Über Taubersche Bedingungen, *Proc. London Math. Soc.*, (2) **23** (1925), pp. 172–184, espacially p. 180.

second it can be seen that there exist series summable by Abel's means, to which the theorem of O. Szász can not be applied, while the conditions of our theorem are satisfied. The third example illustrates that the opposite case is also possible, viz. a series for which our theorem does not work, while that of O. Szász can be applied.

We prove the following

Theorem A. *If* $A(x)=\sum_0^\infty a_k x^k$ *is convergent for* $|x|<1$ *and* $\lim_{x\to 1-0} A(x)=s$ *exists, the series* $\sum_0^\infty a_k$ *converges to the sum s, provided that* V_n*, defined in* (1), *converges to a limit V.*

Proof. Write

$$S_n=a_0+a_1+\ldots+a_n. \tag{2}$$

We prove first that $|S_n|$ is bounded. This follows already from $A(x)$ and V_n being bounded. In fact, let us suppose

$$|A(x)|\leqq c_1, \qquad V_n\leqq c_2.$$

Evidently the Abel sums of the sequence $t_n=n|a_n|$ have the same upper bound as the arithmetic means V_n, i. e.

$$(1-x)\sum_0^\infty k|a_k|x^k\leqq c_2, \qquad 0\leqq x<1.$$

From the identity

$$S_n=\sum_0^n a_k(1-x^k)-\sum_{n+1}^\infty a_k x^k+A(x),$$

combined with the inequality $1-x^k\leqq k(1-x)$, $0\leqq x<1$, it follows that

$$|S_n|\leqq(1-x)nc_2+\frac{c_2}{(1-x)n}+c_1.$$

Putting $x=1-\frac{1}{n}$ we get

$$|S_n|\leqq c_1+2c_2.$$

Now Karamata's following lemma[4]) will be required:

If the sequence S_n is bounded from below, $S_n\geqq -M$ $(M\geqq 0)$, and the function $f(t)$ is bounded and integrable in Riemann's sense over the interval (0, 1), then

$$\lim_{x\to 1-0}(1-x)\sum_0^\infty S_k x^k=S$$

implies that

$$\lim_{x\to 1-0}(1-x)\sum_0^\infty S_k x^k f(x^k)=S\int_0^1 f(t)\,dt.$$

[4]) J. Karamata, Über die Hardy—Littlewoodschen Umkehrungen des Abelschen Stetigkeitssatzes, *Math. Zeitschrift*, 32 (1930), pp. 319–320.

After having shown the boundedness of S_n, this lemma can be applied to the sequence S_n. It can also be applied to the sequence $t_n = n|a_n|$ which, by the suppositions made, evidently satisfies the conditions of Karamata's lemma. In both cases $f(t)$ shall be defined as follows:

$$f(t) = \frac{1}{t} \quad \text{for} \quad e^{-(1+\varrho)} \leq t \leq e^{-1}$$

and $f(t) = 0$ in the remaining parts of the interval (0, 1). In this definition ϱ is an arbitrary positive number. We shall denote the integral part of $n(1+\varrho)$ by n'. Using the relation

$$\lim_{n\to\infty} n\left(1 - e^{-\frac{1}{n}}\right) = 1,$$

applying Karamata's lemma to the sequences S_n and t_n and choosing for $f(t)$ the function defined above, we have

(3)
$$\lim_{n\to\infty} \frac{1}{n} \sum_{n}^{n'} S_k = S\varrho$$

and

(4)
$$\lim_{n\to\infty} \frac{1}{n} \sum_{n}^{n'} k|a_k| = V\varrho.$$

Further, it follows from (4), that

(5)
$$\overline{\lim_{n\to\infty}} \sum_{n}^{n'} |a_k| \leq V\varrho.$$

Let us consider the difference $S_n - S$. We have

$$|S_n - S| = \left| \frac{\sum_{n}^{n'} (S_n - S_k)}{n' - n} + \frac{\sum_{n}^{n'} (S_k - S)}{n' - n} \right| \leq \frac{\sum_{n}^{n'} |S_n - S_k|}{n' - n} + \left| \frac{n}{n' - n} \frac{\sum_{n}^{n'} S_k}{n} - S \right|.$$

Since by (3) and because of $\lim_{n\to\infty} \frac{n}{n' - n} = \frac{1}{\varrho}$ the second member on the right tends to 0, it follows from (5) that

$$\overline{\lim_{n\to\infty}} |S_n - S| \leq V\varrho.$$

ϱ being arbitrary it follows that

$$\lim_{n\to\infty} S_n = S.$$

This proves Theorem A. We can state it in a slightly generalised form, which however is a simple consequence of its original form. Let σ_n^r denote the CESÀRO means of order r of the series $\sum_{0}^{\infty} |a_k|$. We have evidently $V_n = \sigma_n^0 - \sigma_n^1$. We prove the following

Theorem B. *If the series $\sum_0^\infty a_k$ is summable by Abel's method, and $(\sigma_n^r - \sigma_n^{r+1})$ tends to a limit V_r, then the series $\sum_0^\infty a_k$ is convergent.* (r is an integral number)

Theorem B follows easily from Theorem A. In fact, let τ_n^r denote the CESÀRO means of order r of the sequence $t_k = k\,|a_k|$ (i. e. of the series $\sum_1^\infty [k|a_k| - (k-1)|a_{k-1}|]$). It can be easily verified that

$$\sigma_n^r - \sigma_n^{r+1} = \frac{\tau_n^{r+1}}{r+1} \tag{6}$$
$$(r=0, 1, 2, \ldots;\ n=0, 1, 2, \ldots).$$

But it is well known[5]) that if a positive sequence is summable by CESÀRO means of any order greater than 1, it is also summable by CESÀRO means of the first order, i. e. by arithmetic means. Thus the hypothesis that $\sigma_n^r - \sigma_n^{r+1}$ converges to a limit ensures also the convergence of V_n, and Theorem A can be applied.

The following examples may serve for illustrating the mutual relation of our Theorem A and the theorem of O. SZÁSZ mentioned above.

Example 1. There exist divergent series, summable by ABEL means, with V_n bounded. For instance the series:

$$\left.\begin{array}{llll} a_k = & 1 & \text{if} & k=2^n \\ a_k = & -1 & \text{if} & k=2^n+1 \end{array}\right\} \quad n=1, 2, 3, \ldots,$$
$$a_k = 0 \quad \text{for every other value of the index } k.$$

Example 2. There exist series for which V_n converges, while $\frac{1}{n}\sum_1^n k^p|a_k|^p$ is unbounded for every value of p greater than 1. For instance the series:

$$a_k = \frac{(-1)^n}{n} \quad \text{if} \quad k=2^n,\ n=1, 2, 3, \ldots,$$
$$a_k = 0 \qquad \text{for every other value of the index } k.$$

Example 3. There exist convergent series for which V_n does not converge, while

$$\frac{1}{n}\sum_1^n k^p|a_k|^p \tag{7}$$

is bounded, moreover convergent, for some $p>1$. (In this connection it must be observed that according to the inequalities of SCHWARZ—

5) J. KARAMATA, loc. cit., p. 320.

HÖLDER, V_n is bounded provided that (7) is bounded with some $p>1$. This is the reason why the following example is a little bit more intricate.) We define first the absolute values of the numbers a_k. Let us have

$$|a_k|=\frac{5}{k} \quad \text{if} \quad k=1,2,\ldots,n_1;$$

$$|a_k|=\frac{1}{k} \quad \text{if} \quad k=n_1+(2m-1),\ m=1,2,\ldots,n_2;$$

$$|a_k|=\frac{7}{k} \quad \text{if} \quad k=n_1+2m,\ m=1,2,\ldots,n_2;$$

$$|a_k|=\frac{5}{k} \quad \text{if} \quad k=n_1+2n_2+m,\ m=1,2,\ldots,n_3;$$

$$|a_k|=\frac{1}{k} \quad \text{if} \quad k=n_1+2n_2+n_3+(2m-1),\ m=1,2,\ldots,n_4;$$

$$|a_k|=\frac{7}{k} \quad \text{if} \quad k=n_1+2n_2+n_3+2m,\ m=1,2,\ldots,n_4;$$

$$\ldots \qquad \qquad \ldots$$

The sequence of integers $n_1, n_2, n_3, \ldots$ can be chosen so as to cause V_n to oscillate between 4 and 5. After having chosen the numbers $n_1, n_2, n_3, \ldots$ in that way, the signs of the numbers a_k can be fixed so as to render convergent the series $\sum\limits_0^\infty a_k$. The choice of the numbers 1, 7, 5, serves to ensure the convergence of

$$\frac{1}{n}\sum_1^n k^2|a_k|^2$$

which tends to 5^2 in view of $\frac{1}{2}(1^2+7^2)=5^2$.

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