On the coefficients of schlicht functions.

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Let $f(z)=z+a_2z^2+\ldots+a_nz^n+\ldots$ denote a function which is analytic and schlicht in the unit circle. There are still many unsolved questions, regarding the estimation of the coefficients of such functions, of which the best known is the conjecture of BIEBERBACH: $|a_n|\leqq n$. This has been proved in general only for $n=2$ (KOEBE) and $n=3$ (LÖWNER), but for every n it is proved only under some additional conditions, among which we mention a theorem of R. NEVANLINNA: $|a_n|\leqq n$ for $n=2, 3, 4,\ldots$ if the unit circle is mapped by $w=f(z)$ on a domain in the w-plane which is star-shaped with respect to the origin. It follows from this theorem that in case the unit circle is mapped by $w=f(z)$ on a convex domain, we have $|a_n|\leqq 1$; as a matter of fact it is easy to see that in this case the unit circle is mapped by the function $w_1=zf'(z)$ on a star-shaped domain, and thus $|na_n|\leqq n$ and therefore $|a_n|\leqq 1$ for $n=2, 3,\ldots$ The theorem of NEVANLINNA has been generalized by N. G. DE BRUIJN [3] as follows: the conjecture of Bieberbach is valid if $w=f(z)$ maps the unit circle on a domain G which has the property that there exists a point A in the w-plane such that every straight line through A and cutting G has only one segment in common with G. The theorem of de Bruijn includes the case also when A is a point at infinity, when his condition means that there exists a direction L such that every straight line parallel to L and cutting G has only one segment in common with G.[1])

The object of the present paper is to prove a new theorem on the coefficients of schlicht functions showing the dependence of the estimation of the coefficients on certain geometrical data of the domain on which the unit circle is mapped by the schlicht function in question. We prove first the following

Theorem I. *Let* $f(z)=z+a_2z^2+\ldots+a_nz^n+\ldots$ *be analytic and schlicht in the unit circle. Let us put* $z=re^{i\varphi}$ *and*

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[1]) This special case which will be applied below, was proved earlier by M. S. ROBERTSON [8].

$$(1) \qquad 1+\frac{zf''(z)}{f'(z)}=u(r,\varphi)+iv(r,\varphi).$$

If we suppose

$$(2) \qquad U(r)=\int_0^{2\pi}|u(r,\varphi)|\,d\varphi\leqq\alpha$$

for $0\leqq r<1$, *it follows*

$$(3) \qquad |a_n|\leqq n^{\frac{\alpha-2\pi}{\pi}} \qquad \textit{for } n=2,3,\ldots$$

Before proving Theorem I, let us discuss the geometrical meaning of the condition (2). Let us suppose, that we have chosen for α the least possible value, i. e. that α is the least upper bound of $U(r)$ for $0\leqq r<1$. It has been proved by V. PAATERO [5] that α is equal to the "boundary-rotation" (Randdrehung) of the domain on which the unit circle is mapped by the function $w=f(z)$. The boundary-rotation of a simply connected domain G can be defined, following PAATERO as follows: If the boundary of G has a continuous tangent in every of its points, the boundary-rotation of G is defined as the total variation, for a full turn, of the angle of direction of the tangent to G. In the general case let us consider a sequence G_n of closed domains, G_n being contained in G_{n+1} and each G_n contained in G, further let us suppose that G is exhausted by the sequence G_n, i. e. that if G' is any closed subdomain of G, G' is contained in G_n for n sufficiently large. Let us define α_n as the lower bound of the boundary-rotations (as defined above) of all closed JORDAN curves with a continuous tangent which lie in G and contain G_n in their interior. Evidently the sequence α_n is non-desreasing. The finite or infinite limit of the sequence α_n shall be called the boundary-rotation of G. It has been proved by PAATERO that the function $U(r)$ defined by (2), which, according to a general theorem of F. RIESZ on subharmonic functions [6] is an increasing function of r, tends to the boundary-rotation of the domain G (on which the unit circle is mapped by $w=f(z)$) for $r\to 1$. Thus our theorem can be announced also in the following equivalent form:

Theorem II. *If the function* $f(z)=z+a_2z^2+\ldots+a_nz^n+\ldots$ *which is analytic and schlicht in the unit circle, maps the unit circle on a domain* G, *having the boundary-rotation* α, *we have*

$$|a_n|\leqq n^{\frac{\alpha-2\pi}{\pi}} \qquad n=2,3,\ldots$$

Naturally our result is interesting only for $\alpha\leqq 3\pi$. In the special case $\alpha=2\pi$, we obtain the well known estimation $|a_n|\leqq 1$. As a matter of fact the condition $\alpha=2\pi$ is equivalent to the domain G being convex (see PAATERO l. c.). For $\alpha=3\pi$ we obtain from Theorem II. exactly the BIEBERBACH estimation $|a_n|\leqq n$, but we shall prove that a more precise estimation of the coefficients can be given, which gives for $\alpha = 3\pi$ $|a_n| \leqq \frac{n+1}{2}$. This follows from

Theorem III. *Under the conditions of Theorem II. we have*

$$|a_n| \leqq \prod_{k=2}^{n}\left(1+\frac{\alpha-2\pi}{k\pi}\right) \qquad n=2,3,\ldots \tag{4}$$

It is evident, using the inequalities $\sum_{k=1}^{n}\frac{1}{k} < \log n$ and $1+x<e^x$ that

$$\prod_{k=2}^{n}\left(1+\frac{\alpha-2\pi}{k\pi}\right) < \left(\frac{\alpha-2\pi}{\pi}\sum_{k=2}^{n}\frac{1}{k}\right) < n^{\frac{\alpha-2\pi}{\pi}} \tag{5}$$

Thus Theorem II follows from Theorem III. Let us prove now Theorem III. We start from the following theorem of A. OSTROWSKI [4]:

If $g(z) = u(r,\varphi) + iv(r,\varphi)$ is analytic in the unit circle, $(z = re^{i\varphi})$ and if $\int_0^{2\pi} |u(r,\varphi)|\, d\varphi \leqq \alpha$ for $0 \leqq r < 1$, there exists a function $\psi(\varphi)$ of bounded variation, for which we have

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\psi}-z}\, d\psi(\varphi) + i \,.\, v(0). \tag{6}$$

Further we have

$$\int_0^{2\pi} d\psi(\varphi) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\psi(\varphi)| \leqq \alpha. \tag{7}$$

This theorem is a generalization of a theorem of F. RIESZ, [7] who proved it for the special case $u(r,\varphi) \geqq 0$, in which case $\psi(\varphi)$ is a monotonic function. Applying this theorem of OSTROWSKI to the function $g(z) = 1 + \frac{zf''(z)}{f'(z)}$ we obtain

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, d\psi(\varphi) \tag{8}$$

and thus

$$f''(z) = \frac{f'(z)}{\pi}\int_0^{2\pi} \frac{d\psi(\varphi)}{e^{i\varphi}-z} \tag{9}$$

further we have

$$\int_0^{2\pi} d\psi(\varphi) = 2\pi, \int_0^{2\pi} |d\psi(\varphi)| \leqq \alpha. \tag{10}$$

Let us differentiate both sides of (9) $n-2$ times, $n \geqq 2$, we obtain that

$$(11)\qquad f^{(n)}(z)=\frac{(n-2)!}{\pi}\sum_{k=0}^{n-2}\frac{f^{(k+1)}(z)}{k!}\int_0^{2\pi}\frac{d\psi(\varphi)}{(e^{i\varphi}-z)^{n-k-1}}.$$

Substituting in (11) $z=0$ we obtain, using (10) that

$$(12)\qquad |f^{(n)}(0)|\leqq\frac{\alpha(n-2)!}{\pi}\sum_{k=0}^{n-2}\frac{|f^{(k+1)}(0)|}{k!}$$

or

$$(13)\qquad |na_n|\leqq\frac{\frac{\alpha}{\pi}\sum_{k=0}^{n-2}|(k+1)a_{k+1}|}{n-1}.$$

Now if we find a sequence c_n of positive numbers satisfying the recursion formula

$$(14)\qquad c_n=\frac{\alpha}{\pi}\frac{\sum_{k=0}^{n-2}c_{k+1}}{n-1},\quad c_1=1$$

we have by induction $|na_n|\leqq c_n$ for $n=2,3,\ldots,$ as we have $a_1=c_1$. The sequence c_n in question can be found as follows: Let us put

$$(15)\qquad y=\sum_{k=1}^{\infty}c_nx^{n-1}=\frac{\alpha}{\pi}\sum_{n=2}^{\infty}\frac{\sum_{k=0}^{n-2}c_{k+1}}{n-1}x^{n-1}+c_1.$$

We have

$$(16)\qquad y'=\frac{\alpha}{\pi}\sum_{n=2}^{\infty}\left(\sum_{k=0}^{n-2}c_{k+1}\right)x^{n-2}=\frac{\alpha}{\pi}\frac{y}{1-x}.$$

Solving the differential equation $y'=\frac{\alpha}{\pi}\frac{y}{1-x}$ we obtain

$$(17)\qquad y=\frac{1}{(1-x)^{\frac{\alpha}{\pi}}}.$$

Thus we have

$$(18)\qquad c_n=\binom{-\frac{\alpha}{\pi}}{n-1}(-1)^{n-1}=n\prod_{k=2}^{n}\left(1+\frac{\alpha-2\pi}{k\pi}\right).$$

As it has been indicated above, it follows

$$(19)\qquad |a_n|\leqq\frac{c_n}{n}=\prod_{k=2}^{n}\left(1+\frac{\alpha-2\pi}{k\pi}\right)$$

which proves Theorem III.

As remarked before, the estimation given by Theorem III is of no interest for $\alpha>3\pi$. Using the theorem of ROBERTSON [8] mentioned in § 1, we prove the following theorem for this case:

Theorem IV. *Under the conditions of Theorem II we have*

$$|a_n| \leq n, \qquad n = 2, 3, \ldots$$

for $\alpha \leq 4\pi$.

The possibility of proving a theorem of this type has been suggested to me by N. G. DE BRUIJN, to whom I am thankful also for some valuable remarks concerning the proof.

Theorem IV is a consequence of the following simple geometrical lemma:

Lemma 1. *If the simply connected domain G is bounded by the Jordan curve C having a continuous tangent in every of its points and having its boundary rotation* $\alpha < 4\pi$, *there is at least one direction such that every straight line parallel to this direction has at most one segment in common with the domain G, i. e. at most two common points with C. In this case, for the sake of brevity we shall say that G is convex with respect to this direction.*

Lemma 1 can be proved using the following result, due to S. BANACH [1]:

Lemma 2. *Let* $f(x)$ *denote a function which is continuous and of bounded variation in the interval* (a, b). *If* $N(y)$ *denotes the number of roots* x *of the equation* $y = f(x)$, *we have*

$$\int_a^b |df(x)| = \int_{-\infty}^{+\infty} N(y)\, dy \tag{20}$$

i. e. the total variation of $f(x)$ *is equal to the Lebesgue integral of the function* $N(y)$ [2].

Let us take for $f(x)$ the angle of direction of the tangent to the curve C mentioned in Lemma 1, where x is some suitable parameter, for instance the arc length. In this case $N(y)$ denotes the number of tangents with the angle of direction congruent to y mod π, $0 \leq y < \pi$. Let us denote by e_k $k = 1, 2, \ldots$ the measure of the set of values of t for which $N(y) = k$. It is easy to see that $e_{2k+1} = 0$, $k = 0, 1, 2, \ldots$, because if the value of $N(y)$ is odd, at least one of the points where the tangent has the direction t is a point of inflexion. Thus we have

$$\int_0^\pi N(y)\, dy = 2e_2 + 4e_4 + 6e_6 + \ldots \tag{21}$$

We have further evidently

$$e_2 + e_4 + e_6 + \ldots = \pi. \tag{22}$$

On the other hand, according to the definition of the boundary rotation, the total variation of $f(x)$ is equal to the boundary rotation α of C. Thus, according to our supposition $\alpha < 4\pi$, and by (20) and (21), we obtain

$$2e_2 + 4e_4 + 6e_6 + \ldots < 4\pi. \tag{23}$$

Comparing (22) and (23) we obtain $e_2 > 0$, i. e. there exists a direction y such that the curve C has only two tangents parallel to the direction y, which implies that the domain G is convex with respect to this direction. Thus Lemma 1 is proved.[2])

Now it is easy to complete the proof of Theorem IV. Let us consider the function

$$f_r(z) = \frac{1}{r} f(rz) = z + a_2 r z^2 + \ldots + a_n r^{n-1} z^n + \ldots$$

$(0 < r < 1)$, and let us denote by G_r the domain on which the unit circle is mapped by $f_r(z)$. Evidently the boundary of G_r is an analytical curve. The boundary rotation $\alpha(r)$ of G_r is evidently equal to $U(r)$ defined by (2). Now it is easy to see, that it follows from our hypothesis $\alpha \leq 4\pi$ that $U(r) < 4\pi$ for $r < 1$. As a matter of fact this is evident if $\alpha < 4\pi$, and if $\alpha = 4\pi$ it follows from the fact that $\log U(r)$ is a convex function of $\log r$ (see F. RIESZ [7]) and thus is strictly increasing for those values of r for which $U(r) < 2\pi$. Thus Lemma 1 can be applied and we obtain that the domain G_r is convex with respect to a certain direction. Using the theorem of ROBERTSON mentioned above we obtain

(24) $$r^{n-1} |a_n| \leq n.$$

As (24) holds for any $r < 1$, our theorem is proved.

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[2]) Lemma 1 can also be proved in an elementary way. Elementary proofs have been communicated to me by VERONICA SÓS and N. G. DE BRUIJN.