Integral formulae in the theory of convex curves.

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Introduction.

The definition of external parallel-curves of a convex curve can be formulated in many ways. For example, let us shift all supporting lines by the same distance outwards; the external parallel curve can be defined as the curve envelopped by these lines, or as the boundary of the domain which is the common part of all the negative half-planes of these lines. The first definition fails when using it to define internal parallel curves, because generally the curves thus obtained will not be convex, in fact not even simple Jordan-curves.

On the second way mentioned above, however, a useful definition of internal parallel curves can be obtained. The method of internal parallel curves has first been applied to isoperimetric problems by BÉLA V. SZ. NAGY[1]) by making use of an idea of F. RIESZ[2]) developed in connection with some other problems. Later on, G. BOL used the same method to give an extraordinary simple proof of the isoperimetric inequality[3]).

In the present paper the theory of internal parallel curves shall be developed further. Our main result is an explicit positive integral representation of the isoperimetric deficiency[4]). This is obtained by

1) B. v. SZ. NAGY, Über ein geometrisches Extremalproblem, *these Acta*, 9 (1939), pp. 253–257.

2) F. RIESZ, Sur une inégalité intégrale, *Journal of the London Math. Society*, 5 (1930), pp. 162–168.

3) G. BOL, Einfache Isoperimetriebeweise für Kreis und Kugel, *Abhandlungen aus dem Math. Seminar der Hamburgischen Universität*, 1 (1943), pp. 27–36.

4) As far as I am aware the only known explicit representation of the isoperimetric deficiency is that of SANTALÓ, which also contains the results of BONNESEN. In spite of the apparent coincidence of the consequences, the investigations of SANTALÓ are built on a totally different ground, — the integral-geometry of BLASCHKE — and have nothing in common with this paper. See W. BLASCHKE, *Vorlesungen über Integralgeometrie*, I (Leipzig, Berlin, 1936), pp. 25–36.

introducing a function which we call the characteristic function of the curve. From our formula, besides other inequalities, there follows easily an improvement of the isoperimetric inequality given by Bonnesen[5]), [6]),

$$P^2-4\pi A \geqq (P-2\pi\varrho)^2, \tag{2}$$

P denoting the periphery, A the area of the curve and ϱ the radius of the greatest inscribable circle.

BOL proves the isoperimetric inequality by showing that the isoperimetric deficiency of the internal parallel curves decreases when proceeding inwards. This is a consequence of the decrease of size only, and it would be false to conclude that the internal parallel curves show a gradually increasing resemblance to the circle. In fact, the very opposite of this is the case: we prove that the relative deficiency,

$$\frac{P^2-4\pi A}{A}, \tag{3}$$

increases monotonously.

BONNESEN gave also a second improvement[7]) of the isoperimetric inequality, namely:

$$P^2-4\pi A \geqq (2\pi R-P)^2, \tag{4}$$

where R denotes the radius of the least circumscribable circle. (4) can also be proved by the method of internal parallel-curves. For this purpose the theory has to be generalized by employing internal "relative-parallel-curves"[8]). The method furnishes an explicit integral representation of Minkowski's deficiency,

$$A_{12}^2-A_{11}A_{22}, \tag{5}$$

where A_{11}, A_{22} denote the areas of the convex curves and A_{12} their "mixed area," as introduced by MINKOWSKI. In full analogy to the special case an improvement of Minkowski's inequality is obtained, and by a simple lemma on quadratic equations, (4) follows therefrom. It is remarkable, that here generalization supplies fuller knowledge of the special case.

The introduction of the characteristic of a convex curve, and of its internal parallel curves, i. e., of the characteristic function, is the most important feature of these investigations. The isoperimetric deficiency of a curve is determined exclusively by its characteristic func-

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[5]) T. BONNESEN—W. FENCHEL, *Theorie der konvexen Körper* (Berlin, 1934), p. 113.

[6]) T. BONNESEN, *Les problèmes des isoperimètres* (Paris, 1929), pp. 59—63.

[7]) BONNESEN, l. c., p. 86; BONNESEN—FENCHEL, l c., p. 97.

[8]) G. BOL, Beweis einer Vermutung von H. MINKOWSKI, *Abhandlungen aus dem Math. Seminar der Hamburgischen Universität,* **1** (1943), pp. 37—56.

tion, though the curve itself is far from being determined by it. The characteristic of a polygon can be evaluated by a simple trigonometric sum. For general curves the characteristic is defined by passing to the limit. An explicit representation of the characteristic for general curves can be obtained by using integrals of "non-additive functions of interval"[9]). This may only be mentioned here; the detailed discussion of this question would lead beyond the scope of this paper.

Part I.

Let us denote the area and periphery of the internal parallel curves $C(\mu)$ at the distance μ of a given convex curve C by $A(\mu)$ and $P(\mu)$. First let us consider the internal parallel curves of a polygon, which are polygons themselves, obtained by shifting each of the sides of the original polygon inwards by the distance μ. For sufficiently small values of μ the parallel polygons will have the same number of sides, and angles equal to those of the original polygon. By increasing μ, a "critical value" of μ is reached, at which one of the sides will shrink to a point, and thus the number of sides will be diminished. The angles of the parallel polygons after passing this critical value will be equal to those of the polygon, obtained from the original polygon by prolonging, until their point of intersection, the two sides of the latter, adjacent to the shrunken side. After passing the first critical value, the number of sides, and the angles, do not change until the second critical value is reached, and so on.

A simple calculation shows, that

$$-\frac{dA(\mu)}{d\mu}=P(\mu) \tag{6}$$

and

$$-\frac{dP(\mu)}{d\mu}=\varkappa(\mu) \tag{7}$$

where

$$\varkappa(\mu)=2\sum \operatorname{tg}\frac{\beta_k}{2} \tag{8}$$

(the β_k denote the external angles of $C(\mu)$). (7) holds, except at a finite number of points, viz. the critical values mentioned above, which are the points of discontinuity of $\varkappa(\mu)$.

$\varkappa(\mu)$, called characterestric function of the polygon $C(\mu)$, has a simple geometrical interpretation: it is equal to the double area of the polygon

[9]) F. Riesz, Sur l'existence de la dérivée des fonctions d'une variable réelle et des fonctions d'intervalles, *Verhandlungen des internationalen Math.-Kongresses, Zürich,* 1932, pp. 267—269.

— called, according to TH. KALUZA, the "form-figure," — having angles equal to those of $C(\mu)$ and circumscribed to the unit circle.

It follows, that $\varkappa(\mu)$ is increasing with μ, because between two critical values of μ the form-figure remains unaltered, and by passing a critical value the form-figure increases by the prolongation, until their point of intersection, of the two sides, adjacent to the side corresponding to that of $C(\mu)$ which, at the critical value in question, has shrunk to a point.

$\varkappa(\mu)$ being an increasing function, it follows by (7) that $-P(\mu)$ is a convex function.

Further, the form-figure being circumscribed to the unit circle, we have

$$\varkappa(\mu) \geqq 2\pi. \tag{9}$$

The characteristic $\varkappa = \varkappa(0)$ of a general convex curve C is defined as the greatest lower bound of the characteristics of the polygons circumscribed to C and formed by some of the tangents of the curve C. Similarly, $\varkappa(\mu)$ is defined as the characteristic of the curve $C(\mu)$.

The formulae (6) and (7) can be generalized for arbritrary convex curves, by passing to the limit, without any essential difficulty. A lemma of F. RIESZ[10]) is the only tool required. It follows from this lemma, that if a sequence of convex functions converges to a limit function — which is naturally convex itself too — the derivatives of the functions of the sequence converge to the derivative of the limit function, provided that the latter exists, that is, almost everywhere (precisely, with exception at most of an enumerable set of points.) Let us consider a sequence of polygons C_n converging to the curve C, each polygon being formed by some of thetangents of C and each polygon containing — besides new ones — the tangents forming the preceding polygon of the sequence. Let $A_n(\mu)$, $P_n(\mu)$ and $\varkappa_n(\mu)$ denote the area, periphery and characteristic function of the internal parallel polygons of C_n, it follows that

$$\lim_{n\to\infty} A_n(\mu) = A(\mu); \quad \lim_{n\to\infty} P_n(\mu) = P(\mu); \quad \lim_{n\to\infty} \varkappa_n(\mu) = \varkappa(\mu).$$

Applying the lemma of F. RIESZ, mentioned above, (6) and (7) are proved to be valid, for arbitrary convex curves, almost everywhere.

It may be mentioned that if a curve has a tangent in every of its points, its characteristic is equal to 2π. In fact, for curves of this kind, circumscribed polygons, formed by tangents of the curve and having each of its external angles equal to $\frac{2\pi}{n}$ can be drawn, n being

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10) F. RIESZ, Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel, II, *Acta Math.*, **54** (1930), pp. 321–360, especially p. 353.

11

any integer. Thus, by the definition of the characteristic and taking (9) into account, we have

(10) $$2\pi \leqq \varkappa \leqq 2n \operatorname{tg} \frac{\pi}{n}$$

and owing to

$$\lim_{n \to \infty} 2n \operatorname{tg} \frac{\pi}{n} = 2\pi,$$

$\varkappa = 2\pi$ follows.

It can easily be seen that if ϱ denotes the radius of the greatest circle inscribable in the curve C, $C(\mu)$ shrunks for $\mu = \varrho$ to a point or to an interval, called the "kernel" of the curve. If the kernel is a point, it follows from (6) and (7) that

(11) $$P = \int_0^{\varrho} \varkappa(\mu) d\mu$$

and

(12) $$A = \int_0^{\varrho} P(\mu) d\mu.$$

If the kernel is an interval of the length K, we have, instead of (11),

(11a) $$P = 2K + \int_0^{\varrho} \varkappa(\mu) d\mu.$$

In what follows, we suppose for the sake of brevity that the first case takes place, i. e., the kernel is a point. The second case offers no additional difficulties and can be treated in the same way, and — with obvious modifications — the same conclusions can be drawn.

In (11) the integral is taken in Lebesgue's sense, so that the exceptional set of measure zero, for which (7) does not hold, can be neglected.

Taking $C(\mu)$ for the original curve, we have from (11)

(13) $$P(\mu) = \int_{\mu}^{\varrho} \varkappa(\lambda) d\lambda.$$

By substituting (13) into (12), it results

(14) $$A = \int_0^{\varrho} \int_{\mu}^{\varrho} \varkappa(\lambda) d\lambda \cdot d\mu = \int_0^{\varrho} \mu \varkappa(\mu) d\mu.$$

By a well known transformation of the double integral,

(15) $$P^2 = \left(\int_0^{\varrho} \varkappa(\mu) d\mu \right)^2 = \int_0^{\varrho} \int_0^{\varrho} \varkappa(\mu) \varkappa(\lambda) d\mu d\lambda = 2 \int_0^{\varrho} \varkappa(\mu) \int_{\mu}^{\varrho} \varkappa(\lambda) d\lambda \cdot d\mu.$$

Combining (15) with (14) our explicit formula for the isoperimetric deficiency is obtained:

$$(16)\qquad P^2-4\pi A=2\int_0^\varrho(\varkappa(\mu)-2\pi)\int_\mu^\varrho\varkappa(\lambda)\,d\lambda\cdot d\mu.$$

Owing to $\varkappa(\mu)\geqq 2\pi$ the positivity of the representation is obvious.

The inequality of BONNESEN can be obtained as follows:

$$2\int_0^\varrho(\varkappa(\mu)-2\pi)\int_\mu^\varrho\varkappa(\lambda)\,d\lambda\,d\mu\geqq 2\int_0^\varrho(\varkappa(\mu)-2\pi)\int_\mu^\varrho(\varkappa(\lambda)-2\pi)\,d\lambda\cdot d\mu=$$

$$=\left[\int_0^\varrho(\varkappa(\mu)-2\pi)\,d\mu\right]^2$$

and thus

$$(17)\qquad P^2-4\pi A\geqq(P-2\pi\varrho)^2.$$

Another improvement of the isoperimetric inequality follows from the monotonity of $\varkappa(\mu)$; $\varkappa(\mu)\geqq\varkappa(0)=\varkappa$ and therefrom

$$P^2-4\pi A\geqq 2(\varkappa-2\pi)\int_0^\varrho\int_\mu^\varrho\varkappa(\lambda)\,d\lambda\cdot d\mu=2(\varkappa-2\pi)A,$$

that is

$$(18)\qquad P^2-2\varkappa A\geqq 0.$$

This inequality has already been proved for polygons by LHUILIER[11]).

Another group of inequalities can be obtained from (14). We have

$$(19)\qquad A-\frac{\varrho}{2}P=\int_0^\varrho\left(\lambda-\frac{\varrho}{2}\right)\varkappa(\lambda)\,d\lambda=\int_{\frac{\varrho}{2}}^\varrho\left(\lambda-\frac{\varrho}{2}\right)(\varkappa(\lambda)-\varkappa(\varrho-\lambda))\,d\lambda.$$

$\varkappa(\lambda)$ being an increasing function it follows

$$(20)\qquad A-\frac{\varrho}{2}P\geqq 0.$$

On the other hand, an upper estimation of (19) can also be effected:

$$A-\frac{\varrho}{2}P\leqq\frac{\varrho}{2}\int_{\frac{\varrho}{2}}^\varrho[\varkappa(\lambda)-\varkappa(\varrho-\lambda)]\,d\lambda\leqq\frac{\varrho}{2}\left(2P\left(\frac{\varrho}{2}\right)-P\right);$$

that gives, combined with (20)

$$(21)\qquad \frac{\varrho}{2}P\leqq A\leqq\varrho P\left(\frac{\varrho}{2}\right).$$

(20) combined with (17) gives the range of variation of ϱ if A and P are given[12]):

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[11]) See L. FEJES, *Extremális pontrendszerek a síkban, a gömbfelületen és a térben* (Kolozsvár, 1944), p. 19.

[12]) BONNESEN-FENCHEL, l. c., p. 82.

$$\frac{P-\sqrt{P^2-4\pi A}}{2\pi} \leqq \varrho \leqq \frac{2A}{P}. \tag{22}$$

The decrease of

$$D(\mu) = P^2(\mu) - 4\pi A(\mu)$$

observed by Bol follows from

$$D'(\mu) = -2P(\mu)(\varkappa(\mu) - 2\pi)'; \tag{23}$$

the increase of $d(\mu) = \dfrac{P^2(\mu) - 4\pi A'(\mu)}{A(\mu)}$ from

$$d'(\mu) = \frac{P(\mu)}{A^2(\mu)}(P^2(\mu) - 2\varkappa(\mu) A(\mu)) \tag{24}$$

with respect to (18). In this connection it is worth mentioning that for external parallel curves the isoperimetric deficiency is constant, so that these problems do not occur there[13]).

Part II.

We now turn to the generalization of the results of the first part. Let $p(\varphi)$ denote the "supporting function"[14]) of the convex curve C. We have

$$P = \int_0^{2\pi} p(\varphi)\, d\varphi \tag{25}$$

and

$$A = \frac{1}{2}\int_0^{2\pi} (p^2(\varphi) - p'^2(\varphi))\, d\varphi. \tag{26}$$

The "mixed area" of two curves C_1, C_2 with supporting functions $p_1(\varphi)$, $p_2(\varphi)$ is defined by[15])

$$A_{12} = \frac{1}{2}\int_0^{2\pi} \left[p_1(\varphi)\, p_2(\varphi) - p_1'(\varphi)\, p_2'(\varphi)\right] d\varphi. \tag{27}$$

If $C_2 = C_1$ we have $A_{12} = A_1 = A_2$, therefore A_1 and A_2 can be denoted by A_{11}, A_{22}, respectively. The inequality

$$A_{12}^2 - A_{11} A_{22} \geqq 0 \tag{28}$$

is called the inequality of Minkowski. (28) contains the isoperimetric inequality, corresponding to the case when C_2 is the unit circle. The internal parallel curve, denoted by $C_1(\mu)$, of C_1 relatively to C_2, at the

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[13]) H. Hadwiger, Eine elementare Ableitung der isoperimetrischen Ungleichungen für Polygone, *Commentarii Math. Helvetici,* 16 (1943–44), pp. 305–309.

[14]) "Stützfnnktion"; see Bonnesen-Fenchel, l. c, p. 23.

[15]) Blaschke, l. c., p 34.

distance μ, is obtained by shifting the supporting line of C_1 belonging to the normal direction φ, inwards by the distance $\mu p_2(\varphi)$. Let $A_{11}(\mu)$ denote the area of $C_1(\mu)$, and $A_{12}(\mu)$ the mixed area of $C_1(\mu)$ and $C_2(\mu)$. The whole discussion of Part I can be repeated with $A_{11}(\mu)$ and $2A_{12}(\mu)$ instead of $A(\mu)$ and $P(\mu)$. If ϱ_{12} denotes the greatest number for which $\varrho_{13} C_2$ can be placed within C_1, we obtain instead of (11), (12), (16),

$$(29) \qquad A_{12} = \frac{1}{2} \int_0^{\varrho_{12}} \varkappa_{12}(\mu)\, d\mu,$$

$$(30) \qquad A_{11} = 2 \int_0^{\varrho_{13}} A_{12}(\mu)\, d\mu = \int_0^{\varrho_{13}} \mu \varkappa_{12}(\mu)\, d\mu$$

$$(31) \qquad A_{12}^2 - A_{11} A_{22} = \frac{1}{2} \int_0^{\varrho_{12}} (\varkappa_{12}(\mu) - 2A_{22}) \int_\mu^{\varrho_{12}} \varkappa_{12}(\lambda)\, d\lambda \cdot d\mu.$$

$\varkappa_{12}(\mu)$ is called the mixed characteristic function of the two curves C_1, C_2. Its definition is an obvious generalization of the definition of $\varkappa(\mu)$. We have — as a generalization of (9) —

$$(32) \qquad \varkappa_{12}(\mu) \geqq 2A_{22}\,.$$

It follows from (32) that

$$(33) \qquad A_{12}^2 - A_{11} A_{22} \geqq (A_{12} - A_{22} \varrho_{12})^2\,;$$

(33) contains (17) when C_2 is the unit circle. But (33) is by no means symmetrical. If the unit circle is taken for C_1 and for C_2 any convex curve C, we obtain a new inequality:

$$(34) \qquad P^2 - 4\pi A \geqq \left(P - \frac{2A}{R}\right)^2.$$

Now we use the following elementary

Lemma: *If $b^2 - 4ac \geqq (b-2a)^2$ then $b^2 - 4ac \geqq (b-2c)^2$ provided that $\frac{c}{a} \geqq 0$.*

This results from both inequalities being equivalent to $b \geqq a + c$, or $b \leqq a + c$ according to the common sign of a and c. This lemma expresses the following property of quadratic equations: if the equation $ax^2 + bx + c = 0$ has real roots α, β, the two pairs of points (α, β) and $\left(1, \frac{c}{a}\right)$ separate each other or not, according to $\frac{c}{a}$ being negative or positive.

Applying this lemma to (34), the second inequality of BONNESEN

(35) $$P^2-4\pi A \gtreqqless (P-2\pi R)^2$$
follows.

The other inequalities of Part I have also their counterpart; for instance we have, as a generalization of (18),

(36) $$A_{12}^2-A_{11}\varkappa_{12} \gtreqqless 0,$$
etc.

G. Bol extended the method of internal parallel curves to the case of three or more dimensions. Our results can also be generalized in this direction.

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