ON COMPOSED POISSON DISTRIBUTIONS, I

By

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Introduction

The present paper consists of four parts. In the first part (§ 1) the classical stochastic process of POISSON is deduced from assumptions much weaker than usual. As a matter of fact, the derivability of the functions $W_k(t)$ $(k = 0,1,2,\ldots)$ denoting the probability of the occurrence of exactly k events in a time interval of length t, is not supposed, and the »rarity« of the events considered is introduced through the rather weak condition:

$$\lim_{t \to 0} \frac{W_1(t)}{1 - W_0(t)} = 1 \tag{1}$$

(instead of the condition $W_0'(0) + W_1'(0) = 0$). The proof makes use of functional equations instead of differential equations, which have been used formerly.[1]

The second part (§ 2) contains the deduction of the most general form of a discontinuous, integer-valued, additive Markoff process (differential process) of random events. Thus we consider processes as follows:

The process is homogeneous in time, further the numbers of events in non-overlapping time intervals are independent random variables. Concerning the rarity of the events nothing is supposed. The general form of such processes is deduced by the same method as in § 1. It is shown that the most general process of this type is the sum of an enumerable set of independent processes $P_k (k = 1, 2, 3, \ldots)$, P_k being an ordinary Poisson process, but here an event means the simultaneous occurrence of a k-tuple of simple events. In other words, if

$$f(u, t) = \sum_{k=0}^{\infty} W_k(t)\, e^{iku} \tag{2}$$

is the characteristic function of the process, we have

$$f(u, t) = \prod_{k=1}^{\infty} \exp\left(tc_k\, (e^{iku} - 1)\right) \tag{3}$$

[1] A. KHINTCHINE, *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Ergebnisse d. Math. II. 4, 1933.

with $c_k \geqq 0$ and $\sum_{k=1}^{\infty} c_k < \infty$. Such a discrete distribution $\{W_k(t)\}$ — in case $\sum_{k=1}^{\infty} k\, c_k$ is finite — is called throughout this paper a *composed Poisson distribution.*

The third part (§ 3) contains the deduction of these composed Poisson distributions, in a still more elementary way, without introducing time. It is shown that if a family of discrete (integer-valued) distributions, depending on a single parameter p (the mean value of the distribution), is closed under convolution (i. e. if the convolution of two members of the family also belongs to the family), then it is a family of composed Poisson distributions. It is also shown that the variance of every member of such a family is not less than its mean value, equality holding only in the case of the family of ordinary Poisson distributions. In other words, the family of Poisson distributions can be characterized — instead of making use of the rarity of the events — as a family of integer-valued distributions, which is closed under convolution, with the further property that for a fixed mean value, the variance is minimal. This condition of the minimality of variance clearly replaces the condition of the rarity of the events. The fourth part (§ 4) contains some remarks on composed Poisson distributions. It is shown that these distributions are contained in the generalized Poisson distributions of A. KHINTCHINE[2], further we show that the class of composed Poisson distributions contains the contagious distributions of PÓLYA-EGGENBERGER[3] (also called »negative-binomial« distributions), the contagious distributions of I. NEYMAN[4] and the generalizations of the Poisson distribution given by H. POLLACZEK-GEIRINGER[5].

The method as well as the results of § 1 are due to A. RÉNYI. Starting from these, J. ACZÉL developed the results of § 2. The interpretation of the generalized Poisson processes considered in § 2, in terms of the superposition of independent Poisson processes of k-tuples of events, has been kindly suggested to the above mentioned two authors by A. N. KOLMOGOROFF, to whom they are

[2] Loc. cit. [1], pp. 21—24. Cf. Also the following papers: F. E. SATHERTHWAITE, Generalized Poisson distribution, *Annals of Math. Stat.*, **13** (1942), pp. 410—417; W. FELLER, On a general class of contagious distributions, *Annals of Math. Stat.*, **14** (1943), pp. 389—400, and E. CANSADO MACEDA, On the composed and generalized Poisson distributions, *Annals of Math. Stat.*, **19** (1948), pp. 414—416.

[3] F. EGGENBERGER und G. PÓLYA, Über die Statistik verketteter Vorgänge, *Zeitschrift für angewandte Math. u. Mech.*, **3** (1923), pp. 279—289.

[4] I. NEYMAN, On a new class of »contagious« distributions applicable in entomology aud bacteriology, *Annals of Math. Stat.*, **10** (1939), pp. 35—57.

[5] H. POLLACZEK-GEIRINGER, Über die Poissonsche Verteilung und die Entwicklung willkürlicher Verteilungen, *Zeitschrift für angewandte Math. u. Mech.*, **8** (1928), pp. 292—309.

grateful for his valuable remarks.[6] The results of § 3 (including the interpretation of these distributions as the convolution of an enumerable set of Poisson distributions of k-tuples of events) are due entirely to L. JÁNOSSY who found them independently and some months earlier. The authors are indebted to Á. CSÁSZÁR for a valuable remark.

§ 1. The classical stochastic process of Poisson

Let us consider events occurring in time (for example impacts of particles, telephone calls, etc.) and let us impose the following conditions:

A) *The process is homogeneous in time*, i. e., we assume that the probability of exactly k events occurring in the time interval (t_1, t_2) depends only on the lenght $t = t_2 - t_1$ of this interval; this probability will be denoted by $W_k(t)$, $(k = 0, 1, 2, \ldots)$. Evidently we have

$$(1.1) \qquad W_k(t) \geqq 0 \quad \text{and} \quad \sum_{k=0}^{\infty} W_k(t) = 1 \quad \text{for any} \quad t \geqq 0,$$

further

$$(1.2) \qquad W_0(0) = 1 \quad \text{and thus} \quad W_k(0) = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots$$

B) *The process is of Markoff's type*, i. e., the number of events occurring during the time interval (t_1, t_2) is independent of the number of events occurring during the time interval (t_3, t_4) provided that $t_1 < t_2 \leqq t_3 < t_4$.

C) *The events are rare.* We mean by this that

$$(1.3) \qquad \lim_{t \to 0} \frac{W_1(t)}{1 - W_0(t)} = 1.$$

In other words, if t tends to 0, the probability of *one* event occurring in the time interval $(0, t)$ is asymptotically equal to the probability of *at least one* event occurring in the same time interval.

Clearly (1.3) implies that

$$(1.4) \qquad \lim_{t \to 0} \frac{\sum_{k=2}^{\infty} W_k(t)}{W_1(t)} = 0,$$

i. e., (1.3) really means that in a short interval the probability of the occurrence of two or more events becomes arbitrarily small when compared with the probability of the occurrence of exactly one event in the same time interval. Instead

[6] Verbal communication at the Ist Hungarian Mathematical Congress in Budapest, 27 August—3 September 1950.

of (1.3) it would also be sufficient — as will be seen from the proof — to suppose only

$$\limsup_{t \to 0} \frac{W_1(t)}{1 - W_0(t)} = 1. \tag{1.3a}$$

We assert that conditions **A**, **B** and **C** imply that

$$W_k(t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \tag{1.5}$$

i. e., that the process is that of POISSON: here $\lambda > 0$ is the mean value of the number of events during a time unit (»density« of events).

Let us prove our assertion. First of all it follows from **B** that the probability of no event occurring in the time interval $(0, t + s)$ is equal to the product of the probabilities of no event occurring in the time intervals $(0, t)$ and $(t, t + s)$. Thus, with respect to **A**, we obtain

$$W_0(t + s) = W_0(t)\, W_0(s). \tag{1.6}$$

According to (1.1) $0 \leqq W_0(t) \leqq 1$, and thus, taking into account that the only bounded solutions of this functional equation are[7] $W_0(t) = q^t$; in our case we must have evidently $0 < q < 1$. Consequently, we can put

$$W_0(t) = e^{-\lambda t} \quad \text{with} \quad \lambda > 0. \tag{1.7}$$

Similarly, if in the time interval $(0, t + s)$ there occurs exactly one event, this is possible in two ways: either there occurs an event in the time interval $(0, t)$ and no event in $(t, t + s)$, or there occurs no event in $(0, t)$ and one event in $(t, t + s)$. Therefore, in view of **A** and **B**, we obtain

$$W_1(t + s) = W_1(t)\, W_0(s) + W_1(s)\, W_0(t). \tag{1.8}$$

Substituting the expression (1.7) of $W_0(s)$ and $W_0(t)$ into (1.8), it follows

$$W_1(t + s) = W_1(t)\, e^{-\lambda s} + W_1(s)\, e^{-\lambda t}. \tag{1.9}$$

Let us put

$$f(t) = e^{\lambda t}\, W_1(t), \tag{1.10}$$

then

$$f(t + s) = f(t) + f(s). \tag{1.11}$$

It is well known that the only bounded solutions of (1.11) are of the form

$$f(t) = c_1 t, \tag{1.12}$$

[7] See J. L. W. V. JENSEN, Sur les fonctions convexes et les inégalités entre les valeurs moyennes, *Acta Math.*, **30** (1906), p. 189, where it is proved that the only one-sidedly bounded solutions of $f(x + y) = f(x) + f(y)$ are $f(x) = Cx$, C a constant. This implies that the only bounded solutions of $f(x + y) = f(x) f(y)$ are $f(x) = q^x$.

hence we find

(1.13) $$W_1(t) = c_1\, t\, e^{-\lambda t}.$$

Substituting the expressions for $W_0(t)$ and $W_1(t)$ from (1.7) and (1.13) into (1.3), it follows that

(1.14) $$\lim_{t\to 0} \frac{c_1 t\, e^{-\lambda t}}{1 - e^{-\lambda t}} = \frac{c_1}{\lambda} = 1,$$

i. e., $$c_1 = \lambda\,.$$

Consequently, (1.5) is proved for $k = 0$ and 1. Supposing that (1.5) holds for $k = 1, 2, \ldots, n - 1$, we can show that it holds also for n. Clearly we have

(1.15) $$W_n(t + s) = \sum_{k=0}^{n} W_k(t)\, W_{n-k}(s)$$

and, as we have supposed (1.5) to hold for $k \leqq n - 1$, it follows

(1.16) $$W_n(t + s) = W_n(t)\, e^{-\lambda s} + W_n(s)\, e^{-\lambda t} + \frac{\lambda^n e^{-\lambda(t+s)}}{n!} ((t + s)^n - t^n - s^n)\,.$$

Putting

(1.17) $$f(t) = e^{\lambda t}\, W_n(t) - \frac{(\lambda t)^n}{n!}$$

we obtain

(1.18) $$f(t + s) = f(t) + f(s).$$

According to (1.17) $f(t)$ is bounded, hence[7] $f(t) = c_n t$ and

(1.19) $$W_n(t) = \left(\frac{(\lambda t)^n}{n!} + c_n t \right) e^{-\lambda t}\,.$$

According to (1.19) $W_n(t)$ is derivable and

(1.20) $$W_n'(0) = c_n\,.$$

But it follows from (1.4) that $W_n'(0) = 0$ for $n \geqq 2$ and hence $c_n = 0$. Thus from (1.19) and (1.20) we conclude that (1.5) holds for $k = n$, provided it holds for $k \leqq n - 1$. Thus our assertion (1.5) is proved by induction.

§ 2. The general homogeneous Markoff process of random events

In this § we drop the postulate of rarity of events, i. e., we assume the validity of **A** and **B** only, but do not claim the validity of **C**. We obtain exactly as in § 1, that

(2.1) $$W_0(t) = e^{-\lambda t}$$

and

$$W_1(t) = c_1 t e^{-\lambda t}, \tag{2.2}$$

but now we cannot conclude that $c_1 = \lambda$. We shall prove by induction that

$$W_k(t) = e^{-\lambda t} \sum_{\substack{r_1 + 2r_2 + \ldots + kr_k = k \\ r_i \geq 0\ (i=1, 2, \ldots k)}} \frac{(c_1 t)^{r_1} (c_2 t)^{r_2} \ldots (c_k t)^{r_k}}{r_1! \cdot r_2! \ldots r_k!}. \tag{2.3}$$

Clearly (2.3) holds for $k = 1$ by virtue of (2.2). For $k = 2$ we have by (1.15)

$$W_2(t + s) = W_2(t) e^{-\lambda s} + W_2(s) e^{-\lambda t} + c_1^2 ts e^{-\lambda(t+s)}.$$

Putting $f(t) = e^{\lambda t} W_2(t) - \frac{(c_1 t)^2}{2}$ we obtain $f(t + s) = f(t) + f(s)$. Thus (as $f(t)$ is bounded) $f(t) = c_2 t$ and therefore $W_2(t) = \left(\frac{(c_1 t)^2}{2!} + c_2 t\right) e^{-\lambda t}$. Similarly we obtain $W_k(t)$ for $k = 3, 4$, etc. To prove (2.3) by induction, let us suppose that (2.3) holds for $k = 0, 1, \ldots, n-1$. Substituting the formula (2.3) for $W_k(t)$ with $k = 0, 1, \ldots, n-1$ into (1.15), we obtain

$$W_n(t + s) = W_n(t) e^{-\lambda s} + W_n(s) e^{-\lambda t} + \tag{2.4}$$

$$+ e^{-\lambda (t+s)} \sum_{k=1}^{n} \sum_{\substack{r_1 + 2r_2 + \ldots + kr_k = k \\ \varrho_1 + 2\varrho_2 + \ldots + (n-k)\varrho_{n-k} = n-k}} \frac{(c_1 t)^{r_1} \ldots (c_k t)^{r_k} \cdot (c_1 s)^{\varrho_1} \ldots (c_{n-k} s)^{\varrho_{n-k}}}{r_1! \ldots r_k! \cdot \varrho_1! \ldots \varrho_{n-k}!}.$$

Putting

$$f(t) = e^{\lambda t} W_n(t) - \sum_{r_1 + 2r_2 + \ldots + (n-1)r_{n-1} = n} \frac{(c_1 t)^{r_1} (c_2 t)^{r_2} \ldots (c_{n-1} t)^{r_{n-1}}}{r_1!\ r_2! \ldots r_{n-1}!}, \tag{2.5}$$

it follows from (2.4) that

$$f(t + s) = f(t) + f(s) \tag{2.6}$$

and thus

$$f(t) = c_n t, \tag{2.7}$$

which proves (2.3) for $k = n$.

Clearly it follows from (2.3) that

$$c_k = W_k'(0) \qquad (k \geqq 1) \tag{2.8}$$

As $W_k(0) = 0$ and $W_k(t) \geqq 0$ for $t > 0$, we have

$$W_k'(0) = \lim_{t \to 0} \frac{W_k(t)}{t} \geqq 0 \tag{2.9}$$

and thus

$$c_k \geqq 0. \tag{2.10}$$

We shall prove that the series $\sum_{k=1}^{\infty} c_k$ converges and its sum is equal to λ. As a matter of fact, it follows from (2.3) that

$$e^{-\lambda t} \prod_{k=1}^{M} \left(\sum_{r=0}^{N} \frac{(c_k t)^r}{r!} \right) \leqq \sum_{k=0}^{n} W_k(t) \leqq 1, \tag{2.11}$$

if $n > NM^2$, and therefore for every $M \geqq 1$ and $N \geqq 0$ we have

$$\prod_{k=1}^{M} \left(\sum_{r=0}^{N} \frac{(c_k t)^r}{r!} \right) \leqq e^{\lambda t}. \tag{2.12}$$

As (2.12) holds for every value of N, it follows that

$$\exp \left(t \sum_{k=1}^{M} c_k \right) \leqq e^{\lambda t} \tag{2.13}$$

and thus

$$\sum_{k=1}^{M} c_k \leqq \lambda. \tag{2.14}$$

With (2.10) we find that $\sum_{k=1}^{\infty} c_k$ converges. Let us put

$$\mu = \sum_{k=1}^{\infty} c_k. \tag{2.15}$$

It follows that

$$1 = \sum_{k=0}^{\infty} W_k(t) = e^{-\lambda t} \prod_{k=1}^{\infty} \left(\sum_{r=0}^{\infty} \frac{(c_k t)_r}{r!} \right) = e^{(\mu - \lambda) t} \tag{2.16}$$

whence

$$\mu = \lambda = \sum_{k=1}^{\infty} c_k. \tag{2.7}$$

Substituting this value for λ into (2.3), we obtain the final formula

$$W_k(t) = \exp \left(- \sum_{n=1}^{\infty} c_n t \right) \cdot \sum_{r_1 + 2r_2 + \ldots + kr_k = k} \frac{(c_1 t)^{r_1} \cdot (c_2 t)^{r_2} \ldots (c_k t)^{r_k}}{r_1! \cdot r_2! \ldots r_k!}. \tag{2.18}$$

Thus the most general homogeneous Markoff process of random events depends on an enumerable sequence c_n of non-negative numbers, and the corresponding probability of k events holding in a time interval of lenght t is given by (2.18).

From (2.18) it is easily seen that putting

$$v_k^{(n)}(t) = \frac{(c_n t)^k}{k!} e^{-c_n t} \quad (k = 0, 1, \ldots; \quad n = 1, 2, \ldots), \tag{2.19}$$

we have

$$W_k(t) = \sum_{r_1 + 2r_2 + \ldots + kr_k = k} v_{r_1}^{(1)}(t) \cdot v_{r_2}^{(2)}(t) \ldots v_{r_k}^{(k)}(t) \cdot \prod_{n=k+1}^{\infty} v_0^{(n)}(t). \tag{2.20}$$

This can be interpreted as follows: let $\xi_n(t)$ $(n = 1, 2, \ldots ; t > 0)$ denote independent Poisson-distributed random variables, the mean value of $\xi_n(t)$ being $c_n t$, and let us put

$$\xi(t) = \xi_1(t) + 2\,\xi_2(t) + \ldots + n\,\xi_n(t) + \ldots \tag{2.21}$$

Clearly (2.20) expresses the fact that the probability of $\xi(t) = k$ is given by (2.18) and thus $\xi(t)$ is the stochastic process considered in this §. As $n\,\xi_n(t)$ is a Poisson process in which an »event« means the simultaneous occurrence of n simple events, our result may also be expressed by saying that *the most general homogeneous Markoff process of random events is the sum of an infinity of independent Poisson processes, the n-th process consisting in the random occurrence of n-tuples of events with the mean value (density)* $c_n t$, *where* $c_n \geqq 0$ *and* $\sum_{n=1}^{\infty} c_n$ *converges.*

Alternatively, $W_k(t)$ can be expressed as follows:

$$W_k(t) = \exp\left(-\sum_{n=1}^{\infty} c_n t\right) \cdot \sum_{\substack{r_1+r_2+\ldots+r_n=k,\, n\leq k \\ r_i \geq 1;\; i=1,2,\ldots,n}} \frac{c_{r_1} c_{r_2} \ldots c_{r_n}}{n!} t^n \tag{2.22}$$

where the summation is extended over all *ordered* n-tuples of positive integers $(r_1, r_2, \ldots, r_n)$ satisfying $r_1 + r_2 + \ldots + r_n = k$ $(n \leqq k)$.

Let us consider now the characteristic function of the distribution (2.18). Putting

$$f(u, t) = \sum_{k=0}^{\infty} W_k(t)\, e^{iuk} \tag{2.23}$$

we obtain easily

$$f(u, t) = \exp\left(t \sum_{k=1}^{\infty} c_k (e^{iuk} - 1)\right), \tag{2.24}$$

hence

$$f(u, t) = \prod_{k=1}^{\infty} \exp\left(t c_k (e^{iuk} - 1)\right). \tag{2.25}$$

Let us denote by $\varphi(u, \lambda)$ the characteristic function of an ordinary Poisson process:

$$\varphi(u, \lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda} e^{iku}}{k!} = \exp\left(\lambda (e^{iu} - 1)\right). \tag{2.26}$$

With this notation (2.25) can be written in the form

$$f(u, t) = \prod_{k=1}^{\infty} \varphi(ku, t c_k) \tag{2.27}$$

which expresses the fact, already emphasized, that (2.8) is the distribution of (2.21). In virtue of (2.24) we see that a distribution of the form (2.18) can be characterized by the following property of its characteristic function : The logarithm of the characteristic function of such a distribution is of the form $g(e^{iu})$ where $g(z)$ is an analytic function which is regular in the unit circle and satisfies the conditions : $g^{(n)}(0) \geqq 0$ for $n = 1, 2, \ldots,$ and

$$\lim_{z \to 1} g(z) = 0 .$$

The mean value of the number of events during a time interval of length t produced by the process (2.21) is given by

$$M(t) = t \sum_{k=1}^{\infty} kc_k \ , \tag{2.28}$$

i. e., it is finite or infinite, according as the series $\sum_{k=1}^{\infty} kc_k$ converges or diverges. Naturally only the first case is of interest. In this case the distribution (2.18) will be called a *composed Poisson distribution.*

§ 3. Descriptive characterisation of families of composed Poisson distributions

Let us consider a random variable ξ which assumes only non-negative integer values, and let us call the distribution of such a variable an *integral-valued* (abbreviated i. v.) *distribution.* Let us consider a family of i. v. distributions depending on a single parameter p, the mean value of the distribution. Let us denote by $P(k, p)$ the probability of the value k ($k = 0, 1, 2, \ldots$) and by $\{P(k, p)\}$ the distribution itself.

Let us suppose that this family is *closed under convolution.* By this we mean that if ξ_1 and ξ_2 are two independent random variables with the distributions $\{P(k, p_1)\}$ and $\{P(k, p_2)\}$ respectively, then the distribution of $\xi_1 + \xi_2$ also belongs to the family considered, i. e. it is equal to $\{P(k, p_3)\}$. Clearly, we must have $p_3 = p_1 + p_2$, because the mean value of $\xi_1 + \xi_2$ must be equal to the sum of the mean values of ξ_1 and ξ_2 . The well-known family of Poisson distributions

$$P(k, p) = \frac{p^k e^{-p}}{k!} \quad (k = 0, 1, 2, \ldots) \tag{3.1}$$

clearly has this property, i. e.

$$\sum_{k=0}^{n} P(k, p_1) \ P(n-k, p_2) = P(n, p_1 + p_2). \tag{3.2}$$

The question arises as to whether there exist even other families of distributions having the same property? The answer to this question is given by the following

THEOREM. *Let* $\{P(k, p)\}$ *be a family of i. v. distributions, closed under convolution, where p (denoting the mean value of the distribution) runs over all non-negative real numbers. Then* $\{P(k, p)\}$ *is a composed Poisson distribution, i. e.*

$$(3.3)\quad P(k,p) = \exp\left(-p\sum_{n=1}^{\infty} d_n\right) \sum_{r_1+2r_2+\ldots+kr_k=k} \frac{(pd_1)^{r_1}(pd_2)^{r_2}\ldots(pd_k)^{r_k}}{r_1!\,r_2!\ldots r_k!}$$

where

$$(3.4)\qquad \sum_{k=1}^{\infty} kd_k = 1.$$

The family of Poisson distributions may be characterized by the following property : For every fixed value of p the variance of the distribution is minimal in comparison with the other distributions of the family (3.3).

Let us denote the generating function of the distribution $\{P(k, p)\}$ by $\Pi(z, p)$, i. e. let us put

$$(3.5)\qquad \Pi(z,p) = \sum_{k=0}^{\infty} P(k,p)\, z^k \quad (z \text{ complex}, |z| \leqq 1)\ .$$

As we have supposed that the family is closed under convolution, i. e. that (3.2) holds, we conclude

$$(3.6)\qquad \Pi(z,p_1)\,\Pi(z,p_2) = \Pi(z,p_1+p_2)\ ,$$

which implies that

$$(3.7)\qquad \Pi(z,p) = f(z)^p\ .$$

As we clearly have

$$(3.8)\qquad \sum_{k=0}^{\infty} P(k,p) = 1$$

and

$$(3.9)\qquad \sum_{k=1}^{\infty} k\,P(k,p) = p\ ,$$

it follows that

$$(3.10)\qquad f(1) = f'(1) = 1\ .$$

Now we prove that $\Pi(z, p)$ can not vanish in the unit circle if p is $< \frac{1}{2}$. As a matter of fact, from (3.8) and (3.9) we conclude that

$$(3.11)\qquad P(0,p) = 1 - p + \sum_{k=2}^{\infty} (k-1)\,P(k,p),$$

and therefore

$$(3.12)\qquad P(0,p) \geqq 1 - p$$

and by virtue of (3.8)

$$\sum_{k=1}^{\infty}{}' P(k, p) \leqq p \,. \tag{3.13}$$

This implies that for $|z| \leqq 1$ we have

$$|\Pi(z, p)| \geqq P(0, p) - \sum_{k=1}^{\infty}{}' P(k, p) \geqq 1-2p \,; \tag{3.14}$$

therefore $\Pi(z, p)| > 0$ if $p < \frac{1}{2}$. But if a regular analytic function does not vanish in the closed unit circle, its logarithm is also a regular analytic function in the same circle, consequently

$$g(z) = \frac{1}{p} \log \Pi(z, p) = \log f(z) \tag{3.15}$$

is also an analytic function, and we can put

$$\Pi(z, p) = e^{pg(z)} \tag{3.16}$$

with

$$g(z) = \sum_{n=0}^{\infty}{}' d_n z^n \qquad (|z| \leqq 1). \tag{3.17}$$

Clearly, if (3.16) holds for $p < \frac{1}{2}$, it holds even for all values of $p > 0$.

Now (3.10) implies $g(1) = 0$, that is,

$$\sum_{n=0}^{\infty}{}' d_n = 0 \tag{3.18}$$

and we may write

$$g(z) = \sum_{n=1}^{\infty}{}' d_n (z^n - 1) \,. \tag{3.19}$$

We shall prove that the coefficients d_n are real and

$$d_n \geqq 0 \,. \tag{3.20}$$

This may be done as follows: let us denote $M = \operatorname{Max}_{|z| \leq 1} |g(z)|$. Clearly we have from (3.16)

$$\Pi(z, p) = \sum_{k=0}^{\infty}{}' P(k, p) z^k = 1 + pg(z) + \frac{p^2 g^2(z)}{2!} + \dots + \frac{p^k g^k(z)}{k!} + \dots \tag{3.21}$$

Let us differentiate both sides n times, and substitute $z=0$. Then we obtain

$$n!\,P(n,p)=n!\,p\,d_n+\sum_{k=2}^{\infty}\frac{p^k D_{kn}}{k!}, \tag{3.22}$$

where

$$D_{kn}=\left(\frac{d^n g^k(z)}{dz^n}\right)_{z=0}. \tag{3.23}$$

By virtue of CAUCHY's inequality for the coefficients of power series we obtain

$$|D_{kn}|\leqq n!\,M^k. \tag{3.24}$$

Dividing both sides of (3.22) by $n!\,p$, it follows that

$$\left|\frac{P(n,p)}{p}-d_n\right|\leq p\,M^2 e^{Mp}, \tag{3.25}$$

which implies that

$$\lim_{p\to 0}\frac{P(n,p)}{p}=d_n\,. \tag{3.26}$$

As $\dfrac{P(n,p)}{p}$ is clearly non-negative, (3.20) follows at once. Substituting $z=e^{it}$ into (3.16) and comparing it with (2.21), taking (3.19) and (3.20) into account, further in view of

$$\sum_{n=1}^{\infty} nd_n=g'(1)=1 \tag{3.27}$$

it follows that $P(k,p)$ has the form (3.3), i. e. $\{P(k,p)\}$ is a composed Poisson distribution, and the characteristic function of the distribution $\{P(k,p)\}$ is

$$\Pi(z,p)=\exp\left(p\sum_{n=1}^{\infty}d_n(z^n-1)\right) \tag{3.28}$$

with $d_n\geqq 0$ and $\sum_{n=1}^{\infty} nd_n=1$.

Otherwise the coefficients d_n are arbitrary, and thus there exists an infinity of families satisfying the condition of the theorem of this § and any distribution contained in such a family is a composed Poisson distribution. We note that according to (3.28) we have

$$P(0, p) = e^{-p}$$

(compare this with the inequality). Let us now calculate (3.12), the variance of the distribution $\{P(k, p)\}$. We evidently have

$$\sigma^2 = p \sum_{n=1}^{\infty} n^2 d_n . \tag{3.29}$$

Thus we have

$$\sigma^2 = p \sum_{n=1}^{\infty} n^2 d_n \geqq p \sum_{n=1}^{\infty} n d_n = p, \tag{3.30}$$

equality standing if and only if $d_1 = 1$ and thus $d_k = 0$ for $k = 2, 3, \ldots$. Therefore, the second assertion of the above theorem is also proved, and we see that the Poisson distribution can be characterized as having the smallest variance (equal to the mean) among all distributions $\{P(k, p)\}$ satisfying the conditions of the above theorem, provided that the value of p is fixed.

§ 4. Special composed Poisson distributions

Let us consider a composed Poisson distribution with the generating function

$$g(z) = \exp\left(p \sum_{k=1}^{\infty} d_k (z^k - 1)\right) \tag{4.1}$$

where

$$d_k \geqq 0 \ (k = 1, 2, \ldots) \text{ and } \sum_{k=1}^{\infty} k\, d_k = 1. \tag{4.2}$$

Let us put $\sum_{k=1}^{\infty} d_k = d$ and $\frac{d_k}{d} = e_k$, then we obtain

$$g(z) = \exp\left(pd \sum_{k=1}^{\infty} e_k (z^k - 1)\right) \tag{4.3}$$

where

$$e_k \geqq 0 \ (k = 1, 2, \ldots) \text{ and } \sum_{k=1}^{\infty} e_k = 1. \tag{4.4}$$

Let $\xi_1, \xi_2, \ldots, \xi_n$ denote independent random variables, each taking the values 1, 2, 3, ... with the corresponding probabilities $e_1, e_2, e_3, \ldots$. Let us denote by $\Phi_n(k)$ the probability of $\xi_1 + \xi_2 + \ldots + \xi_n = k$ for $n \geqq 1$ and let us put $\Phi_0(0) = 1$ and $\Phi_0(k) = 0$ for $k = 1, 2, \ldots$. We evidently have $\Phi_1(k) = e_k$ and

(4.5) $$\Phi_n(k) = \sum_{j=0}^{k} \Phi_{n-1}(j)\, e_{k-j}\,.$$

Putting $g_0(z) = 1$ and

(4.6) $$g_n(z) = \sum_{k=1}^{\infty} \Phi_n(k)\, z^k \qquad \text{for } n = 1, 2, \ldots ,$$

we obtain

(4.7) $$g_n(z) = (g_1(z))^n\,.$$

Making use of (4.7) it follows from (4.1) that

(4.8) $$g(z) = e^{-pd}\, e^{pdg_1(z)} = e^{-pd} \sum_{n=0}^{\infty} \frac{(pd)^n\,(g_1(z))^n}{n\,!} = e^{-pd} \sum_{n=0}^{\infty} \frac{(pd)^n\, g_n(z)}{n!}\,.$$

Comparing the coefficients on both sides of (4.6) we obtain

(4.9) $$P(k, p) = \sum_{n=0}^{\infty} \frac{(pd)^n\, e^{-pd}}{n!}\, \Phi_n(k).$$

Thus we find that the composed Poisson distributions, considered in the preceding §§, are contained in the generalized Poisson distribution of A. KHINTCHINE (loc. cit.[2]). The formula (4.9) can be used also (instead of (3.3) to calculate explicitly the probabilities $P(k, p)$, and especially to obtain an asymptotic formula for $p \to \infty$. Let us now consider some special composed Poisson distributions.

a) *The limit case of the Pólya-Eggenberger contagious distribution. (The »negative-binomial« distribution.)* In this case

(4.10) $$P(k, p) = (1 + p\delta)^{-\frac{1}{\delta}} \binom{-\frac{1}{\delta}}{k} \left(\frac{p\delta}{1 + p\delta}\right)^k (-1)^k \qquad (\delta > 0)$$

and

(4.11) $$g(z) = \sum_{k=0}^{\infty} P(k, p)\, z^k = \exp\left(p \sum_{n=1}^{\infty} d_n\,(z^n - 1)\right)$$

with

(4.12) $$d_n = \frac{1}{p\delta n} \left(\frac{p\delta}{1 + p\delta}\right)^n \qquad (n = 1, 2, \ldots).$$

We note that, if $\delta \to 0$, this distribution tends to the ordinary Poisson distribution.

b) *The contagious distribution of* I. NEYMAN. The characteristic function of the »contagious« distribution introduced by I. NEYMAN (loc. cit.[4]) has the form

$$(4.13)\quad g(z) = \exp\left[p \left(\iint_A \sum_{n=1}^{\infty} p_n \left(F(\xi, \eta)\, z + 1 - F(\xi, \eta)\right)^n d\xi\, d\eta - A \right) \right]$$

where A stands for a domain of the ξ, η plane and at the same time for the area of this domain ; in this domain $0 \leqq F(\xi, \eta) \leqq 1$, further $p > 0$, $p_n \geqq 0$ and $\sum_{n=1}^{\infty} p_n = 1$.

A simple calculation gives

$$(4.14)\qquad g(z) = \exp\left(p \sum_{k=1}^{\infty} d_k (z^k - 1) \right)$$

with

$$(4.15)\qquad d_k = \sum_{n=k}^{\infty} \binom{n}{k} \iint_A F^k(\xi, \eta)\, (1 - F(\xi, \eta))^{n-k}\, d\xi\, d\eta .$$

As clearly $d_k \geqq 0$, these distributions are also contained in the class of composed Poisson distributions.

c) *The generalized Poisson distributions of* H. POLLACZEK-GEIRINGER. These distributions (see loc. cit.[5]) are simply the composed Poisson distributions for which $c_n = 0$ for $n \geqq N$, i. e., for which the logarithm of the generating function is a polynomial.

Finally we mention that the composed Poisson distributions may be characterized also as those discrete infinitely divisible distributions which have jumps only at the points $n = 1, 2, \ldots$. The well-known general formula of B. DE FINETTI[8] (which is a special case of the formulae of A. N. KOLMOGOROFF[9] resp. of P. LÉVY[10] and A. KHINTCHINE)

$$(4.16)\qquad \log f(u) = p \int_{-\infty}^{+\infty} (e^{iux} - 1)\, d\Phi(x)$$

reduces to

$$(4.17)\qquad \log f(u) = p \sum_{n=1}^{\infty} d_n (e^{inu} - 1)$$

if $\Phi(x)$ is a step function with the discontinuity points $n = 1, 2, \ldots$ and jumps $d_n = \Phi(n + 0) - \Phi(n - 0)$.

[8] B. DE FINETTI, Sulla possibilita di valori eccessionali per una legge di incrementi aleatori, *Atti d. R. Accademia Naz. dei Lincei*, Rendiconti, Cl. sc. fis. mat., **10** (1929), pp. 325—230.

[9] A. KOLMOGOROFF, Sulla forme generale di un processo stocastico omogeneo (Un problema di Bruno de Finetti), *Atti d. R. Accademia Naz. dei Lincei*, Rendiconti, Cl. sc. fis. mat., **15** (1932), pp. 805—808.

[10] P. LÉVY, *Théorie de l'addition des variables aléatoires* (Paris, 1937).

The above examples **a)** — **c)** show that the class of composed Poisson distributions furnishes a large variety of widely different types of distributions. Of course it is an intricate problem to find the precise member of the class which gives the best fit to certain statistical data. If a (theoretically) given distribution is known to belong to the class of composed Poisson distributions, the values of the d_n may successively be calculated directly from the distribution by using the well-known formulae for the semi-invariants. As a matter of fact, if we put

(4.18) $\alpha_k = k!\, P(k, p)\, e^p$ for $k = 0, 1, 2, \ldots$ and $H_k = k!\, d_k\, p$ for $k = 1, 2,$

then it follows

$$\sum_{k=0}^{\infty} \frac{\alpha_k z^k}{k!} = \exp\left(\sum_{k=1}^{\infty} \frac{H_k z^k}{k!}\right) \tag{4.19}$$

and hence we have

$$\begin{aligned} H_1 &= \alpha_1 \\ H_2 &= \alpha_2 - \alpha_1^2 \\ H_3 &= \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^2 \\ H_4 &= \alpha_4 - 3\,\alpha_2^2 - 4\,\alpha_1\alpha_3 + 12\,\alpha_1^2\,\alpha_2 - 6\,\alpha_1^4 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\text{etc.} \end{aligned} \tag{4.20}$$

Taking into account that $p = \log \dfrac{1}{P(0,p)}$, the values of the d_k can be deteımined.

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ОБОБЩЕННЫЕ РАСПРЕДЕЛЕНИЯ ТИПА ПУАССОНА, I

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(*Резюме*)

В § 1 дается вывод обыкновенного случайного процесса Пуассона, т. е. однородного по времени марковского процесса »редких« событий, исходя из возможно слабых предположений, и особенно из слабого определения »редкости« событий (см. (1.3)). Метод доказательства состоит в применении функциональных уравнений вместо дифференциальных уравнений. В § 2 тем же методом определен общий вид (см. (2.18)) однородных по времени марковских процессов случайных событий ; распределения, полученные при этом называются авторами обобщенными распределениями типа Пуассона. Сказывается (на это обратил внимание двух из авторов статьи А. Н. Колмогоров, кому они выражают благодарность за это), что эти распределения являются композициями счётного множества обыкновенных распределений Пуассона. В § 3 дана характеристика классов обобщенных распределений типа Пуассона как однопараметрических групп (по композиции) целочисленных распределений. Доказывается, что среди всех обобщенных распределений типа Пуассона обыкновенные распределеня Пуассона отличаются минимальной дисперсией при данном среднем значении. В § 4 изучаются некоторые специальные распределения расматриваемого типа, например распределения Эгенвергер-Поля, »заразительные« распределения Неймана, и т. д. Обобщенные распределения типа Пуассона могут быть охарактеризованы также как дискретные целочисленные безгранично делимые распределения.