# ON A CONJECTURE OF H. STEINHAUS

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**Introduction.** In a lecture [1] in 1937 H. Steinhaus formulated the following conjecture: if a system

$$(1) \qquad \{f_n(x)\} \qquad (n=1,2,\ldots)$$

consisting of a finite or enumerably infinite number of stochastically independent [1]) measurable functions defined in a finite interval $(a,b)$, *is saturated with respect to independence*, i. e. if there exists no measurable function $g(x)$, which is not constant almost everywhere, and which could be added to the system (1) without violating the independence of the system, then the system of functions

$$(2) \qquad \{f_1^{m_1}(x)\cdot f_2^{m_2}(x)\cdot\ldots\cdot f_n^{m_n}(x)\}$$

where $m_1, m_2, \ldots, m_n$ run independently over all non-negative integers, and $n=1,2,3,\ldots$, is complete in the interval $(a,b)$. He mentioned some suggestive examples, in which the above

[1]) In what follows the term "stochastically" will be generally omitted; when we speak of independence we always mean stochastic independence. The independence of measurable functions was defined first by Steinhaus [2], the equivalence of his definition with the definition of A. N. Kolmogoroff (*Über die Grundlagen der Wahrscheinlichkeitsrechnung*, Ergebnisse d. Math. (1933)), in answer to a question of E. Marczewski, for the case of Lebesgue measure has recently been proved by S. Hartman (Colloquium Math. I, 1948, p. 19-22). In the general case the two definitions do not agree (see J. L. Doob, ibidem, p. 216-217). By the assertion that (1) is a system of independent functions, we mean that these functions are independent in their totality, i. e. not only every pair of these functions, but also every 3, every 4,...., every $n$-tuple of functions of the systems are independent of one another.

assertion is valid, for instance when (1) is the system of Rademacher functions

$$R_n(x) = \text{sg} \sin(2^{n-1}\pi x) \qquad (n=1,2,3,\dots) \tag{3}$$

in which case (2) is the well known Walsh system which is known to be complete in the interval (0,1). Another example is as follows: a function which itself forms a system, saturated with respect to independence, will be called a *universal function*. Clearly $f_1(x)=x$ is a universal function; in this case (2) is the system $1, x, x^2, \dots, x^n, \dots$ which is known to be complete.

In spite of these suggestive examples, the conjecture is not true in general. Let us consider for instance the function

$$h(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \leqslant 1. \end{cases}$$

As $h(x)$ is monotonic in $(0,\frac{1}{2})$ and outside this interval it does not again take on the values which it takes on in $(0,\frac{1}{2})$, according to a theorem of Ottaviani [3], $h(x)$ is a universal function. On the other hand it is easy to see that the system $\{h^n(x)\}$ is not complete, as $h^n(x)$ is for every value of $n$ orthogonal to the function $g(x)$, which is defined as follows:

$$g(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \leqslant \frac{3}{4}, \\ -1 & \text{if } \frac{3}{4} < x \leqslant 1. \end{cases}$$

Nevertheless, the conjecture of Steinhaus is valid for a very general class of saturated systems of independent functions.

The purpose of the present paper is to prove the above mentioned conjecture of Steinhaus under very general conditions. In a lecture [4] held at the First Hungarian Mathematical Congress in 1950 at Budapest, I proved the same conjecture under more restrictive conditions; in the meantime I succeeded in eliminating some of these conditions. In § 1 I shall prove a general theorem which makes it possible to establish the completeness of system (2) under a condition in which the notion of independence does not figure at all; this is a theorem of the theory of real functions, nevertheless it contains all known cases in which the conjecture of Steinhaus is valid. We introduce the notion of *maximal systems*;

the system (1) of real measurable functions defined in $(a,b)$ is called maximal, if for different values of $x$ the sequences $\{f_n(x)\}$ are different (except for a set of measure 0 of values of $x$). Thus we call the system (1) maximal if there exists a set $Z$ of measure 0, such that if $x_1$ and $x_2$ do not belong to $Z$ and $f_n(x_1)=f_n(x_2)$ for all values of $n=1,2,3,\ldots$, then $x_1=x_2$. The content of the theorem is that if system (1) is maximal, system (2) is complete. All examples of systems (1) of independent functions, for which system (2) is complete, are maximal systems. It is also interesting to note that a maximal system of independent functions is always saturated with respect to independence (Lemma 3). An interesting example of a maximal system in the interval $(0,2\pi)$, is the system consisting of the two functions $\sin x$, $\cos x$; in this case system (2) — after omitting those functions which are linearly dependent on the others — is the system $\{\cos^n x, \sin x\cdot\cos^n x;\ n=0,1,2,\ldots\}$ which is equivalent [2]) to the system $\{\cos nx, \sin(n+1)x;\ n=0,1,2,\ldots\}$, i. e. to the well known trigonometrical system, which is known to be complete. Further $\cos x$ in itself forms a maximal system in $(0,\pi)$ and thus the system $\{\cos^n x\}$, — or, which is the same, the system $\{\cos nx\}$ — is complete in $(0,\pi)$ (see [5]).

The example of the system $\{\sin x, \cos x\}$ may be generalized as follows: if $u=f(x)$ and $v=g(x)$ $(a\leqslant x\leqslant b)$ are the parametric equations of a curve in the $(u,v)$-plane, which does not intersect itself (or the set of multiple points of which corresponds to a set of measure zero of values of $x$) then the system $\{f^n(x)\cdot g^m(x);\ n,m=0,1,2,\ldots\}$ is complete (we suppose $f(x)$ and $g(x)$ to be bounded and measurable). The case in which system (1) consists only of a single function, is not completely trivial either: in this case the theorem asserts that if $f(x)$ is bounded and measurable in $(a,b)$, further if $y=f(x)$ establishes a one-to-one mapping of the interval $(a,b)$ onto a set on the $y$-axis, then the system $\{f^n(x)\}$ is complete; this is Lemma 2 of the present paper; it is stated in the form of a Lemma, though it is a special case of our theorem, because

[2]) We call two systems of functions *equivalent*, if the sets of linear combinations of the two systems are identical. Clearly if a system is complete, any system equivalent to it is also complete.

the proof of the general case is based on this special case. A function which forms a maximal system in itself, will be called a *maximal function.*

The most important step which leads from Lemma 2 to our theorem is the application of an idea of A. N. Kolmogoroff; he used this idea to prove a theorem on conditional mean values in [6] in which he generalizes an earlier theorem of mine [7] on the invariance of the central limit theorem of probability theory, with respect to the change of measure on the underlying field of probability[3]).

**§ 1. Proof of the theorem on complete systems.** We begin by proving the following

**Lemma 1.** *Let $f(x)$ denote a measurable, bounded function, defined in the interval $(0,1)$[4]), with values belonging to the same interval. If $g(x)$ is any bounded Baire function in $(0,1)$ and $A\geqslant 1$, for any $\varepsilon>0$ a polynomial $P(x)$ can be found such that*

$$\int_0^1 |g(f(x))-P(f(x))|^A\,dx<\varepsilon. \tag{1.1}$$

Proof. First we prove (1.1) for the case in which $g(x)=1$ if $0\leqslant x\leqslant u<1$ and $g(x)=0$ for $u<x<1$. Given $\varepsilon>0$, we can find a polynomial $P(x)$ with the following properties: 1. $0\leqslant P(x)\leqslant 1$, 2. $|g(x)-P(x)|<\varepsilon$ if $0\leqslant x\leqslant u$ and if $u+\varepsilon\leqslant x\leqslant 1$.

For instance, the polynomial

$$P(x)=\frac{\int\limits_0^{u+1/\sqrt[3]{n}}(1-(x-t)^2)^n\,dt}{\int\limits_{-1}^{+1}(1-t^2)^n\,dt} \tag{1.2}$$

if $n$ is chosen sufficiently large, has all the required properties. It follows, that

$$\int_0^1 |g(f(x))-P(f(x))|^A\,dx\leqslant\varepsilon^A+|E(\varepsilon)|\cdot C, \tag{1.3}$$

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[3]) I am thankful to Á. Császár for a valuable remark, which helped me to simplify my proof.

[4]) Throughout this paper we consider the interval (0,1), but evidently all our theorems are valid for any finite interval.

where $E(\varepsilon)$ denotes the set of those points for which $u<f(x)<u+\varepsilon$ and $|E(\varepsilon)|$ its Lebesgue measure, and $C$ is a positive constant. As evidently $\lim_{\varepsilon\to 0} E(\varepsilon)=0$, (1) is proved for the functions mentioned. Applying the well known inequality of H. Minkowski:

$$(1.4)\quad \int_0^1 |a(x)+b(x)|^A\,dx \leqslant \Big[\Big(\int_0^1 |a(x)|^A\,dx\Big)^{\frac{1}{A}}+\Big(\int_0^1 |b(x)|^A\,dx\Big)^{\frac{1}{A}}\Big]^A$$

it follows that if Lemma 1 holds for $g(x)=g_k(x)$, $k=1,2,\dots,m$, it also holds for $g(x)=\sum_{k=1}^{m} c_k g_k(x)$. Thus Lemma 1 holds for any step function. Using the same inequality it follows that if Lemma 1 holds for $g(x)=g_n(x)$, $n=1,2,\dots$, and

$$(1.5)\qquad \lim_{n\to\infty}\int_0^1 |g_n(f(x))-g^*(f(x))|^A\,dx=0$$

it holds for $g^*(x)$ also. But as for any continuous function, and therefore for any bounded Baire function $g^*(x)$, a sequence of step-functions $g_n(x)$ can be found such that (1.5) holds, it follows that Lemma 1 holds for any bounded Baire function $g(x)$.

Now we can prove

**Lemma 2.** *If $f(x)$ is a measurable, bounded and maximal function in the interval $(0,1)$ and $0\leqslant f(x)\leqslant 1$, the set of functions*

$$(1.6)\qquad \{f^n(x)\}\qquad (n=0,1,2,\dots)$$

*is closed in the space $L^2$.*

Proof. Let us suppose first that $f(x)$ is a Baire function. Let $G(x)$ denote any bounded Baire function, in the interval $(0,1)$. Since $f(x)$ is maximal, it follows that $f^{-1}(y)$ and thus $G(f^{-1}(y))=g(y)$ are also bounded Baire functions [5]), and applying Lemma 1 with $A=2$ to $g(x)$, since $g(f(x))=G(x)$, it follows that for any $\varepsilon>0$ a polynomial $P(x)$ can be found such that

$$(1.7)\qquad \int_0^1 |G(x)-P(f(x))|^2\,dx<\varepsilon.$$

[5]) For those values of $y$ which do not belong to the set of values of $f(x)$, we define $f^{-1}(y)=0$.

As to any measurable function $F(x)$ of the class $L^2$ a bounded Baire function $G(x)$ can be found such that $\int_0^1 (F(x)-G(x))^2 dx < \varepsilon$, Theorem 1 holds if $f(x)$ is a Baire function. But as the integral (1.7) is not changed if the value of $f(x)$ is changed on a set of measure zero, Theorem 1 is proved.

Now we prove our

**Theorem.** *Let $\{f_n(x)\}$ be a finite or enumerably infinite maximal system of bounded measurable functions in the interval $(0,1)$, then the set of functions*

$$f_1^{m_1}(x) \cdot f_2^{m_2}(x) \cdot \ldots \cdot f_n^{m_n}(x) \quad (m_k = 0,1,2,\ldots;\ k=1,2,\ldots,n;\ n=1,2,\ldots) \tag{1.8}$$

*is complete in $L^2$.*

**Proof.** We may evidently suppose that $0 \leqslant f_n(x) \leqslant 1$, further that all $f_n(x)$ are Baire functions. Let us put

$$\varphi_1(x) = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{2^{2n-1}}, \quad \text{where} \quad x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{2^n}, \tag{1.9}$$

is the dyadic expansion of $x$, i. e.

$$\varepsilon_n(x) = \begin{cases} 1 & \text{if } \frac{1}{2} \leqslant (2^{n-1}x) < 1, \\ 0 & \text{if } 0 \leqslant (2^{n-1}x) < \frac{1}{2}, \end{cases}$$

where $(z)$ denotes the fractional part of $z$.

Further let us define $\varphi_k(x) = \varphi_{k-1}(\varphi_1(x))$ for $k=2,3,\ldots,$ and

$$f(x) = \sum_{k=1}^{\infty} \frac{\varphi_k(f_k(x))}{2^{2^{k-1}-1}}. \tag{1.10}$$

As $\varphi_1(x)$ is a Baire function, $\varphi_k(f_k(x))$ and thus $f(x)$ is also a Baire function. It is easy to see further that $f(x)$ is a maximal function [6]. As a matter of fact, if $N = 2^{r-1}(2s-1)$ we have $\varepsilon_N(f(x)) = \varepsilon_s(f_r(x))$; thus if $x_1 \neq x_2$, there exists (except when $x_1$ or $x_2$

[6]) The introduction of a single function of one real variable, which is maximal if and only if system (1) is maximal, is the idea of A. N. Kolmogoroff referred to in the introduction.

belongs to a certain set $Z$ of measure 0) at least one value of $r$ for which $f_r(x_1) \neq f_r(x_2)$, and thus at least one value of $s$ for which $\varepsilon_N(f(x_1)) = \varepsilon_s(f_r(x_1)) \neq \varepsilon_s(f_r(x_2)) = \varepsilon_N(f(x_2))$, where $N = 2^{r-1}(2s-1)$; thus we obtain $f(x_1) \neq f(x_2)$. It follows by Lemma 2, that for an arbitrary $F(x) \in L^2$ for any $\varepsilon > 0$ a polynomial $P(x)$ can be found such that

$$\int_0^1 (F(x) - P(f(x)))^2 dx < \varepsilon. \tag{1.11}$$

Let us put

$$S_N(x) = \sum_{k=1}^{N} \frac{\varphi_k(f_k(x))}{2^{2^{k-1}-1}}.$$

As we have $0 \leqslant f(x) \leqslant 1$, $0 \leqslant S_N(x) \leqslant 1$ and

$$|f(x) - S_N(x)| < \frac{1}{2^{2^N}} \tag{1.12}$$

for all values of $x$, it follows, that for any $\varepsilon > 0$ a polynomial $P(x)$ and an integer $N$ can be found, such that

$$\int_0^1 (F(x) - P(S_N(x)))^2 dx < 4\varepsilon. \tag{1.13}$$

But again using Lemma 1, we can find for any $A > 0$ and $\delta > 0$ polynomials $P_k(x)$ $(k = 1, 2, \ldots, n)$ such that

$$\int_0^1 |\varphi_k(f_k(x)) - P_k(f_k(x))|^A dx < \delta^A \quad \text{for } k = 1, 2, \ldots, N \tag{1.14}$$

and thus

$$\int_0^1 |\varphi_k(f_k(x)) - P_k(f_k(x))|^r dx < \delta \quad \text{for } r \leqslant A \quad (k = 1, 2, \ldots, N). \tag{1.15}$$

Choosing for $A$ the degree of $P(x)$, after some calculation by putting

$$\sigma_N(x) = \sum_{k=1}^{N} \frac{P_k(f_k(x))}{2^{2^{k-1}-1}}, \tag{1.16}$$

we obtain:

$$\int_0^1 |P(s_N(x)) - P(\sigma_N(x))|^2 dx < C\delta, \tag{1.17}$$

where $C$ is a constant which depends only on $P(x)$. Choosing $\delta=\varepsilon/c$ we obtain

$$(1.18)\qquad \int_0^1 |F(x)-P(\sigma_N(x))|^2\,dx<9\varepsilon.$$

As $P(\sigma_N(x))$ is of the form

$$(1.19)\qquad \sum_{m_1=0}\sum_{m_2=0}\cdots\sum_{m_N=0} c_{m_1 m_2 \ldots m_N} f_1^{m_1}(x)\cdot f_2^{m_2}(x)\cdot\ldots\cdot f_N^{m_N}(x),$$

i. e. is a finite linear combination of the functions (1.8) it fol lows that the system (1.8) is closed, and thus complete in $L^2$.

**§ 2. Some remarks on independent functions.** We first prove

**Lemma 3.** *If the system of independent functions* $\{f_n(x)\}$ *is maximal, it is saturated with respect to independence.*

**Proof.** We may suppose here also that the functions $f_n(x)$ are Baire functions. Let us suppose, that there exists a function $g(x)$ of the class $L^2$ such that the system $\{g(x); f_n(x)\}$ is a system of independent functions. Then the function $g(x)$ is independent of $\varphi_k(f_k(x))$ for $k=1,2,\ldots$ (note that $\varphi_k(x)$ is a monotonic function!) and thus $g(x)$ is independent of

$$(2.1)\qquad f(x)=\sum_{k=1}^{\infty}\frac{\varphi_k(f_k(x))}{2^{2^{k-1}-1}}$$

also. Thus the functions $f^n(x)$ $(n=0,1,2,\ldots)$ are all independent of $g(x)$, and we have (see [2]) putting $\gamma=\int_0^1 g(x)\,dx$

$$(2.2)\qquad \int_0^1 f^n(x)(g(x)-\gamma)\,dx=\int_0^1 f^n(x)\,dx\cdot\int_0^1 (g(x)-\gamma)\,dx=0$$

for $n=0,1,2,\ldots$ Clearly $f(x)$ is a maximal function and therefore the system $\{f^n(x)\}$ is closed, and thus complete. This implies that $g(x)-\gamma$ is almost everywhere equal to 0, and thus $g(x)\equiv\gamma$ almost everywhere. This proves Lemma 3.

Thus we have deduced the completeness of system (2) from an assumption (that of maximality) regarding system (1), which in the case of independent functions is somewhat stronger than the assumption that (1) is saturated whith respect to independence.

The following problem still remains unsolved:

What are the necessary and sufficient conditions regarding the system (1) of independent functions, which ensure the completeness of system (2)?

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