On the geometry of conformal mapping.

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Introduction.

Let us denote by S the class of analytic functions

$$f(z)=z+a_2z^2+\ldots+a_nz^n+\ldots \tag{1}$$

which are regular and schlicht in the circle $|z|<1$. Let us denote by $D(r)$ the domain of the w-plane onto which the circle $|z|<r$ $(r<1)$ is mapped by the function $w=f(z)$, and by $C(r)$ the boundary of $D(r)$. Let $A(r)$ denote the area of $D(r)$ and $L(r)$ the length of the curve $C(r)$. We put $z=re^{i\varphi}$ and denote by $s=s(r,\varphi)$ the length along $C(r)$ from the point $w=f(r)$ to the point $f(re^{i\varphi})$ in the positive direction. We have evidently

$$\frac{ds}{d\varphi}=r|f'(z)|. \tag{2}$$

Let us put $\arg f'(z)=\chi$ and $\psi=\chi+\varphi+\frac{\pi}{2}$; clearly ψ denotes the angle between the tangent to $C(r)$ in the point $f(re^{i\varphi})$ and the real axis of the w-plane. Let us denote by $R=R(r,\varphi)$ the radius of curvature of $C(r)$ in the point $f(re^{i\varphi})$ and let us put $\gamma=\gamma(r,\varphi)=\frac{1}{R(r,\varphi)}$; it follows

$$\gamma=\frac{d\psi}{ds}=\frac{1+R\left(\frac{zf''(z)}{f'(z)}\right)}{r|f'(z)|}. \tag{3}$$

Here and in what follows we denote by $R(\zeta)$ the real part, and by $I(\zeta)$ the imaginary part of the complex number ζ. We denote by $S(f)=S(f(z))$ the invariant of SCHWARZ[1])

$$S(f)=\frac{f'''(z)}{f'(z)}-\frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{4}$$

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[1]) The invariant of SCHWARZ is the differential form of least order which remains invariant with respect to every linear transformation effected on $f(z)$; cf. H. A. SCHWARZ, *Gesammelte Math. Abhandlungen.* II (Berlin, 1890), pp. 351—355.

It is known, that for any $f(z)$, belonging to S, $C(r)$ is convex for $0 < r < r_c = 2 - \sqrt{3} = 0.26\ldots$, and star-like with respect to the point $w = 0$ for $r < r_s = \tanh \frac{\pi}{4} = 0.65\ldots$[2]). In Part I we shall investigate in detail the form of $C(r)$ for $r < r_c$; it is evident that when r decreases, the form of $C(r)$ approaches more and more the form of a circle, our aim is to express this fact in a precise manner. For this purpose we have to introduce some quantity measuring the degree of dissemblance between $C(r)$ and a circle; for this quantity we choose the total variation of γ along $C(r)$, i. e. we put

(5) $$\delta(r) = \int_{C(r)} |d\gamma| = \int_0^{2\pi} \left| \frac{d\gamma}{d\varphi} \right| d\varphi.^{3)}$$

The following theorem will be proved:

Theorem 1. *For any function belonging to S we have*

(6) $$\delta(r) < \frac{12\pi r(1+r)}{(1-r)^3}.$$

As a consequence of Theorem 1 (Corollary II) we shall prove that $C(r)$ is contained between two circles with radii $r - O(r^3)$ and $r + O(r^3)$. This is an improvement compared with the distortion theorem[4]), from which it follows only that $C(r)$ is contained between two circles with radii $r \pm O(r^2)$. (Here and in what follows we denote by $O(r^2)$, $O(r^3)$ etc. quantities which are bounded uniformly (i. e. independently of r as well as of $f(z)$) when divided by r^2, r^3, etc.) To prove the mentioned result we need the following

Lemma 1. *For any $f(z)$ belonging to S we have*

(8) $$L(r) = 2\pi r + O(r^3).$$

Clearly Lemma 1 can be expressed also by stating that

(9) $$\left(\frac{d^2L}{dr^2}\right)_{r=0} = 0.$$

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[2]) The radius of convexity has been determined by R. NEVANLINNA, Über die schlichte Abbildungen des Einheitskreises, *Oversigt av Finska Vet. Soc. Forhandlingen*, **62** (1920), pp. 1—14; the exact radius of starlikeness has been found, after long series of trials, by H. GRUNSKY, Zwei Bemerkungen zur konformen Abbildung, *Jahresbericht der Deutschen Math. Vereinigung*, 5 (1933), pp. 140—143

[3]) The use of this quantity has been kindly suggested to me by Dr. István FÁRY. It must be added, that the quantity defined by (5) gives a measure of the dissemblance of a curve from the circle only if the knowledge of the size of the curve (i. e. its length is presupposed; an absolute measure of dissemblance is furnished by the product of (5) with the length of the curve.

[4]) The distortion theorem asserts that $C(r)$ is contained between the two concentric circles with centre at the origin having the radii $\frac{r}{(1+r)^2}$ and $\frac{r}{(1-r)^2}$.

It may be mentioned that (8) is by no means evident, as from the distortion theorem[6]) applied to the formula

$$L(r) = r \int_0^{2\pi} |f'(z)| \, d\varphi \tag{10}$$

it follows at the first sight only $L(r) = 2\pi r + O(r^2)$.

In Part II we investigate the form of $C(r)$ for $r_c < r < r_s$. We define $K(r)$, the set of those (interior) points of $D(r)$, with respect to which $C(r)$ is star-like; we shall call $K(r)$ the star-kernel of $D(r)$[5]). According to the theorems mentioned above, and taking into account that a convex domain is star-like with respect to every of its interior points, it follows that $K(r) = D(r)$ for $r < r_c$ and $K(r)$ not void for $r < r_s$. The question arises what can be said regarding the size of $K(r)$ for $r_c < r < r_s$. Theorem 2 is a first attempt to answer this question.

In the present paper we do not consider the range of values $r_s < r < 1$, we refer only to the interesting results obtained by GOLUSIN[6]).

Part I.

We shall need the following

L e m m a 2. *For any function $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ belonging to S we have $|a_2^2 - a_3| \leq 1$. This inequality is best possible as equality stands for $f(z) = \frac{z}{(1-z)^2}$.* This lemma has been proved by GOLUSIN[7]) and SCHIFFER[8]).

Using Lemma 2 we obtain the following estimation of the invariant of SCHWARZ:

L e m m a 3. *For any $f(z)$ belonging to S we have*

$$|S(f)| = \left| \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \right| \leq \frac{6}{(1-r^2)^2}. \tag{15}$$

This is a "best possible" result as for $f(z) = \frac{z}{(1-z)^2}$ and $z = r$ we have equality in (15).

[5]) It is easy to see that $K(r)$ is a convex domain. This has been mentioned first by Thekla LUKÁCS; cf. G. PÓLYA and G. SZEGŐ, *Aufgaben und Lehrsätze aus der Analysis*, I. (Berlin, 1925), p. 277.

[6]) For the mentioned results and for further literature we refer to the excellent survey article of G. M. GOLUSIN, Interior problems of the theory of schlicht functions, *Uspekhi Mat. Nauk*, **6** (1939), pp. 26–89.

[7]) G. M. GOLUSIN, Einige Koeffizientenabschätzungen für schlichte Funktionen, *Mat. Sbornik*, 3 (1938), pp. 321–330.

[8]) M. SCHIFFER, Sur un problème d'extremum de la représentation conforme, *Bulletin de la Société Math. de France*, **66** (1938), pp. 48–55.

To prove Lemma 3 let us introduce the function

(16) $$h(\zeta)=\frac{f\left(\frac{\zeta+z}{1+\bar{z}\zeta}\right)-f(z)}{f'(z)(1-r^2)}.$$

A simple calculation gives

(17) $$(1-r^2)^2 S(f)=h'''(0)-\frac{3}{2}(h''(0))^2.$$

As $h(\zeta)$ belongs evidently to S, putting $(\zeta)=\zeta+c_2\zeta^2+c_3\zeta^3+\ldots$ and applying Lemma 2, we have $(1-r^2)^2|S(f)|=6|c_3-c_2^2|\leq 6$ which proves Lemma 3. Lemma 3 has been proved recently in another way (without using Lemma 1) by NEHARI[9]).

Let us calculate now the variation of γ along $C(r)$. We have by some calculations

(18) $$\frac{d\gamma}{d\varphi}=\frac{I[z^2S(f)]}{r|f'(z)|}.$$

Using the distortion theorem, according to which

(19) $$\frac{1-r}{(1+r)^3}\leq|f'(z)|\leq\frac{1+r}{(1-r)^3}$$

and using Lemma 3, Theorem 1 follows immediately.

Before considering the consequences of Theorem 1 we prove Lemma 1. We start by the decomposition

(20) $$L(r)=r\int_0^{2\pi}f'(z)e^{-i\chi}d\varphi=r\int_0^{2\pi}f'(z)\,d\varphi+r\int_0^{2\pi}(e^{-i\chi}-1)\,d\varphi+ + r\int_0^{2\pi}(f'(z)-1)(e^{-i\chi}-1)d\varphi.$$

Evidently

(21) $$r\int_0^{2\pi}f'(z)\,d\varphi=2\pi r\frac{1}{2\pi i}\int\frac{f'(z)\,dz}{z}=2\pi r,$$

further

(22) $$r\int_0^{2\pi}(e^{-i\chi}-1)\,d\varphi=-ir\int_0^{2\pi}\chi\,d\varphi+r\int_0^{2\pi}(e^{-i\chi}-1+i\chi)\,d\varphi.$$

As $\log f'(z)$ is regular in $|z|<1$, $\chi=I(\log f'(z))$ is a harmonic function, and thus $\int_0^{2\pi}\chi d\varphi=\chi(0)=0$; using the elementary inequality $|e^{-ix}+ix-1|=O(x^2)$ and the rotation theorem:

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[9]) Z. NEHARI, The Schwarzian derivative and schlicht functions, *Bulletin of the American Math. Society*, **55** (1949), pp. 545–551.

(23) $$|\chi| \leq 2\log\frac{1+r}{1-r}$$

we obtain

(24) $$r\int_0^{2\pi}(e^{-i\chi}-1)\,d\varphi = O(r^3).$$

As regards the third term of (20), we have by (19) and (23)

(25) $$r\left|\int_0^{2\pi}(f'(z)-1)(e^{-i\chi}-1)\,d\varphi\right| \leq r\int_0^{2\pi}|f'(z)-1|\,|\chi|\,d\varphi = O(r^3).$$

Thus, using (20), (21), (24) and (25), it follows

(26) $$L(r) = 2\pi r + O(r^3)$$

which is Lemma 1.

As an immediate consequence of Lemma 1 we mention that the isoperimetric deficiency of $C(r)$ is $O(r^4)$. As a matter of fact, we have by a well known formula

$$A(r) = \pi r^2 + \sum_{n=2}^{\infty} n^2 |a_n|^2 r^{2n}$$

from which, combined with (26) it follows

(27) $$L^2(r) - 4\pi A(r) = O(r^4).$$

Now let us consider some consequences of Theorem 1. We start by the formula

(28) $$L(r) = \int_0^{2\pi} R\,d\psi.$$

Let us denote by R_0 the mean value of R on $C(r)$, i. e. we put

(29) $$R_0 = \frac{1}{2\pi}\int_0^{2\pi} R\,d\psi = \frac{L(r)}{2\pi}.$$

For any value of R we have evidently

(30) $$\left|\frac{1}{R} - \frac{1}{R_0}\right| < \delta(r)$$

and thus, for sufficiently small r,

(31) $$\frac{R_0}{1+R_0\delta(r)} \leq R \leq \frac{R_0}{1-R_0\delta(r)}.$$

According to (26) we have $R_0 = r + O(r^3)$ and by Theorem 1 it follows $\delta(r) = O(r)$, thus we have from (31):

Corollary I. *If $R(r, \varphi)$ denotes the radius of curvature of $C(r)$ in the point $f(re^{i\varphi})$ we have*

$$R(r, \varphi) = r + O(r^3) \tag{32}$$

uniformly in φ, r and $f(z) \in S$.

Let us denote by R_M and r_m the maximal resp. minimal value of R on $C(r)$. According to a theorem of BLASCHKE[10]), if the convex curves C_1 and C_2 have a common tangent in one point and the radius of curvature of C_1 exceeds the radius of curvature of C_2 in points with parallel directions, it follows that C_2 is contained in C_1. Thus $C(r)$ contains a circle with radius R_m and is contained in a circle with radius R_M; if ϱ_M resp. ϱ_m denote the radii of the least circumscribable resp. the greatest inscribable circle of $C(r)$ it follows $R_m \le \varrho_m < \varrho_M \le R_M$, and thus, using (32) we obtain

Corollary II.

$$\varrho_M - \varrho_m = O(r^3). \tag{33}$$

As remarked in the introduction, the distortion theorem gives only $\varrho_M - \varrho_m = O(r^2)$. (Of course the least circumscribable and the greatest inscribable circle are generally not concentric.)

Finally we mention that R_M always exceeds r. This follows from the fact, that $\frac{R(r, \varphi)}{r}$ is a subharmonic function. As a matter of fact, it suffices to show that $\log \frac{R}{r}$ is subharmonic. As regards the latter function, we have

$$\log \frac{R}{r} = R(\log f'(z)) - \log R\left(1 + \frac{zf''(z)}{f'(z)}\right). \tag{34}$$

The first term on the right of (34) is a harmonic function, and the second — being the negative logarithm of a harmonic function — is subharmonic, and thus $\log \frac{R}{r}$, and therefore also $\frac{R}{r}$ itself are subharmonic; as the maximal value of a subharmonic function can not be taken in an interior point and as $\frac{R}{r} = 1$ for $r = 0$, it follows $R_M > r$.

Part II.

Let $r(\lambda)$ $(0 \le \lambda \le 1)$ denote the least upper bound of those values of r for which, for any $f(z) \in S$, the star-kernel $K(r)$ contains $D(\lambda r)$. According to the theorems on the radii of convexity and starlikeness, cited in the introduction, we have $r(1) = 2 - \sqrt{3}$ and $r(0) = \tanh \frac{\pi}{4}$. Evidently $r(\lambda)$ is a continuous decreasing function of λ. In what follows we shall prove the following estimate for $r(\lambda)$:

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[10]) W. BLASCHKE, *Kreis und Kugel* (Leipzig, 1916), p. 115.

Theorem 2. *We have for* $0<\lambda<\frac{\pi-\log 3}{2e^{\pi/2}}$

(35) $$r(\lambda)>\tanh\left(\frac{\pi}{4}-\frac{e^{\pi/2}}{2}\lambda\right).$$

Proof. It is easy to see that

(36) $$\left|\arg\frac{zf'(z)}{f(z)-f(a)}\right|<\frac{\pi}{2}$$

for $z=re^{i\varphi}$, $0\leqq\varphi<2\pi$, is the necessary and sufficient condition for $C(r)$ being star-like with respect to the point $w=f(a)$. Let us put $\zeta=\frac{a-z}{1-az}$, it follows $a=\frac{\zeta+z}{1+\bar{z}\zeta}$.

We need the following theorem, valid for any $f(z)\in S$, which has been proved first by GRUNSKY:[11])

(37) $$\left|\arg\frac{f(z)}{z}\right|<\log\frac{1+|z|}{1-|z|}.$$

Let us apply (37) to the function $h(\zeta)$ defined by (16), we obtain

(38) $$\left|\arg\frac{f(a)-f(z)}{f'(z)\zeta}\right|<\log\frac{1+|\zeta|}{1-|\zeta|}$$

and thus

(39) $$\left|\arg\frac{zf'(z)}{f(z)-f(a)}\right|=\left|\arg\left(\frac{f(a)-f(z)}{f'(z)\zeta}\right)\cdot\left(-\frac{\zeta}{z}\right)\right|\leqq\log\frac{1+|\zeta|}{1-|\zeta|}+\left|\arg\left(-\frac{\zeta}{z}\right)\right|.$$

The circle $|z|=r$ is mapped by $\zeta_1=\frac{z-a}{1-\bar{a}z}$ onto the circle with centre $-a\frac{1-r^2}{1-r^2|a|^2}$ and radius $\frac{r(1-|a|^2)}{1-r^2|a|^2}$. As $|\zeta_1|=|\zeta|$ it follows that for $|z|=r$ and for any a with $|a|=\varrho$ we have $|\zeta|\leqq\frac{\varrho+r}{1+\varrho r}$. We have further for $|z|=r$ and $|a|=\varrho$

$$\left|\arg\left(-\frac{\zeta}{z}\right)\right|=\left|\arg\frac{1-\frac{a}{z}}{1-az}\right|\leqq\operatorname{arctg}\frac{\varrho}{r}+\operatorname{arctg}\varrho r=\operatorname{arctg}\frac{\varrho\left(r+\frac{1}{r}\right)}{1-\varrho^2}.$$

Thus it follows that for $|z|=r$ and $|a|\leqq\lambda r$ $(\lambda>0)$

(40) $$\left|\arg\frac{zf'(z)}{f(z)-f(a)}\right|\leqq\log\frac{1+r}{1-r}+\log\frac{1+\lambda r}{1-\lambda r}+\operatorname{arctg}\frac{\lambda(r^2+1)}{1-\lambda^2r^2}.$$

Using the elementary inequalities $\log\frac{1+x}{1-x}\leqq\frac{2x}{1-x^2}$ and $\operatorname{arctg}x\leqq x$ we obtain

[11]) See H. GRUNSKY, l. c. [2]).

$$\left|\arg\frac{zf'(z)}{f(z)-f(a)}\right| \le \log\frac{1+r}{1-r}+\frac{\lambda(r+1)^2}{1-\lambda^2r^2}\le\log\frac{1+r}{1-r}+\lambda\frac{1+r}{1-r}. \tag{41}$$

Taking into account that we may suppose $r<\tanh\frac{\pi}{4}$, we obtain

$$\left|\arg\frac{zf'(z)}{f(z)-f(a)}\right| \le \log\frac{1+r}{1-r}+\lambda e^{\frac{\pi}{2}} \tag{42}$$

thus if $r\leq\tanh\left(\frac{\pi}{4}-\frac{\lambda e^{\frac{\pi}{2}}}{2}\right)$, i. e. if $\log\frac{1+r}{1-r}\leq\frac{\pi}{2}-\lambda e^{\frac{\pi}{2}}$, we have for any a with $|a|\leq\lambda r$

$$\left|\arg\frac{zf'(z)}{f(z)-f(a)}\right| \le \frac{\pi}{2} \tag{43}$$

which proves Theorem 2.

We may deduce from (41) also the slightly more precise result

$$r(\lambda)\geq\frac{x(\lambda)-1}{x(\lambda)+1} \tag{44}$$

where $x(\lambda)$ is the only positive root of the equation

$$\log x+\lambda x=\frac{\pi}{2}. \tag{45}$$

These estimations are not the best possible, nevertheless they give rather good approximation for small values of λ.

(Received August 6, 1949, revised November 10, 1949.)