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MATHEMATIQUES

REMARKS CONCERNING THE ZEROS OF CERTAIN INTEGRAL FUNCTIONS

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We consider the integral functions

(1) $$F(z)=\int_0^1 f(t)\cos zt\,dt$$

and

(2) $$\Phi(z)=\int_0^1 f(t)\sin zt\,dt$$

where $f(t)$ is a real integrable function. It has been proved by G. Pólya[1], that (1) and (2) have only real roots, provided that $f(t)$ is non-negative and non-decreasing. In two papers,[2, 3] published in this journal, L. Ilieff proved that (1) has only real roots if $f(t)$ belongs to a certain class of non-negative and non-increasing functions. The purpose of the present paper is to give some theorems, based on very simple principles, which generalize the results of L. Ilieff mentioned above.

Theorem A. Let n and m denote non-negative integers, and let us suppose that $n+m$ is odd. Let $f(t)$ denote a real function, which is n times derivable in the interval (0, 1) and which satisfies the following conditions:

a) $f(1)=0$ further $^{(k)}(1)=0$ for $k=1, 2, \cdots (n-1)$;

b) $f^{(2k+1)}(0)=0$ for $1\leq 2k+1<n$;

c) $g(t)=\dfrac{f^{(n)}(t)}{t^m}$ is integrable, non-negative and non-decreasing in (0, 1).

It follows that (1) and (2) have only real roots.

Proof: We begin by proving that (1) has only real roots. Let us put

(3) $$G(z)=\int_0^1 g(t)\sin zt\,dt$$

According to the theorem of Pólya mentioned above $G(z)$ has only real roots. From a well known result[4] which states that if an integral function of order ≤ 1 takes real values on the real axis and has only real roots, its derivative has also only real roots, it follows that $G^{(m)}(z)$ has only real roots. Using conditions a) and b), by iterated partial integration we obtain that $G^{(m)}(z)$ is — up to a factor $(-1)^s$ where the parity of the integer s depends on n and m — is equal to $z^n F(z)$, which proves that $F(z)$ has only real roots. As regards (2) the proof is the same with the only difference that we consider

$$G(z) = \int_0^1 g(t) \cos zt \, dt \tag{4}$$

instead of (3).

By the same method we obtain

Theorem B. Let n and m denote non-negative integers and let us suppose that $n+m$ is even. Let $f(t)$ denote a real function, which is n-times derivable in (0, 1), satisfies conditions a) and c) of theorem A, but instead of b) satisfies the following condition:

b') $f^{(2k)}(0)=0$ for $2 \leq 2k < n$.

It follows that (1) and (2) have only real roots. Theorem I of Ilieff's paper[3]) which states that if $\alpha \geqq 1$, $x=1-t^{\alpha}$, $\varphi_0(x)=f_0(t)$ and $f_1(t)=\int_0^x \varphi_0(x)\,dx$ where $f_0(t)$ is integrable, non-negative and non-decreasing, then $\int_0^1 f_1(t)\cos zt\,dt$ has only real roots, follows from theorem A of the present paper by putting $n=1$, $m=0$, $f(t)=f_1(t)$ and $g(t)=f_0(t)\,t^{\alpha-1}$.

Finally let us mention that instead of condition c) of theorems. A and B it can be supposed that (3) resp. (4) have only real zeros Theorem II. of Ilieff's paper[3] follows from this remark.

ЗАМЕЧАНИЯ ОТНОСИТЕЛЬНО НУЛЕЙ НЕКОТОРЫХ ЦЕЛЫХ ФУНКЦИЯХ

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Резюме

Автор выводит и обобщает результаты Л. Илиева (см. [2] и [3] исходя из известных результатов Г. Полиа о целых функциях вида (1) и (2). При этом он пользуется тем, что нули производной целой действительной функции порядка ≤ 1 имеюшей лишь действительные нули, так же действительны.

LITERATURE

1. G. Pólya, Mathematische Zeitschrift, 2 (1918), p. 352—383.
2. L. Ilieff, these Comptes Rendus, Vol. 1 (1948) № 2 and 3, p. 15—18.
3. L. Ilieff, these Comptes Rendus, Vol. 2 (1949) № 1, p. 17—20.
4. E. C. Titchmarch, The theory of functions, Oxford 1932, 266.