ON PROJECTIONS OF PROBABILITY DISTRIBUTIONS

By

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J. RADON, in a paper [5] published in the year 1917, solved the following problem: It is to determine a continuous function $f(x, y)$, defined in a bounded domain K of the (x, y) plane, if given the values of the integral of this function along every chord of the domain K. From his results it follows, in particular, the following

THEOREM R. *If K is a bounded domain of the (x, y) plane, and the integral of the continuous function $f(x, y)$ vanishes along every chord of the domain K, then $f(x, y)$ is identically equal to zero.*

Since that this theorem has been independently re-discovered by many authors. Thus, it has been proved by H. STEINHAUS in 1941 in a lecture held at the conference of the University of Lwow;[1] at that time, Prof. H. STEINHAUS was unaware of the results of J. RADON. Recently he found this paper and kindly called my attention to it. My attention was called to the present problem by G. HAJÓS who raised the same problem in connection with a conjecture of S. TARSKI [7], which has been proved in the meantime by TH. BANG [1], [2]. The theorem of BANG reads as follows: If a *convex* domain K is covered by n parallel strips $S_1, S_2, \ldots, S_n$, the breadths of which are $d_1, d_2, \ldots, d_n$, respectively, then the sum $\sum_{k=1}^{n} d_k$ is greater than or equal to the breadth d of the domain K. Before making clear the connection between Theorem R and that of BANG, let us make the following remark: if for some domain K there exists a non-negative integrable function $f(x, y)$ whose integral along every chord of K is equal to 1, the statement of BANG's theorem follows easily for this domain, because if the strips $S_1, S_2, \ldots, S_n$ cover K, we have

$$(1) \qquad \sum_{k=1}^{n} d_k = \sum_{k=1}^{n} \iint_{S_k} f(x, y)\, dx\, dy \geqq \iint_{K} f(x, y)\, dx\, dy = d.$$

[1] Cf. [6] where a short summary of the lecture can be found.

Such a function is known only for the circle; if K is the circle $x^2+y^2=1$, the function

$$f(x, y)=\frac{1}{\pi\sqrt{1-x^2-y^2}} \tag{2}$$

has the required properties. To prove this, it suffices, by reasons of symmetry, to consider only chords which are parallel to the y-axis; for the chord $x=a$, for example, we have

$$\frac{1}{\pi}\int\limits_{-\sqrt{1-a^2}}^{+\sqrt{1-a^2}}\frac{dy}{\sqrt{1-a^2-y^2}}=\frac{1}{\pi}\int\limits_{-1}^{+1}\frac{dz}{\sqrt{1-z^2}}=1. \tag{3}$$

This can be proved also without calculations by means of the well known geometrical fact that the surface of a segment of the sphere of radius 1 depends only on the height of the segment.

It is easy to see that such a function can exist only for domains K of constant breadth. As a matter of fact, for every position of the coordinate system we have

$$\iint\limits_K f(x, y)\,dx\,dy=\int\limits_K dy=\int\limits_K dx, \tag{4}$$

i. e. the breadth of the domain in the direction of the coordinate axes is equal to the constant $\iint\limits_K f(x, y)\,dx\,dy$ which is independent of the direction of the coordinate axes. It is not known whether there exists actually such a function for domains of constant breadth other than the circle.

Now we turn to considering the connection of Theorem R with TARSKI's conjecture. This consists in that if a function $f(x, y)$ having the property that its integral is constant along every chord of K exists at all for some domain K, one may ask whether it is unique or not. Theorem R shows that (if the continuity of $f(x, y)$ is also required) $f(x, y)$ is unique.[2] Theorem R has been independently re-discovered and generalized by I. SZARSKI and T. WAZEWSKI in their paper [8], in which a very simple proof of this theorem can be found. Further generalizations of Theorem R will be included in a paper of I. MIKUSINSKI and C. RYLL-NARDZEWSKI to be published in the *Studia Mathematica*, and in a paper of W. WOLIBNER under preparation. The purpose of the present paper is also to give a new proof and some generalizations (different from those already mentioned) of Theorem R.

It will be shown that the whole problem belongs essentially to probability theory, and can be attacked by analytical methods of probability theory, namely by the application of the theorem of unicity concerning characteristic functions.

[2] If continuity is not supposed, $f(x, y)$ can be modified on an arbitrary set, the common part of which with every straight line has the linear measure 0.

We show namely that Theorem R is a consequence of a theorem of H. CRAMÉR and H. WOLD [4], which can be formulated as follows:

THEOREM CW. *Every probability distribution on the plane is uniquely determined by the totality of its linear projections.*

Clearly, this theorem can be formulated also for mass distributions instead of probability distributions.

Chapter I contains the proof — which is essentially that of H. CRAMÉR and H. WOLD, and is included in the paper only to make it self-contained — of Theorem CW mentioned just now, as well as of a generalization of this theorem for spaces of 3 or more dimensions (Theorem CW′), further the proof of the theorem (Theorem 1) that, for a broad class of distributions, the knowledge of an arbitrary infinite set of different projections already determines the distribution uniquely. It follows from this theorem that to ensure that the continuous function $f(x, y)$ defined in the bounded domain K vanishes identically, it suffices to suppose that its integral vanishes along every chord parallel to some line belonging to an arbitrary enumerable infinite set of straight lines (Theorem 2). The paper leaves open the question whether or not this is true for every distribution.

Chapter II is devoted to the study of finite distributions, i. e. — using the terminology of mass distributions — of distributions consisting of a finite number of mass points, that is, points in which positive masses are concentrated. It has been conjectured by the author and proved by G. HAJÓS that a distribution consisting of n mass points in the plane is uniquely determined by $n+1$ arbitrary projections, but n projections are not always sufficient to determine the distribution (Theorem 3). The proof of this theorem is included in the paper with the kind permission of G. HAJÓS. We shall show that the same is true for n equal mass points in the space (Theorem 4).

The author expresses his sincere thanks to H. STEINHAUS, T. WAZEWSKI, M. FISZ and G. HAJÓS for their valuable remarks.

Chapter I

The mentioned theorem of J. RADON can be formulated also in the following equivalent form:

THEOREM R′. *A continuous and non-negative function $f(x, y)$ defined in the convex domain K, is uniquely determined if the value of its integral along every chord of K is given.*

Let us show that Theorem R′ follows from Theorem R, and conversely. If the value of the integral of the non-negative and continuous function $f(x, y)$ is the same along every chord as the value of the integral of the continuous and non-negative function $g(x, y)$, then the integral of $f(x, y)-g(x, y)$ vanishes along every chord and thus, according to Theorem R, we have

$$f(x, y) \equiv g(x, y).$$

Thus Theorem R′ follows from Theorem R. On the other hand, if the integral of the continuous function $f(x, y)$ vanishes along every chord of K, let us put

$$f_1(x, y) = \begin{cases} f(x, y) & \text{if } f(x, y) \geqq 0 \\ 0 & \text{if } f(x, y) < 0 \end{cases}$$

and

$$f_2(x, y) = f_1(x, y) - f(x, y);$$

it follows that $f_1(x, y)$ and $f_2(x, y)$ are continuous and non-negative, and the integrals of $f_1(x, y)$ and $f_2(x, y)$ have the same value for every chord of K. Hence from Theorem R′ we conclude that $f_1(x, y) \equiv f_2(x, y)$ and therefore we have $f(x, y) \equiv 0$; that is, Theorem R follows from Theorem R′.

Now instead of supposing that the integral of $f(x, y)$ is known along every straight line, we may suppose that the value of the integral

$$(1.2) \qquad I(H) = \iint_{HK} f(x, y)\, dx\, dy$$

is known for every half-plane H where HK denotes the common part of the domain K and the half-plane H. In fact, if $I(H)$ is known for every half-plane, then the value of the integral

$$(1.3) \qquad I(S) = \iint_{SK} f(x, y)\, dx\, dy$$

is known for every parallel strip S, and thus the value of the integral of $f(x, y)$ along every chord l can be calculated by means of the limit relation

$$(1.4) \qquad i(l) = \int_l f(x, y)\, ds = \lim_{\Delta \to 0} \frac{1}{\Delta} \iint_{S_\Delta K} f(x, y)\, dx\, dy$$

where S_Δ is a parallel strip whose mid-line is l and whose breadth is Δ. Conversely, if $i(l)$ is known for every chord l, $I(H)$ can be calculated for every half-plane as $I(H) = \int_{-\infty}^{+\infty} i(l_x)\, dx$ where l_x denotes a chord which is parallel to the boundary line of H and which cuts the perpendicular to this line through the origin at a point having the abscissa x on this line. Thus we may suppose, instead of that $i(l)$ is known for every chord l, that $I(H)$ is known for every half-plane H.

Now the first step of generalization consists in that we omit the restriction of $f(x, y)$ being defined in a bounded domain, and consider functions $f(x, y)$ defined on the whole plane, but suppose that the integral $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)\, dx\, dy$ is finite. Without restricting generality we may suppose that

(1.5) $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1.$$

The second step in the generalization consists in that we omit the restriction that the non-negative function $f(x,y)$ should be continuous, and suppose only that it is L-integrable. Thus we may consider $f(x,y)$ as the density function of a probability distribution, and ask whether the values of the integral of this density function for every half-plane determine uniquely the density function, — or what is the same — the corresponding distribution function

(1.6). $$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,dv\,du.$$

Let l denote an arbitrary straight line through the origin, and let H_p denote the half-plane whose boundary line is perpendicular to l and cuts l in a point whose coordinate on l is equal to p. Clearly,

(1.7) $$F_l(p) = I(H_p) = \iint_{H_p} f(x,y)\,dx\,dy$$

as a function of p is nothing else than the distribution function of the projection on l of a random point of the plane whose distribution function is defined by (1.6). In what follows we shall call the linear distribution on l with the distribution function $F_l(p)$ the projection on l of the distribution on the plane with the distribution function $F(x,y)$. The last step of generalization consists in that we consider also distributions which have no density function and prove the following

THEOREM CW. *Let $F(x,y)$ denote the distribution function of an arbitrary probability distribution on the plane, and let us suppose that the projection of this distribution is known on every straight line l through the origin, i. e. that*

(1.8) $$F_{l_\varphi}(p) = \iint_{x\cos\varphi + y\sin\varphi \leqq p} dF(x,y)$$

is known as a function of p for every value of φ $(0 \leqq \varphi < \pi)$ where φ denotes the angle between the straight line l_φ and the x-axis. Then $F(x,y)$ is uniquely determined for every value of x and y.[3]

As it has been remarked in the introduction, this theorem is due to H. CRAMÉR and H. WOLD. We reproduce the simple proof of this theorem to make the paper self-contained.

PROOF. In what follows we shall denote by $M(\zeta)$ the mean value of a random variable ζ, and by $\Pr(A)$ the probability of the event A.

[3] The uncertainity of the value of $F(x,y)$ at its points of discontinuity does not come in, since we suppose (as usual) that $F(x,y)$ is continuous to the left as a function of x as well as a function of y.

Let (ξ, η) denote the coordinates of a random point on the plane having the distribution function $F(x, y)$. Let us denote by $\psi(u, v)$ the characteristic function of the point (ξ, η) (or, in other words, of the probability distribution with the distribution function $F(x, y)$), i. e. we put

$$(1.9) \qquad \psi(u, v) = M(e^{i(u\xi + v\eta)}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(ux+vy)} dF(x, y).$$

The projection of the point (ξ, η) on the line l_φ has the coordinate $\xi \cos\varphi + \eta \sin\varphi = \zeta_\varphi$ and thus $F_{l_\varphi}(p)$ is the distribution function of ζ_φ. If $F_{l_\varphi}(p)$ is known as a function of p, its characteristic function

$$(1.10) \qquad \psi_\varphi(t) = M(e^{it\zeta_\varphi}) = \int_{-\infty}^{+\infty} e^{itp} dF_{l_\varphi}(p)$$

is also known. But by (1.9) and (1.10) we have

$$(1.11) \qquad \psi_\varphi(t) = M(e^{it\zeta_\varphi}) = M(e^{it(\xi\cos\varphi + \eta\sin\varphi)}) = \psi(t\cos\varphi, t\sin\varphi).$$

Thus it follows that, for every real value of u and v, we have

$$(1.12) \qquad \psi(u, v) = \psi_{\operatorname{arc\,tg}\frac{v}{u}}(\sqrt{u^2+v^2}).$$

Hence $\psi(u, v)$ is known for every real value of u and v. As it is well known that a distribution function is uniquely determined by means of its characteristic function,[4] Theorem CW is completely proved.

Using the same method, the following theorem can also be proved.

THEOREM CW'. *A probability distribution in the n dimensional space is uniquely determined by its projections on such a set of subspaces of* $1, 2, \ldots, (n-1)$ *dimensions which together cover the whole space.*

Thus, for instance, a probability distribution in the space of 3 dimensions is uniquely determined by its projections on every straight line passing through the origin, or by its projections on every plane passing through a given line, or else by its projections on any collection of straight lines and planes which cover together the whole space. The proof of Theorem CW gives at the same time a criterion for a set of distribution functions $F_{l_\varphi}(p)$ to be the projections of a plane distribution. Clearly, the necessary and sufficient condition of this consists in that $\psi_{\operatorname{arc\,tg}\frac{v}{u}}(\sqrt{u^2+v^2})$ shall be a characteristic function where

$$(1.13) \qquad \psi_\varphi(t) = \int_{-\infty}^{+\infty} e^{itp} dF_{l_\varphi}(p).$$

By the same method we can prove also the following

THEOREM 1. *If the point (ξ, η) is contained with probability 1 in a circle $\xi^2 + \eta^2 \leq R^2$ and the distribution of the projection of the point (ξ, η) is given*

[4] See e. g. [8], p. 101.

on an arbitrary infinite set of straight lines through the origin, i. e. the distribution function $F_{\zeta_\varphi}(p)$ of the random variable $\zeta_\varphi = \xi\cos\varphi + \eta\sin\varphi$ is given for an infinity of mod π *different values of φ, then the distribution function $F(x, y) = \Pr(\xi < x, \eta < y)$ of the random point (ξ, η) is uniquely determined.*

Before proving Theorem 1, let us formulate as a theorem a corollary of this theorem which is a straightforward generalization of Theorem R.

THEOREM 2. *Let $f(x, y)$ denote a continuous function which is equal to* 0 *if $x^2+y^2 \geqq R^2$ for some $R > 0$. If the integral of $f(x, y)$ vanishes along every line parallel to some line belonging to an arbitrary infinite set of lines passing through the origin, then $f(x, y) \equiv 0$.*[5]

PROOF of THEOREM 1. Let us denote by $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ the values of φ for which $F_{\zeta_\varphi}(p)$ is known. Let φ_0 denote a limit point of the sequence φ_n. According to (1. 11) $\psi(t\cos\varphi, t\sin\varphi)$ is known for every value of t and for every $\varphi = \varphi_n$. As $\psi(t\cos\varphi, t\sin\varphi)$ is an analytic function of φ for every fixed value of t, it follows that $\psi(t\cos\varphi, t\sin\varphi)$ is known for any fixed value of t for values $\varphi = \varphi_{n_k}$ where $\lim_{k\to\infty} \varphi_{n_k} = \varphi_0$. Hence we conclude that $\psi(t\cos\varphi, t\sin\varphi) = \psi_\varphi(t)$ is known for *every* value of t and φ, and thus Theorem 1 follows in the same way as Theorem CW was proved. The analyticity of $\psi_\varphi(t)$ is clear from the existence of the derivative

$$(1.\ 15) \qquad \frac{\partial\psi_\varphi(t)}{\partial\varphi} = it \iint\limits_{x^2+y^2 \leqq R^2} (-x\sin\varphi + y\cos\varphi) e^{it(x\cos\varphi + y\sin\varphi)} dF(x, y)$$

for every (complex) value of φ.

Chapter II

In this chapter we consider discrete distributions. For the sake of simplicity we shall use the terminology of mass distributions. Let us consider a discrete mass distribution on the plane, consisting of n mass points, i. e. a distribution consisting of the masses $m_k > 0$ situated in the points (x_k, y_k) $(k = 1, 2, \ldots, n)$. We shall prove first the following

THEOREM 3. *A discrete mass distribution consisting of n distinct mass points with masses $m_1, m_2, \ldots, m_n$ situated in the points $(x_1, y_1), (x_2, y_2), \ldots,$*

[5] It will be seen from the proof of Theorem 1 that Theorem 2 also holds if instead of supposing $f(x, y) = 0$ for $x^2+y^2 \geqq R^2$ we suppose only that $f(x, y)$ is sufficiently small for large values of x^2+y^2, for example, if for every $\lambda > 0$ we have

$$(1.\ 14) \qquad |f(x, y)| \leqq e^{-\lambda\sqrt{x^2+y^2}}$$

provided that $\sqrt{x^2+y^2} \geqq R(\lambda)$ where $R(\lambda)$ is an arbitrary positive function of λ. This will become clear by taking into account that, in the proof of Theorem 1, the condition that $\xi^2+\eta^2 \leqq R^2$ is fulfilled with probability 1 serves only to ensure that the characteristic function $\psi(p\cos\varphi, p\sin\varphi)$ should be, for any value of p, an analytic function of φ, and this is ensured by the restriction (1. 14).

(x_n, y_n), *respectively, is completely determined if its projections on* $n+1$ *arbitrary different straight lines through the origin are given.*

I have proved Theorem 3 only for the case of equal masses, in which case Theorem 3 is a special case of Theorem 4, and communicated the assertion for unequal masses as a conjecture to G. HAJÓS who succeeded in proving it. I am very thankful to him for his kind permission to publish here his elegant proof.

PROOF OF THEOREM 3. For $n=1$ the theorem is trivial. Let us suppose $n \geqq 2$. Let us mark the two extreme points of each projection and consider the projecting lines (i. e. the perpendiculars to the line on which the projection is considered) through these points; we shall call these lines, for the sake of brevity, *extreme projecting lines*. Thus if $n+1$ projections of the mass distribution are known, we have at least $2n+1$ extreme projecting lines, because at least n projections have two extreme points and only one projection can eventually shrink to a point (and this can happen only in case all points are lying on the same straight line). Each extreme projecting line passes through at least one mass point. As there are n mass points, there must be at least one mass point through which three or more extreme projecting lines are passing. Since all mass points are situated in one of the two closed half-planes determined by every extreme projecting line, we see that if $r \geqq 3$ extreme projecting lines pass through a point P of the plane, these lines divide the plane into $2r$ angular domains, and all mass points must lie in the interior or on the boundary of one of these domains. This domain is bounded by two extreme projecting lines, therefore all other extreme projecting lines, and thus at least one projecting line can have no common point with our mass system other than the point P itself. But since every extreme projecting line passes through at least one mass point, we infer that P itself must be a mass point. Thus we have proved that there is at least one mass point through which three or more projecting lines are passing, and conversely: every point of the plane which is common to three or more extreme projecting lines is a mass point. Consequently, considering only the extreme projecting lines, at least one mass point can be found. The projections of the remaining $n-1$ points on $n+1$ lines being known (by omitting the projection of the point already found), we can apply the same procedure again, and thus find one-by-one all mass points. Therefore Theorem 3 is proved. Clearly, the above proof furnishes an effective method for actually finding all mass points, and the corresponding masses.

It is easy to see that Theorem 3 can not be improved: n projections do not always determine a discrete mass distribution consisting of n points. As a matter of fact, let us consider a regular polygon Π with $2n$ sides, and let the system of n equal mass points, each of mass 1, situated in every second vertex of the polygon Π be called *system A*, and let the system of

n equal mass points, each of mass 1, situated in those n vertices of the polygon Π in which there is no mass point of the system A, be called *system* B. It is easy to see that, denoting by $l_1, l_2, \ldots, l_n$ the perpendiculars to the pairs of opposite sides of the polygon Π, the projection of the system A is the same on each line l_i as the projection of the system B. Analyzing the above proof, it is easy to see that all mass distributions which are not completely determined by n projections are essentially equivalent to that mentioned just now, and can be obtained by replacing the regular polygon of $2n$ sides by some convex polygon of $2n$ sides, with vertices $P_1, P_2, \ldots, P_{2n}$ and having the following property: the lines P_iP_j and P_kP_l are parallel provided that $i+j\equiv 1 \pmod 2$ and $i+j\equiv k+l \pmod{2n}$. Clearly, all polygons obtained from a regular polygon of $2n$ sides by an affine transformation satisfy this condition, but not only these; for instance, the hexagon $P_1P_2P_3P_4P_5P_6$ shown by Fig. 1 has the required property, though it cannot be obtained by an affine transformation from a regular hexagon.

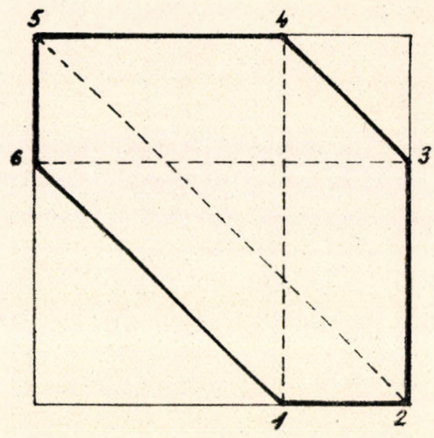

Fig. 1

Instead of speaking of projections *on* some straight line l we can speak of projections *from* the direction perpendicular to l on some fixed straight line L.

Projections from a direction can be considered as projections from some point at infinity of the projective plane. It is easy to see that the proof, and hence the assertion of Theorem 3, remains valid for the more general case when projections from finite points to l are also admitted.

Let us consider now projections of discrete mass systems of the 3-dimensional space. We prove the following

THEOREM 4. *Let us consider a discrete mass distribution M in the 3-dimensional (x, y, z)-space, consisting of n equal masses situated in the points with the rectangular coordinates (x_k, y_k, z_k) $(k = 1, 2, \ldots, n)$. If the (orthogonal)* *projection of the mass distribution M is given on $n+1$ arbitrary planes, no two of which are parallel, then M is completely determined.*

Before proving Theorem 4, let us mention that the theorem can not be improved. As a matter of fact, if the masses are situated in every second vertex of a regular polygon of $2n$ sides in some plane α and the planes on which these masses are projected are all orthogonal to α, we obtain the counter-example, considered in connection with Theorem 3.

PROOF OF THEOREM 4. Let us denote by $A_1, A_2, \ldots, A_{n+1}$ the planes on which the mass distribution is projected. We may suppose that all planes A_k pass through the origin of the rectangular coordinate system (x, y, z) and that they do not pass through the z-axis, and the line of intersection of two of them does not lie in the (y, z)-plane. Let us choose a rectangular coordinate system (u_k, v_k) in each plane A_k such that its origin coincides with the origin of the coordinate system (x, y, z) and let us denote by $\alpha_{1k}, \beta_{1k}, \gamma_{1k}$ the cosines of direction of the straight line $v_k=0$ and by $\alpha_{2k}, \beta_{2k}, \gamma_{2k}$ the cosines of direction of the straight line $u_k=0$. It follows that the projection of the point (x_j, y_j, z_j) on A_k has the coordinates

$$\left.\begin{aligned} u_{jk} &= \alpha_{1k}x_j+\beta_{1k}y_j+\gamma_{1k}z_j \\ v_{jk} &= \alpha_{2k}x_j+\beta_{2k}y_j+\gamma_{2k}z_j \end{aligned}\right\} \quad \begin{aligned} j &= 1, 2, \ldots, n \\ k &= 1, 2, \ldots, n+1 \end{aligned}$$

in the coordinate system (u_k, v_k). Therefore, if these projections are given, the numbers

$$\frac{\alpha_{2k}u_{jk}-\alpha_{1k}v_{jk}}{\beta_{1k}\alpha_{2k}-\beta_{2k}\alpha_{1k}} = y_j+\lambda_k z_j$$

are known where

$$\lambda_k = \frac{\gamma_{1k}\alpha_{2k}-\gamma_{2k}\alpha_{1k}}{\beta_{1k}\alpha_{2k}-\beta_{2k}\alpha_{1k}}.$$

As $\alpha_{2k}\gamma_{1k}-\alpha_{1k}\gamma_{2k}$ and $\alpha_{2k}\beta_{1k}-\alpha_{1k}\beta_{2k}$ are two cosines of direction of the perpendicular to the plane A_k, which does not pass through the z-axis, the second is different from zero, and as the line of intersection of two planes A_k and $A_{k'}$ $(k' \neq k)$ does not lie in the (y, z)-plane, their ratio λ_k is different for different values of k. Thus the numbers $y_j+\lambda z_j$ $(j=1, 2, \ldots, n)$ are known in their totality for $n+1$ different values of λ. Consequently, all elementary symmetric functions of these numbers,

$$S_1(\lambda) = \sum_{j=1}^{n}(y_j+\lambda z_j),$$

$$S_2(\lambda) = \sum_{1\leq j_1<j_2\leq n}(y_{j_1}+\lambda z_{j_1})(y_{j_2}+\lambda z_{j_2}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_n(\lambda) = \prod_{j=1}^{n}(y_j+\lambda z_j),$$

are known for $n+1$ different values of λ. As $S_1(\lambda), S_2(\lambda), \ldots, S_n(\lambda)$ are polynomials of degree not greater than n in λ, it follows that these polynomials are completely determined, and therefore their values can be calculated (e. g. by NEWTON's interpolation formula) for $\lambda = i$. Hence we can obtain the values of $S_1(i), S_2(i), \ldots, S_n(i)$, i. e. the values of the elementary symmetric functions of the complex numbers $w_j = y_j + iz_j$. We conclude that the complex numbers w_j can be determined as the roots of the equation

$$w^n - S_1(i)w^{n-1} + S_2(i)w^{n-2} - \cdots + (-1)^n S_n(i) = 0$$

and hence the pairs of numbers (y_j, z_j) can be obtained. Therefore, starting with the projection of the mass distribution considered onto the planes A_k $(k = 1, 2, \ldots, n+1)$, we can determine the projection of the same mass distribution onto the plane (y, z). As the position of the coordinate system (x, y, z) is arbitrary (we have to take care only of that no plane A_k should pass through the z-axis and the line of intersection of two planes A_j, A_k should not lie in the plane (y, z)), it follows that the projection of the considered distribution can be determined for *every* plane except those planes which pass through the intersecting line of two planes A_j, A_k; but the projections on such planes can also be determined by a limiting process and therefore, according to Theorem CW, the distribution itself is completely determined. Theorem 4 is herewith proved.

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О ПРОЕКЦИЯХ РАСПРЕДЕЛЕНИЙ ВЕРОЯТНОСТЕЙ

А. РЕНЬИ (Будапешт)

(Резюме)

Глава I работы содержит доказательство и обобщение следующей теоремы И. Радона: Если интеграл непрерывной функции $f(x, y)$ ровно 0 на каждом хорде некоторой конечной области K, то $f(x, y) \equiv 0$ на K.

Эта теорема была открыта многими авторами независимо друг от друга. В работе показано, что эта теорема является прямым следствием теоремы Х. Крамера и Х. Волда согласно которому распределение вероятностей в плоскости однозначно определено с помощю совокупности его проекциях на всех прямых плоскости. Дальше доказано, что для определении этого распределении достаточно знание его проекций на бесконечно многих различных прямых, проходящих через данную точку.

В главе II и рассматриваются дискретные распределения вероятностей (или масс) и изложено доказательство данное Г. Гаёшом, теоремы, что если известны проекции дискретного распределения, состоящего из n точечных масс плоскости на $n+1$ различных приямых не параллелных между собой, то это распределения однозначно определено. То же самое имеет место для дискретных распределении в пространстве, если все массы одинаковы; знание n различных проекции для этого вообще не достаточно.