ON COMPOSED POISSON DISTRIBUTIONS, II

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Introduction

The present paper is a continuation of a joint paper of L. JÁNOSSY, J. ACZÉL and the author [1]. It consists of four parts. In § 1 the general form of inhomogeneous stochastic processes of random events is obtained (Theorem 1). This § is a generalization of § 2 of the paper [1] cited above. In § 2 of the present paper, the following problem is solved: let us suppose that every event in a composed Poisson process is the starting point of a happening, which has a definite duration, being also a random variable; it is to be taken into consideration that the distribution law of the duration of a happening may depend on the time when the happening started; we ask about the probability distribution of the number η_t of happenings going on at some time t. We shall prove that this distribution is also a composed Poisson distribution (Theorem 2). This problem for the case the underlying process of random events is an ordinary Poisson process, has been solved recently by the author, in the paper [2], where applications of this problem to several physical and technical questions (radioactive disintegration, telephone engineering, flight of electrons in a vacuum tube) are also mentioned. Another application is mentioned in § 3 of the present paper. In § 3 the general composed Poisson distribution is obtained as limiting distribution of sums of integer-valued independent random variables; as a matter of fact, it is proved (Theorem 3) that if $\xi_{n1}, \xi_{n2}, \dots \xi_{nk_n}$ are independent integer-valued random variables, which are „infinitely small“, i. e. if we suppose

$$(1) \qquad \lim_{n\to\infty} \max_{1\leqq k\leqq k_n} P(\xi_{nk} \neq 0) = 0,$$

and if the distributions of the sums

$$(2) \qquad \eta_n = \xi_{n1} + \xi_{n2} + \cdots + \xi_{nk_n}$$

are tending to a non-degenerated limiting distribution for $n\to\infty$, this limiting distribution is necessarily a composed Poisson distribution. Necessary and sufficient conditions for the convergence of the distributions of the sums (2) are also found, by applying a theorem of B. V. GNEDENKO and A. N. KOLMOGOROFF [3]. This result is closely connected with the fact established in § 4 (Theorem 4) that the class of composed Poisson distributions may be characterized as the class of infinitely divisible distributions of integer-valued random variables, which assume the value 0 with positive probability.

The theory of composed Poisson distributions, as developed in [1] and the present paper, is now in many respects complete; but the possibilities of applications of these distributions are far from being explored.

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§ 1. Non-homogeneous composed Poisson processes

Let the process start at $t=0$, and let us denote by ζ_t $(t>0)$ the number of events which occur in the time interval $(0, t)$. The following assumptions are made:

A) If $s_1 < t_1 \leqq s_2 < t_2 \leqq \cdots \leqq s_r < t_r$, the random variables $\zeta_{t_1}-\zeta_{s_1}$, $\zeta_{t_2}-\zeta_{s_2}, \ldots, \zeta_{t_r}-\zeta_{s_r}$ are independent.

B) Let $W_k(s,t)$ denote the probability of exactly k events occuring in the time interval (s,t), i. e. let us put $(s<t; k=0,1,2,\ldots)$

$$W_k(s,t) = P(\zeta_t - \zeta_s = k); \tag{1.1}$$

we suppose, that for an arbitrary small $\varepsilon > 0$ and any arbitrary large $T>0$, a positive number $\delta > 0$ can be found such that for arbitrary $r=1,2,\ldots$ and $s_1 < t_1 \leqq s_2 < t_2 \leqq \cdots \leqq s_r < t_r < T$ for which

$$\sum_{j=1}^{r} (t_j - s_j) < \delta,$$

we have

$$\prod_{j=1}^{r} W_0(s_j, t_j) > 1-\varepsilon. \tag{1.2}$$

Condition **B)** postulates the „rarity" of the events forming our process in the sense that it is highly probable that no event will take place during a sufficiently short time consisting of an arbitrary number of time intervals. In [2] a second „rarity" condition (Condition C) excluding multiple events, has also been postulated; in the present paper this condition is dropped.

We shall prove the following

THEOREM 1. *Under conditions* **A)** *and* **B)**, *denoting by*

$$\varphi(s,t,z) = \sum_{k=0}^{\infty} W_k(s,t) z^k \tag{1.3}$$

the generating function of our process, we have

(1.4) $$\log \varphi(s,t,z)=\sum_{r=1}^{\infty}(z^r-1)\int_s^t c_r(\tau)\,d\tau$$

where the $c_r(\tau)$ are non-negative integrable functions and $\sum_{r=1}^{\infty} c_r(\tau)$ converges almost everywhere. In other words we have

$$W_k(s,t)=\exp\left(-\sum_{r=1}^{\infty}\int_s^t c_r(\tau)\,d\tau\right)\cdot \sum_{r_1+2r_2+\dots+kr_k=k}\ \prod_{j=1}^{k}\frac{\left(\int_s^t c_j(\tau)\,d\tau\right)^{r_j}}{r_j!}\,.$$

Thus ζ_t is distributed according to a composed Poisson law for every $t>0$.

Proof. Let us put

(1.5) $$-\log \varphi(s,t,z)=\psi(s,t,z);$$

we have evidently

(1.6) $$\varphi(s,\tau,z)\,\varphi(\tau,t,z)=\varphi(s,t,z)$$

and thus

(1.7) $$\psi(s,\tau,z)+\psi(\tau,t,z)=\psi(s,t,z)$$

for $s<\tau<t$.

Taking into account that $0\leqq\varphi(s,t,z)\leqq 1$ for all real positive $z\leqq 1$, it follows that

(1.8) $$\psi(s,t,z)=\psi_z(I)$$

is an additive function of the interval $I=(s,t)$, which is non-negative for $0\leqq z\leqq 1$. We shall prove that for $s\leqq t\leqq T$, $\psi_z(I)$ is absolutely continuous, uniformly for $|z|\leqq 1$. As a matter of fact, let us suppose that $0\leqq s_1<t_1\leqq s_2<$ $<t_2\leqq\dots\leqq s_r<t_r\leqq T$ and $\sum_{j=1}^{r}(t_j-s_j)<\delta$, where δ is chosen so that (1.2) holds for some $\varepsilon>0$. In what follows we shall always suppose that $\varepsilon<\frac{1}{4}$, which implies that $W_0(s_j,t_j)>\frac{3}{4}$; it follows that

(1.9) $$|\varphi(s_j,t_j,z)|\geqq W_0(s_j,t_j)-\sum_{k=1}^{\infty}W_k(s_j,t_j)=2\,W_0(s_j,t_j)-1>\frac{1}{2}>0$$

and

$$|1-\varphi(s_j,t_j,z)|\leqq\sum_{k=1}^{\infty}W_k(s_j,t_j)\,|1-z^k|\leqq 2(1-W_0(s_j,t_j))<\frac{1}{2}\,.$$

Thus $\varphi(s_j,t_j,z)$ is different from zero in the closed unit circle $|z|\leqq 1$ and therefore $\psi_z(I_j)=-\log\varphi(s_j,t_j,z)$ is analytic in the unit circle. Using the inequality $\left|\log \frac{1}{1-\alpha}\right| \leqq 2|\alpha|$ valid for any complex α, with $|\alpha| < \frac{1}{2}$, we obtain

$$|\psi_z(I_j)| = \left|\log \frac{1}{1-(1-\varphi(s_j, t_j, z))}\right| \leqq 2|1-\varphi(s_j, t_j, z)| \leqq 4(1-W_0(s_j, t_j))$$

and taking into account that $\alpha < \log \frac{1}{1-\alpha}$ for $0 < \alpha < 1$, we obtain, using (1. 2), that

$$\sum_{j=1}^{r} |\psi_z(I_j)| \leqq 4 \sum_{j=1}^{r} (1-W_0(s_j, t_j)) \leqq 4 \log \left(\prod_{j=1}^{r} \frac{1}{W_0(s_j, t_j)} \right) \leqq 4 \log \frac{1}{1-\varepsilon}. \tag{1. 10}$$

As we have remarked above, $\psi_z(I_j)$ is analytic in the unit circle; thus we may put

$$\psi_z(I_j) = c_0(I_j) - \sum_{k=1}^{\infty} c_k(I_j) z^k \tag{1. 11}$$

where $c_k(I)$ $(k=0, 1, 2, \ldots)$ are functions of the interval $I=(s, t)$. It follows, by CAUCHY's formula, that

$$c_0(I) = \frac{1}{2\pi i} \int \frac{\psi_\zeta(I)}{\zeta} d\zeta$$

and

$$c_k(I) = -\frac{k!}{2\pi i} \int \frac{\psi_\zeta(I)}{\zeta^{k+1}} d\zeta \qquad \text{for } k=1, 2, \ldots \tag{1. 12}$$

where the integration is extended over the circle $|\zeta| = r < 1$. But using (1. 10) if $\sum_{j=1}^{r} (t_j - s_j) < \delta$ and $I_j = (s_j, t_j)$, we have

$$\sum_{j=1}^{r} |c_k(I_j)| \leqq \frac{8\varepsilon \,.\, k!}{r^k}$$

and thus $c_k(I)$ are also absolutely continuous additive functions of interval. Thus we may put

$$c_k(I) = \int_s^t c_k(\tau)\, d\tau$$

where $c_k(\tau)$ is L-integrable, $k=0, 1, 2, \ldots$. Now we shall prove that $c_k(\tau) \geqq 0$ for $k=1, 2, \ldots$ This can be proved by the same method, as used in [1], by showing the non-negativity of the coefficients c_k figuring there, as follows. We shall prove that

$$c_k(t) = \lim_{\Delta t \to 0} \frac{W_k(t, t+\Delta t)}{\Delta t} \qquad (k=1, 2, \ldots) \tag{1. 13}$$

for almost every t, and thus $c_k(t) \geqq 0$ for $k=1, 2, \ldots$. As a matter of fact, we have

$$\sum_{k=0}^{\infty} W_k(t, t+\Delta t) z^k = e^{-\psi_z(\Delta I)}$$

where $\Delta I=(t, t+\Delta t)$. Differentiating both sides k times with respect to z and substituting $z=0$, we obtain

$$k!\, W_k(t, t+\Delta t)=$$
$$=\Big[-\psi_0^{(k)}(\Delta I)+\sum_{\alpha_1+2\alpha_2+\cdots+(k-1)\alpha_{k-1}=k} c_{\alpha_1\alpha_2\ldots\alpha_{k-1}} \prod_{j=1}^{k-1}\left(-\psi_0^{(j)}(\Delta I)\right)^{\alpha_j}\Big] e^{-\psi_0(\Delta I)}$$

where $c_{\alpha_1\alpha_2\ldots\alpha_{k-1}}$ are numerical coefficients, α_j are non-negative integers, $j=1,2,\ldots,k-1$. As $\psi_0^{(j)}(\Delta I)=-j!\,c_j(\Delta I)$, we have for $|z|\leqq 1$ and almost every t, $|\psi_0^{(j)}(\Delta I)|=o(\Delta t)$ for $j=1,2,\ldots,k-1$, if k is fixed, and thus we obtain, using $\alpha_1+\alpha_2+\cdots+\alpha_{k-1}\geqq 2$ for $k>2$ that

$$k!\, W_k(t, t+\Delta t)=[-\psi_0^{(k)}(\Delta I)+o(\Delta t)]\, e^{-\psi_0(\Delta I)}$$

and therefore (1. 13) follows. Thus $c_0(I)-\psi_z(I)=\sum_{k=1}^{\infty} c_k(I) z^k$ is a power series with non-negative coefficients. Thus taking into account that $\psi_1(I)=0$ and consequently $\lim_{z\to 1}\sum_{k=1}^{\infty} c_k(I) z^k=c_0(I)$, we obtain that $\sum_{k=1}^{\infty} c_k(I)$ converges and its sum equals $c_0(I)$, and therefore we may write

$$\psi_z(I)=\sum_{k=1}^{\infty} c_k(I)(1-z^k).$$

Thus Theorem 1 is proved.

If the process is homogeneous, $c_k(\tau)$ does not depend on τ, $c_k(\tau)\equiv c_k$ $(k=1,2,\ldots)$ and we obtain as a special case the results of [1] § 2. If $c_k=0$ for $k=2,3,\ldots$ we obtain the ordinary Poisson process. Let us mention that in case of an inhomogeneous Poisson process the inhomogeneity is not essential, as it can be eliminated by a change of the scale of time. As a matter of fact, we have only to put $t'=\int_0^t c_1(\tau)\,d\tau$. On the other hand, in the case of inhomogeneous composed Poisson processes this is not possible, because by introducing a new time scale we can make one arbitrarily chosen $c_k(\tau)$ constant, but in general not all of them at the same time. Thus there exist *„genuinely“* inhomogeneous composed Poisson processes.

§ 2. The distribution law of η_t

The following theorem will be proved:

THEOREM 2. *Let us start from an inhomogeneous Poisson process of random events, the characteristic function of which is given by* (1. 3) *and* (1. 4). *We suppose that* $\sum_{k=1}^{\infty} k c_k(t)=\mu(t)$ *converges and is L-integrable, i. e. that the mean value of* ζ_t *exists for every* $t>0$. *Let us suppose that each event of this* *process is the starting point of some happening, the duration of which depends also on chance, and let* $F(\tau, x)$ *denote the distribution function of the duration of a happening starting at time t and let us put* $\Phi(\tau, x) = 1 - F(\tau, x)$; *we suppose that* $\Phi(\tau, x)$ *is continuous, further that* $\Phi(\tau, x) > 0$ *for all* τ *and* x.[1] *Let us denote by* η_t *the number of happenings going on at time t. The law of distribution of* η_t *is a composed Poisson distribution, with generating function*

$$\chi(z, t) = \exp\left(\sum_{k=1}^{\infty} d_k(t)(z^k - 1)\right) \tag{2.1}$$

where

$$d_k(t) = \int_0^t \left[\sum_{n=k}^{\infty} c_n(\tau)\binom{n}{k} \Phi^k(\tau, t-\tau)(1 - \Phi(\tau, t-\tau))^{n-k}\right] d\tau \tag{2.2}$$

and $c_n(\tau)$ *is defined by* (1.4); *evidently we have* $d_k(t) \geqq 0$.

Proof[2]. Let us divide the interval $(0, t)$ into n equal parts by means of the points $t_k = \frac{kt}{n}$ $(k = 0, 1, \ldots, n)$ and let us denote by ΔI_k the interval (t_{k-1}, t_k); let us put further $\Delta t_k = t_k - t_{k-1}$ and

$$M_k = \operatorname*{Max}_{t_{k-1} \leqq \tau \leqq t_k} \Phi(\tau, t-\tau); \qquad m_k = \operatorname*{Min}_{t_{k-1} \leqq \tau \leqq t_k} \Phi(\tau, t-\tau). \tag{2.3}$$

Let $V_k(r)$ denote the probability that there are exactly r such happenings going on at time t which started in the time interval $\Delta I_k (k = 1, 2, \ldots, n)$. First we shall prove the following inequality:

$$\sum_{s=r}^{\infty} \binom{s}{r} W_s^{(k)} m_k^r (1 - M_k)^{s-r} \leqq V_k(r) \leqq \sum_{s=r}^{\infty} \binom{s}{r} W_s^{(k)} M_k^r (1 - m_k)^{s-r} \tag{2.4}$$

where $W_s^{(k)} = W_s(t_{k-1}, t_k)$. In fact, if r happenings are going on at time t, all of which started in the time interval ΔI_k, there must have been $s \geqq r$ events in this interval; now if a happening started exactly at time τ $(t_{k-1} \leqq \tau \leqq t_k)$, the probability that it will continue going on at time t is $\Phi(\tau, t-\tau)$; as we do not know the exact value of τ, only that it lies in ΔI_k, we can state only that this probability lies somewhere between m_k and M_k; similarly the probability that the happening considered is finished before t being equal to $F(\tau, t-\tau)$ with τ in ΔI_k, lies somewhere between $1 - M_k$ and $1 - m_k$.

Now let us introduce the functions

$$\chi^{(k)}(z) = \sum_{r=0}^{\infty} V_k(r) z^r \qquad (k = 1, 2, \ldots, n). \tag{2.5}$$

[1] If $F(\tau, x)$ does not depend on τ, the condition $\Phi(\tau, x) > 0$ can be dropped.

[2] The idea of the proof of Theorem 2 is the same as that applied in § 2 of [2]

It follows from (2. 5) and (1. 4) that for $0 \leqq z \leqq \frac{1}{2}$

$$\exp\left[\sum_{r=1}^{\infty} c_r(\Delta I_k)\left((m_k z+1-M_k)^r-1\right)\right] \leqq \tag{2.6}$$
$$\leqq \chi^{(k)}(z) \leqq \exp\left[\sum_{r=1}^{\infty} c_r(\Delta I_k)\left((M_k z+1-m_k)^r-1\right)\right].$$

(To ensure the convergence of the series in the exponent at the right of (2. 6), we choose n sufficiently large, to obtain[3] $M_k \leqq 2m_k$.) Denoting the mean value of $\zeta_{t_k}-\zeta_{t_{k-1}}$ by μ_k, we have by (1. 4)

$$\mu_k=\sum_{n=1}^{\infty} n W_n(t_{k-1}, t_k)=\left(\frac{\partial \varphi(s, t, z)}{\partial z}\right)_{z=1}=\sum_{r=1}^{\infty} r c_r(\Delta I_k) \tag{2.7}$$

(the existence of μ_k has been postulated!), we obtain easily

$$\chi^{(k)}(z)=\exp\left[-\psi(t_{k-1}, t_k, m_k z+1-M_k)+\vartheta \mu_k(M_k-m_k)\right] \tag{2.8}$$

with $|\vartheta| \leqq 2$.

Now let $p_N(t)$ denote the probability of exactly N happenings going on at time t. We have evidently

$$p_N(t)=\sum_{r_1+r_2+\ldots+r_n=N} V_1(r_1) V_2(r_2) \ldots V_n(r_n) \tag{2.9}$$

where the summation is extended over all *ordered* n-tuples of non-negative integers $(r_1, r_2, \ldots, r_n)$ satisfying $r_1+r_2+\cdots+r_n=N$. Let us put

$$\chi(z, t)=\sum_{N=0}^{\infty} p_N(t) z^N; \tag{2.10}$$

$\chi(z, t)$ is the generating function of the random variable η_t. According to (2. 9) we have

$$\chi(z, t)=\prod_{k=1}^{n} \chi^{(k)}(z) \tag{2.11}$$

and thus, in view of (2. 8)

$$\chi(z, t)=\exp\left[-\sum_{k=1}^{n} \psi(t_{k-1}, t_k, m_k z+1-M_k)+\vartheta' \sum_{k=1}^{n} \mu_k(M_k-m_k)\right] \tag{2.12}$$

$|\vartheta'| \leqq 2$. Now, if $n \rightarrow \infty$, the second member in the exponent at the right of (2. 12) tends to 0, while the limit of the first member is

$$\pi(z, t)=\sum_{r=1}^{\infty} \int_0^t c_r(\tau)\left[\Phi(\tau, t-\tau) z+1-\Phi(\tau, t-\tau)\right]^r d\tau. \tag{2.13}$$

Thus we obtain

$$\chi(z, t)=e^{\pi(z, t)}.$$

[3] We use here that $\Phi(\tau, x)>0$ and that $\Phi(\tau, x)$ is continuous.

By simple rearrangement we obtain

$$\pi(z, t) = \sum_{k=1}^{\infty} d_k(t)(z^k - 1) \tag{2. 15}$$

where $d_k(t)$ is defined by (2. 2); thus Theorem 2 is proved.

Let us mention that if $c_k(\tau) \equiv 0$ for $k = 2, 3, \ldots$, i. e. if the underlying process is an ordinary Poisson process, we have (cf. [2])

$$\pi(z, t) = (z - 1) \int_0^t c_1(\tau)\, \Phi(\tau, t - \tau)\, d\tau, \tag{2. 16}$$

hence η_t is distributed according to an ordinary Poisson distribution. More generally, let us call a composed Poisson distribution to be of degree D if $c_n(\tau) \equiv 0$ for $n > D$; it follows that the degree of the distribution of η_t is equal to that of ζ_t (this holds also for $D = \infty$!).

§ 3. Convergence to composed Poisson distributions

In this § we shall prove the following

THEOREM 3. *Let* $\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}$ *denote non-negative independent integer-valued random variables* ($n = 1, 2, \ldots$) *which are „infinitely small", that is, let us suppose*

$$\lim_{n \to \infty} \max_{1 \leqq k \leqq k_n} P(\xi_{nk} \neq 0) = 0. \tag{3. 1}$$

Let us put

$$\eta_n = \xi_{n1} + \xi_{n2} + \cdots + \xi_{nk_n}, \tag{3. 2}$$

further $p_{nks} = P(\xi_{nk} = s)$ *and* $c_{ns} = \sum_{k=0}^{k_n} p_{nks}$. *The necessary and sufficient condition for the convergence of the distribution functions of the sums* η_n *is the existence of a sequence of non-negative numbers* $c_1, c_2, \ldots, c_s, \ldots$ *with the following properties:* $\sum_{s=1}^{\infty} c_s$ *converges, and* $\sum_{s=1}^{\infty} c_s > 0$, *further*

$$\lim_{n \to \infty} \sum_{s=1}^{\infty} |c_{ns} - c_s| = 0. \tag{3. 3}$$

If (3. 3) *is satisfied, the distribution of* η_n *tends for* $n \to \infty$ *to the composed Poisson distribution function having the generating function*

$$\varphi(z) = \exp\left(\sum_{s=1}^{\infty} c_s (z^s - 1)\right). \tag{3. 4}$$

Proof. We shall deduce Theorem 3 from the following important theorem of B. V. GNEDENKO and A. N. KOLMOGOROFF (Theorem 1 of § 25 of [3]): The necessary and sufficient conditions for the existence of constants $A_n (n = 1, 2, \ldots)$ ensuring the convergence to a limiting distribution of the distributions of the sums $\eta_n = \xi_{n1} + \xi_{n2} + \cdots + \xi_{nk_n} - A_n$ of independent, infinitely small random variables, the distribution function of ξ_{nk} being denoted by $F_{nk}(x)$, are as follows: the existence of non-decreasing functions $M(u)$ $(-\infty < u < 0; M(-\infty) = 0)$ and $N(u)$ $(0 < u < +\infty; N(+\infty) = 0$ of bounded variation and of a constant σ such that

a) in every point of continuity of $M(u)$ resp. of $N(u)$ we have

$$\left\{ \begin{aligned} &\lim_{n\to\infty} \sum_{k=1}^{k_n} F_{nk}(u) = M(u) \qquad &\text{for } u<0, \\ &\lim_{n\to\infty} \sum_{k=1}^{k_n} (F_{nk}(u) - 1) = N(u) \qquad &\text{for } u>0, \end{aligned} \right. \tag{3.5}$$

b)

$$\left\{ \begin{matrix} \lim\limits_{\varepsilon\to\infty} \varlimsup\limits_{n\to\infty} \\ \lim\limits_{\varepsilon\to\infty} \varliminf\limits_{n\to\infty} \end{matrix} \right\} \sum_{k=1}^{k_n} \left[\int\limits_{|x|<\varepsilon} x^2 dF_{nk}(x) - \left(\int\limits_{|x|<\varepsilon} x\, dF_{nk}(x) \right)^2 \right] = \sigma^2. \tag{3.6}$$

In case the above conditions are satisfied, the constants A_n may be chosen so that

$$A_n = \sum_{k=1}^{k_n} \int\limits_{|x|<\beta} x\, dF_{nk}(x)$$

where β is an arbitrary positive number such that $-\beta$ and β are points of continuity of $M(u)$ resp. of $N(u)$. Denoting by $f(t)$ the characteristic function of the limiting distribution of η_n, we have (formula of P. LÉVY)

$$\begin{aligned} \log f(t) = i\gamma t - \frac{\sigma^2 t^2}{2} + \\ + \int_{-\infty}^{0} \left(e^{iut} - 1 - \frac{iut}{1+u^2} \right) dM(u) + \int_{0}^{\infty} \left(e^{iut} - 1 - \frac{iut}{1+u^2} \right) dN(u) \end{aligned} \tag{3.7}$$

where $M(u), N(u)$ and σ are defined as above and γ is a real constant.

In our case the condition that the variables ξ_{nk} are "infinitely small" is equivalent to the condition $\lim\limits_{n\to\infty} \min\limits_{1\le k\le k_n} p_{nk0} = 1$. We have further $F_{nk}(u) = 0$ for $u < 0$, (thus the first condition of **a**) is satisfied with $M(u) \equiv 0$) and $F_{nk}(u) = \sum\limits_{s<u} p_{nks}$ for $u > 0$, and therefore, in view of $\sum\limits_{s=0}^{\infty} p_{nks} = 1$,

$$\sum_{k=1}^{k_n} (F_{nk}(u) - 1) = -\sum_{s\ge u} c_{ns}. \tag{3.8}$$

Hence the second condition of **a**) is equivalent to the existence of the limits

$$\lim_{n\to\infty} D_{nu} = D_u \quad \text{with} \quad \lim_{u\to\infty} D_u = 0 \tag{3.9}$$

for $u=1,2,\ldots$ where $D_{nu}=\sum_{s\geqq u} c_{ns}$. (Clearly the sequence $D_1, D_2, \ldots, D_u, \ldots$ is non-increasing.) Condition **b)** is in our case automatically satisfied with $\sigma=0$, in view of the fact that for $\varepsilon<1$ all integrals figuring in (3. 6) vanish. For similar reasons, in case conditions (3. 9) are satisfied we may choose $A_n=0$ $(n=1,2,\ldots)$. Putting $C_u=D_u-D_{u+1}$ (clearly $c_u\geqq 0$ and $\sum_{u=1}^{\infty} c_u$ converges), it is evident that (3. 9) implies

$$\lim_{n\to\infty}(c_{ns}-c_s)=0 \qquad (s=1,2,\ldots). \tag{3. 10}$$

Of course the contrary of this is not valid, but if we replace the set of conditions (3. 10) by the single one

$$\lim_{n\to\infty}\sum_{s=1}^{\infty}|c_{ns}-c_s|=0, \tag{3. 11}$$

then it is easy to see that (3. 11) is equivalent to the set of conditions (3. 9). As a matter of fact, let us show that (3. 9) follows from (3. 11) and vice versa. If (3. 11) is satisfied, we have

$$|D_{nu}-D_u|=\left|\sum_{s\geqq u}(c_{ns}-c_s)\right|\leqq\sum_{s=1}^{\infty}|c_{ns}-c_s|, \tag{3. 12}$$

thus (3. 9) is valid for every $u=1,2,\ldots$. Conversely, from (3. 9) it follows

$$\sum_{s=1}^{\infty}|c_{ns}-c_s|= \\ =\sum_{s=1}^{\infty}(c_{ns}-c_s)+2\sum{}'(c_s-c_{ns})\leqq D_{n1}-D_1+2\sum_{s=N}^{\infty}c_s+2N\varepsilon_n^{(N)} \tag{3. 13}$$

where Σ' is extended only over those values of s for which $c_s-c_{ns}>0$, and $\varepsilon_n^{(N)}=\max_{s<N}|c_s-c_{ns}|$; thus

$$\sum_{s=1}^{\infty}|c_{ns}-c_s|\leqq D_{n1}-D_1+2D_N+2N\varepsilon_n^{(N)}. \tag{3. 14}$$

Clearly $\lim_{n\to\infty}\varepsilon_n^{(N)}=0$ for every fixed value of N, according to (3. 10). Now let us choose N sufficiently large to obtain $D_N<\frac{\varepsilon}{6}$ and then n_0 sufficiently large so as to ensure $|D_{n1}-D_1|<\frac{\varepsilon}{3}$ and $\varepsilon_n^{(N)}<\frac{\varepsilon}{6N}$ for $n\geqq n_0$. It follows that the right-hand side of (3. 14) is $<\varepsilon$ for $n>n_0$, which is equivalent to (3. 11). Therefore if and only if (3. 3) is valid, the conditions of the theorem of B. V. GNEDENKO and A. N. KOLMOGOROFF are satisfied with $M(u)\equiv 0$ and $N(u)=-\sum_{s\geqq u}c_s$, $\sigma=0$ and $A_n=0$ and thus the distribution function of the sum $\eta_n=\xi_{n1}+\xi_{n2}+\cdots+\xi_{nk_n}$ converges to a limiting distribution having the characteristic function $f(t)$, where

$$\log f(t) = i\gamma t + \int_0^\infty \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dN(u) = \sum_{s=1}^\infty c_s(e^{ist}-1) + it\left(\gamma - \sum_{s=1}^\infty \frac{sc_s}{1+s^2}\right).$$

As η_n assumes only non-negative integes values, and $\varliminf_{n\to\infty} P(\eta_n = 0) > 0$,[4] we must have (see § 4) $\gamma = \sum_{s=1}^\infty \frac{sc_s}{1+s^2}$ and thus

$$f(t) = \exp\left(\sum_{s=1}^\infty c_s(e^{ist}-1)\right)$$

which is equivalent to (3. 4), owing to $f(t) = \varphi(e^{it})$. Thus Theorem 3 is completely proved.

This limit theorem suggests new applications of composed Poisson distribution. For example, let us consider the mixture of two or more grained materials having different specific weights. In particular, suppose that there are only two materials and the ratio of their specific weights is 1 : 2. The specific weight of the mixture will be evidently a mean value of the specific weights of the components, the factors being proportional to the quantities (volumes) of the different components. But if we investigate the specific weight of small parts of the mixture, we shall find that it fluctuates around the mentioned value and we may ask about the distribution law of this specific weight. Clearly we can construct a simple urn-model which describes adequately the mentioned situation, and it follows by Theorem 3 that the distribution of the specific weight of a selected small portion of the mixture is approximately a composed Poisson distribution with generating function $\exp(c_1(z-1) + c_2(z^2-1))$. The same consideration can be applied to the specific weight of small parts of an alloy of two or more different metals, etc.

§ 4. Characterization of composed Poisson distributions

Finally let us prove the following theorem, which throws light on the above results.

[4] We have $P(\eta_n = 0) = \prod_{k=1}^{k_n} p_{nk0} = \prod_{k=n}^{k_n}\left(1 - \sum_{s=1}^\infty p_{nks}\right)$, as by

(3. 1) $\lim \max_{1\leq k\leq k_n}\left(\sum_{s=1}^\infty p_{nks}\right) = 0$ we have $\sum_{s=1}^\infty p_{nks} < \frac{1}{2}$ for $n \geqq n_0$,

and using $1 - x > e^{-2x}$ for $0 < x < \frac{1}{2}$, it follows

$$\varliminf_{n\to\infty} P(\eta_n = 0) \geqq \varliminf_{n\to\infty} e^{-2\sum_{s=1}^\infty c_{ns}} = e^{-2\sum_{s=1}^\infty c_s} > 0.$$

THEOREM 4. *The class of composed Poisson distributions can be characterized as the class of infinitely divisible distributions of non-negative, integer-valued random variables, which assume the value* 0 *with a probability* >0.

Proof. We start from the canonical form

$$\log f(t) = i\gamma t - \frac{\sigma^2 t^2}{2} + \tag{3.15}$$
$$+\int_{-\infty}^{0}\left(e^{iut}-1-\frac{iut}{1+u^2}\right)dM(u)+\int_{0}^{\infty}\left(e^{iut}-1-\frac{iut}{1+u}\right)dN(u)$$

(where γ and σ are real numbers, $M(u)$ and $N(u)$ non-decreasing functions of bounded variation in the intervals $(-\infty, 0)$ resp. $(0, +\infty)$, and $M(-\infty) = N(+\infty) = 0$) of the logarithm of the characteristic function of an infinitely divisible distribution. The logarithm of the characteristic function of a composed Poisson distribution is of the form

$$\log f(t) = \sum_{n=1}^{\infty} c_n (e^{int}-1) \quad \text{with} \quad c_n \geqq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} c_n < \infty. \tag{3.16}$$

Putting $\sigma = 0$, $M(u) \equiv 0$ in (3. 15) and choosing for $N(u)$ the following function: $N(u) = -\sum_{n \geqq u} c_n$ for $u > 0$ and putting $\gamma = \sum_{n=1}^{\infty} \frac{nc_n}{1+n^2}$, we conclude that *every composed Poisson distribution is infinitely divisible.* As a matter of fact this can be seen also by taking into account that any composed Poisson distribution is the convolution of a finite or an enumerably infinite number of Poisson distributions (see [1]). Conversely, let us suppose that the variable ξ assumes only non-negative integer values and its distribution is infinitely divisible, i. e. for any $n = 2, 3, 4, \ldots$ it can be represented in the form $\xi = \xi_1^{(n)} + \xi_2^{(n)} + \cdots + \xi_n^{(n)}$ where the variables $\xi_k^{(n)} (k = 1, 2, \ldots, n)$ are independent and equally distributed. Denoting by $f(t)$ the characteristic function of ξ, we know that $\log f(t)$ can be represented in the form (3. 15). Clearly $\sigma = 0$, because if not, the distribution function of ξ would be continuous. As a matter of fact, let us put

$$f(t) = f_1(t) f_2(t)$$

where $f(t) = e^{-\frac{\sigma^2 t^2}{2}}$. As $f_2(t)$ is the characteristic function of the normal distribution function $F_2(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{x} e^{-\frac{u^2}{2\sigma^2}} du$, if $F(x)$ denotes the distribution function of ξ and $F_1(x)$ the infinitely divisible distribution function whose characteristic function is equal to $f_1(t)$, we have

$$F(x) = \int_{-\infty}^{+\infty} F_2(x-y)\, dF_1(y).$$

Taking into account that $|F_2(a+h)-F_2(a)|\leqq\frac{|h|}{\sqrt{2\pi\sigma}}$, it follows that $|F(x+h)-F(x)|\leqq\frac{|h|}{\sqrt{2\pi\sigma}}$, which means that $F(x)$ is a continuous function; this contradicts our assumption that ξ assumes only non-negative integer values. But since

$$f(t)=\sum_{k=1}^{\infty}P_k e^{ikt} \qquad (P_k=P(\xi=k)), \tag{3.17}$$

we see that $f(2\pi)=1$, and hence the real part of $\log f(2\pi)$ must vanish. Therefore we have

$$\int_{-\infty}^{0}(\cos 2\pi u-1)\,dM(u)+\int_{0}^{\infty}(\cos 2\pi u-1)\,dN(u)=0. \tag{3.18}$$

Using the fact that $M(u)$ and $N(u)$ are non-decreasing and $\cos 2\pi u-1<0$ if $u\not\equiv 0 \bmod 1$, we obtain that $M(u)$ and $N(u)$ can increase only for negative resp. positive integer values of u. Thus putting $c_n=M(n+0)-M(n-0)$ for $n=-1,-2,\ldots$ and $c_n=N(n+0)-N(n-0)$ for $n=1,2,\ldots,$ we obtain

$$\log f(t)=it\gamma'+\sum_{n=-\infty}^{+\infty}{}' c_n(e^{int}-1) \tag{3.19}$$

(here and in what follows Σ' means that the summation is extended for every n except for $n=0$) and we have to put $\gamma'=\gamma-\sum_{n=-\infty}^{+\infty}{}'\frac{nc_n}{1+n^2}$. But it is easy to see that in case $c_{-1},c_{-2},\ldots,c_{-n},\ldots$ were not all equal to 0, ξ would assume also negative integer values; as a matter of fact, we have

$$\begin{aligned}\sum_{n=0}^{\infty}A_n e^{it(n-\gamma')}&=\exp\left(\sum_{n=-\infty}^{+\infty}{}' c_n(e^{int}-1)\right)=\\ &=\left[\sum_{k=0}^{\infty}\frac{\left(\sum_{n=-\infty}^{+\infty}{}' c_n e^{int}\right)^k}{k!}\right]\exp\left(-\sum_{n=-\infty}^{+\infty}{}' c_n\right)\end{aligned} \tag{3.20}$$

since $c_n\geqq 0$ for $n=\pm 1,\pm 2,\ldots,$ if $c_{-n}\neq 0$ for some $n>0$, we should obtain arbitrary large negative integer powers of e^{it} with positive coefficients at the right of (3.20), but not at the left of (3.20), which is a contradiction; thus $c_{-n}=0$ for $n=1,2,\ldots$. As we have supposed $A_0\neq 0$, we obtain that at the left-hand side of (3.20) the non-vanishing term containing the lowest power of e^{it} is $A_0e^{-it\gamma'}$. But at the right-hand side of (3.20) the term of lowest power in e^{it} is the constant $e^{-\sum_{1}^{\infty}c_n}\neq 0$; hence we must have $\gamma'=0$. Accordingly we obtain

$$\log f(t)=\sum_{n=1}^{\infty}c_n(e^{int}-1) \tag{3.21}$$

As we supposed that $\sum_{n=0}^{\infty} c_n = -N(0)$ is finite, it follows that (3. 20) is the characteristic function of a composed Poisson distribution; Theorem 4 is hereby proved. Let us mention the following

COROLLARY. *If a composed Poisson distribution $F(x)$ is the convolution of two infinitely divisible distributions, $F_1(x)$ and $F_2(x)$ which have positive jump at $x=0$, these must also be composed Poisson distributions of degree not exceeding that of $F(x)$.*

This is a generalization of a well-known fact concerning Poisson-distributions. (Cf. [3]). The proof is obvious.

It should be pointed out also that the theorem of § 3 of [1] is a simple consequence of Theorem 4.

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ОБОБЩЕННЫЕ РАСПРЕДЕЛЕНИЯ ТИПА ПУАССОНА, II

А. РЕНЬИ (Будапешт)

(Резюме)

Работа является продолжением совместной работы [1] Л. Яноши, Я. Ацел и автора настоящей статьи. Продолжаются исследования составных раипределений Пуассона, т. е. распределений, характеристическая функция которых имеет вид

$$\exp\left(\sum_{n=1}^{\infty} c_n\,(e^{int}-1)\right), \tag{1}$$

где $c_n \geqq 0\,(n=1,2,\ldots)$ и ряд $\sum_{n=1}^{\infty} c_n$ сходится. В § 1 найден общий вид нестационарных марковских стохастических процессов случайных событий, так как доказывается следующая

Теорема 1. Пусть ζ_t означает число событий, происходящих в интервале времени $(0, t)$, $(\zeta_0 \equiv 0)$, и предположим что выполнены следующие условия:

а) если $s_1 < t_1 \leqq s_2 < t_2 \leqq \cdots \leqq s \ < t_r$, то случайные величины $\zeta_{t_1}-\zeta_{s_1}, \zeta_{t_2}-\zeta_{s_2}, \ldots, \zeta_{t_r}-\zeta_{s_r}$ независимы,

в) пусть $W_k(s,t)$ означает вероятность события $\zeta_t-\zeta_s = k\,(k=0,1,2,\ldots, s<t)$ и предположим что для всякой $\varepsilon>0$ и $T>0$ найдется такая $\delta>0$, что если $s_1 < t_2 \leqq s_2 < t_2 \leqq \cdots \leqq s_r < t_r < T$ и

$$\sum_{j=1}^{r}(t_j-s_j)<\delta, \text{ то имеем } \prod_{j=1}^{r} W_0(s_j,t_j) > 1-\varepsilon.$$

Тогда характеристическая функция

$$\varphi(s,t,u)=\sum_{k=0}^{\infty} W_k(s,t)\,e^{iku}$$

имеет вид

$$\varphi(s,t,u)=\exp\left(\sum_{r=1}^{\infty}(e^{iru}-1)\int_s^t c_r(\tau)\,d\tau\right),$$

где $c_r(\tau)$ — неотрицательная, L-интегрируемая функция, и ряд $\sum_{r=1}^{\infty} c_r(\tau)$ сходится почти всюду, т. е. процесс $\{\zeta_t\}$ является нестационарным составным процессом Пуассона.

В § 2 исследуется следующая проблема: пусть каждое событие некоторого нестационарного составного процесса Пуассона является исходным пунктом некоторого другого события второго типа, которое продолжается в течение некоторого промежутка времени; продолжительность события второго рода, которое началось во время t, является случайной величиной с законом распределения $F(t,T)$. Положим $\Phi(t,T) = 1-F(t,T)$, и обозначим через η_t число событий второго рода, которые происходят в момент t и пусть $p_N(t)$ означает вероятность того что $\eta_t=N$ $(N=0,1,\ldots)$. Тогда имеет место следующая

Теорема 2. Если $\Phi(t,T)$ является непрерывной н положительной функцией, тогда положив

$$\chi(z,t)=\sum_{N=0}^{\infty} p_N(t)\,z^N$$

имеем

$$\chi(z, t) = \exp\left(\sum_{k=1}^{\infty} d_k(t)\,(z^k - 1)\right),$$

где

$$d_k(t) = \int_0^t \sum_{n=k}^{\infty} c_n(\tau) \binom{n}{k} (\Phi(\tau, t-\tau))^k (1 - \Phi(\tau, t-\tau))^{n-k} d\tau,$$

т. е. распределение случайной величины η_t является составным распределением Пуассона Частный случай этой теоремы доказан в работе [2].

В § 3 доказана следующая

Теорема 3. Пусть $\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}$ — независимые случайные величины, принимающие лишь неотрицательные целые значения, и предположим, что величины ξ_{nk} „бесконечно малы", т. е. $\lim\limits_{n\to\infty} \max\limits_{1\leqq k\leqq k_n} P(\xi_{nk} \neq 0) = 0$.

Положим
$$\chi_n = \sum_{k=1}^{k_n} \xi_{nk}, \qquad p_{nks} = P(\xi_{nk} = s) \quad \text{и} \quad c_{ns} = \sum_{k=0}^{k_n} p_{nks}$$

для того чтобы распеределине от χ_n сходилось бы к некоторому предельному распределению при $n \to \infty$, необходимо и достаточно существование таких постоянных $c_s (s = 1, 2, \ldots)$, что $\lim\limits_{n\to\infty} \sum\limits_{=1}^{\infty} |c_{ns} - c_s| = 0$. Если это условие выполняется, то распределие от χ_n является составным распределением Пуассона, характеристическая функция котогоро есть

$$\exp\left(\sum_{s=1}^{\infty} c_s (e^{is} - 1)\right).$$

Доказательство этой теоремы опирается на одну теорему Б. В. Гнеденко и А. Н. Колмогорова [3].

В § 4 составные распределения Пуассона характеризуются как безгранично делимые распределения неотрицательных целочисленных случайных величин, которые принимают значение 0 с положительной вероятностью.

В работе указаны некоторые возможные физические и технические применения теории составных распределений Пуассона, например при исследовании тока в электронных лампах, в области явлений радиоактивного распада, при изучении нагрузки телефонных станций, при определении распределения значений удельного веса в сплавах и т. д.