ON THE ZEROS OF POLYNOMIALS

By

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1. This paper deals with the method of D. BERNOULLI,[1] N. I. LOBATSCHEWSKY[2] and N. GRAEFFE[3] devised for the approximative solution of algebraic equations. In the usual form[4] the method asserts that if

(1. 1) $$f_0(x)=a_{00}+a_{10}x+\ldots+a_{n0}x^n=0 \qquad (a_{n0}=1)$$

is the equation to be solved, with the zeros $z_1, z_2, \ldots, z_n$ where

(1. 2) $$|z_n|<|z_{n-1}|<\ldots<|z_1|,$$

then we have to form the so-called Graeffe-transforms $f_\nu(x)$ defined by

(1. 3) $$f_\nu(x)=(-1)^n f_{\nu-1}(\sqrt{x})f_{\nu-1}(-\sqrt{x}) \qquad (\nu=1,2,\ldots).$$

If

(1. 4) $$f_\nu(x)=a_{0\nu}+a_{1\nu}x+\ldots+a_{n\nu}x^n \qquad (a_{n\nu}=1),$$

then the method asserts that

(1. 5) $$\begin{aligned} |z_1| &= \lim_{\nu\to\infty}\left|\frac{a_{n-1,\nu}}{a_{n,\nu}}\right|^{\frac{1}{2^\nu}} \\ |z_2| &= \lim_{\nu\to\infty}\left|\frac{a_{n-2,\nu}}{a_{n-1,\nu}}\right|^{\frac{1}{2^\nu}} \\ &\ldots\ldots\ldots \\ |z_n| &= \lim_{\nu\to\infty}\left|\frac{a_{0\nu}}{a_{1\nu}}\right|^{\frac{1}{2^\nu}}. \end{aligned}$$

Curiously enough the method was used until 1930 without hesitation for small ν-values and without estimation of the error, nothing said about the condition (1. 2); without this restriction the rule is false in general. Afterwards, in the papers of R. SAN JUAN,[5] A. OSTROWSKI[6] and the second-named

[1] D. BERNOULLI, *Commentationes Petropolitanae,* **3** (1728).

[2] Н. И. Лобачевский, Алгебра или вычисление конечных (Казань, 1834).

[3] N. GRAEFFE, Die Auflösung der höheren numerischen Gleichungen (Zürich, 1837).

[4] See e. g. Я. С. Безикович, Приближенные вычисления.

[5] R. SAN JUAN, Compléments à la méthode de Gräffe pour la résolution des équations algébriques, *Bull. des Sciences Math.*, **59** (1935), pp. 104—109.

[6] A. OSTROWSKI, Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent, *Acta Math.*, **72** (1940), pp. 99—257.

author[7] are contained the first modified forms of this method which work with *finite* ν's, give estimations for the errors and are valid without the restriction (1.2). Replacing (1.2) by[8]

$$|z_n| \leqq |z_{n-1}| \leqq \ldots \leqq |z_2| \leqq |z_1| \tag{1.6}$$

and restricting ourselves to the approximation of $|z_1|$, the rule of OSTROWSKI and the first rule in [7] both form the first ν Graeffe-transforms and obtain approximative values T_ν resp. T'_ν for $|z_1|$ in terms of $a_{j,\nu}$ so that

$$\left(\frac{1}{n}\right)^{2^{-\nu}} \leqq \frac{|z_1|}{T_\nu} \leqq 2^{2^{-\nu}} \tag{1.7}$$

and

$$\left(\frac{1}{n}\right)^{2^{-\nu}} \leqq \frac{|z_1|}{T'_\nu} \leqq 2^{2^{-\nu}}. \tag{1.8}$$

The remarkable fact in both estimations (1.7) and (1.8) is that they depend only upon n and ν, i. e. *not* upon the coefficients of $f_0(x)$ and give exactly the same bounds.

2. In [7] the author has expressed his opinion that his procedure can be imbedded in a chain of procedures (some of which give narrower bounds) *working with the* ν^{th} *Graeffe-transforms.* In this paper we shall show that this opinion was right. Indeed, we shall show the correctness of the following

Rule I. Let us form with the coefficients (1.4) of the ν^{th} Graeffe-transform $f_\nu(x)$ of $f_0(x)$, with $M = [n^2 \log(2n^2)] + 2$ the sequence $s_1, s_2, \ldots, s_M$ successively from the system of recurrent equations

$$\begin{aligned}
s_1 + a_{n-1,\nu} &= 0 \\
s_2 + a_{n-1,\nu}s_1 + 2a_{n-2,\nu} &= 0 \\
&\;\vdots \\
s_n + a_{n-1,\nu}s_{n-1} + \ldots + na_{0,\nu} &= 0 \\
s_{n+1} + a_{n-1,\nu}s_n + \ldots + a_{0\nu}s_1 &= 0 \\
&\;\vdots \\
s_M + a_{n-1,\nu}s_{M-1} + \ldots + a_{0\nu}s_{M-n} &= 0.
\end{aligned} \tag{2.1}$$

Then we have

$$\left(\frac{1}{n}\right)^{2^{-\nu}} \leqq \frac{|z_1|}{\left(\max\limits_{j=1,2,\ldots,M} |s_j|^{\frac{1}{j}}\right)^{2^{-\nu}}} \leqq \left(\frac{1}{1-\frac{1}{n}}\right)^{2^{-\nu}}.$$

[7] P. TURÁN, On approximative solution of algebraic equations, *Publ. Math. Debrecen*, **2** (1951), pp. 26—42.

[8] Obviously (1.6) is a notation only and contains no restrictions, in contrary to (1.2).

With the notation

$$(2.2) \qquad \left(\max_{j=1,2,\ldots,M} |s_j|^{1/j}\right)^{2^{-\nu}} = T''_\nu$$

we have three approximative values for $|z_1|$, namely T_ν, T'_ν and T''_ν. Owing to the rule I, T''_ν gives the closest approximation, but needs obviously the greatest computational work. Rule I in itself gives no scale of rules, such a scale is furnished by the more general

Rule II. With an arbitrary ε in $0<\varepsilon<1$ and

$$(2.3) \qquad N = \left[\frac{n}{\varepsilon}\log\frac{2n}{\varepsilon}\right]+2$$

form the system analogous to (2. 1) but ending with the equation

$$s_N + a_{n-1,\nu}s_{N-1} + \ldots + a_{0,\nu}s_{N-\nu} = 0.$$

Then we can again successively determine $s_1, s_2, \ldots, s_N$ and we have

$$(2.4) \qquad \left(\frac{1}{n}\right)^{2^{-\nu}} \leqq \frac{|z_1|}{\left(\max\limits_{j=1,2,\ldots,N} |s_j|^{1/j}\right)^{2^{-\nu}}} \leqq \left(\frac{1}{1-\varepsilon}\right)^{2^{-\nu}}.$$

Rule I follows from rule II taking $\varepsilon = \frac{1}{n}$; hence it suffices to prove rule II only. For $\varepsilon = \frac{1}{2}$ we have

$$N = [2n \log 4n] + 2$$

and the corresponding bounds in (2. 4) became $\left(\frac{1}{n}\right)^{2^{-\nu}}$, $2^{2^{-\nu}}$. These bounds are identical with those of rule I in [7]. We had there however to form

$$T'_\nu = \left(\max_{j=1,2,\ldots,2n} |s_j|^{\frac{1}{j}}\right)^{2^{-\nu}};$$

this means that by greater computational work we obtained an approximative value, which is not better. The computational work with the approximating value T'_ν is also considerable; we emphasise again the importance of the conjecture expressed in [7] that for a suitable $c>1$, *independent of n*, we have already

$$(2.5) \qquad \left(\frac{1}{n}\right)^{2^{-\nu}} \leqq \frac{|z_1|}{\left(\max\limits_{j=1,2,\ldots,n} |s_j|^{\frac{1}{j}}\right)^{2^{-\nu}}} \leqq c^{2^{-\nu}}.$$

A comparison of these with the rule II shows that probably in (2. 3) one can replace N by an N' of the form

$$N' = [c_1(\varepsilon)n];$$

the truth of this conjecture would diminish considerably the necessary calculations. Further remarks on this subject can be found in **7.**

As the example

$$z_1 = 1,\; z_2 = z_3 = \ldots = z_n = 0$$

shows, we can have for an arbitrary large integer N''

$$\frac{|z_1|}{\left(\max\limits_{j=1,2,\ldots,N''} |s_j|^{\frac{1}{j}}\right)^{2^{-\nu}}} = 1,$$

i. e. one cannot replace in (2.4) the quantity $(1-\varepsilon)^{2^{-\nu}}$ by another one, less than 1.

3. The paper[7] was one of a series of papers dealing with the various applications of a central idea.[9] This idea was to estimate from below sums of the form

$$\left|\sum_{j=1}^{n} b_j w_j^t\right| \tag{3.1}$$

where the numbers b_j and w_j are arbitrary complex numbers, by $\max\limits_{j=1,2,\ldots n} |w_j|^t$ resp. by $\min\limits_{j=1,\ldots,n} |w_j|^t$ when t is an appropriate integer from an interval $(m+1, m+n)$ (m a non-negative integer). The deepest result of this theory is the inequality

$$\max_{\substack{m+1 \leqq t \leqq m+n \\ t \text{ integer}}} |b_1 w_1^t + \ldots + b_n w_n^t| \geqq \left(\frac{n}{250(m+2n)}\right)^n \min_j |b_1 + \ldots + b_j| \tag{3.2}$$

if

$$1 = |w_1| \geqq |w_2| \geqq \ldots \geqq |w_n|. \tag{3.3}$$

The result so obtained for $m=0$ is rather weak, even in the case

$$b_1 = b_2 = \ldots = b_n = 1; \tag{3.4}$$

the main tool of the proofs for the rules in[7] was a direct approach for this case. So we obtained in[7] the inequalities

$$\max_{\substack{1 \leqq t \leqq n \\ t \text{ integer}}} |w_1^t + w_2^t + \ldots + w_n^t| \geqq \frac{\log 2}{\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}}, \tag{3.5}$$

$$\max_{\substack{1 \leqq t \leqq 2n \\ t \text{ integer}}} |w_1^t + \ldots + w_n^t| \geqq \frac{1}{2}. \tag{3.6}$$

The conjecture (2.5) would follow from the proof of

$$\max_{\substack{1 \leqq t \leqq n \\ t \text{ integer}}} |w_1^t + \ldots + w_n^t| \geqq c. \tag{3.7}$$

[9] For a detailed exposition see the forthcoming book of the second-named author entitled *Über eine neue Methode der Analysis und ihre Anwendungen.*

The proof for the rule II of the present paper will be based on the inequality

$$(3.8)\qquad \max_{\substack{1\leqq t\leqq N\\ t \text{ integer}}} |w_1^t+\dots+w_n^t|\geqq 1-\varepsilon$$

where N is defined in (2.3) and (3.3) holds. I. e. the lower estimation in (3.8) is better than in (3.5) and (3.6), but the range of t is bigger. But more generally for the general expression $\sum_{j=1}^{n} b_j w_j^t$ one can obtain for $m=0$ an estimation, in a certain respect better than (3.2), in an important special case which may be called for a certain reason Dirichletian case. This will be given by

THEOREM I. *Putting*

$$f(t)=\sum_{j=1}^{n} b_j w_j^t,$$

we suppose that the w_j's satisfy (3.3) *and the b_j's are positive. Choosing ε satisfying the restriction*

$$(3.9)\qquad 0<\varepsilon<\min\left(1, \frac{2(b_2+b_3+\dots+b_n)}{b_1}\right),$$

we define

$$(3.10)\qquad M_1=\left[\frac{f(0)}{b_1\varepsilon}\log\frac{2f(0)}{b_1\varepsilon}\right]+2.$$

Then we have

$$\max_{\substack{1\leqq t\leqq M_1\\ t \text{ integer}}} |f(t)|\geqq b_1(1-\varepsilon).$$

For $b_1=b_2=\dots=b_n=1$ we have obviously got again (3.8).

Next we deduce from (3.8) the rule II, in the following § we prove theorem I and in the further §§ we treat the similar problem for integrals.

4. Hence we turn to the proof of rule II. The quantity s_j is obviously the j^{th} power-sum of the zeros of $f_\nu(x)$, i. e.

$$s_j=\sum_{l=1}^{n} z_l^{j2^\nu}.$$

Hence, owing to (1.6), for *all* natural j's, we have

$$|s_j|\leqq n|z_1|^{j2^\nu},$$

i. e. this will hold choosing $j=j_0$, where

$$|s_{j_0}|^{\frac{1}{j_0}}=\max_{j=1,2,\dots,N}|s_j|^{\frac{1}{j}}.$$

Thus

$$|s_{j_0}|^{\frac{1}{j_0}} \leqq n^{\frac{1}{j_0}} |z_1|^{2^\nu} \leqq n |z_1|^{2^\nu},$$

$$\frac{|z_1|}{\left(|s_{j_0}|^{\frac{1}{j_0}}\right)^{2^{-\nu}}} \geqq n^{-2^{-\nu}},$$

which gives already the lower limitation of the rule II. To prove the upper limitation, we apply (3. 8) with

$$w_j = \left(\frac{z_j}{z_1}\right)^{2^\nu} \qquad (j = 1, 2, \ldots, n).$$

Then there is an integer t_0 with $1 \leqq t_0 \leqq N$ such that

$$\frac{|s_{t_0}|}{|z_1|^{2^\nu t_0}} \geqq 1 - \varepsilon,$$

$$|z_1|^{2^\nu} \leqq \left(\frac{1}{1-\varepsilon}\right)^{\frac{1}{t_0}} |s_{t_0}|^{\frac{1}{t_0}} \leqq \frac{1}{1-\varepsilon} \max_{j=1,2,\ldots,N} |s_j|^{\frac{1}{j}},$$

i. e.

$$\frac{|z_1|}{\left(\max\limits_{1 \leqq j \leqq N} |s_j|^{\frac{1}{j}}\right)^{2^{-\nu}}} \leqq \left(\frac{1}{1-\varepsilon}\right)^{2^{-\nu}}$$

which completes the proof of the rule II.

5. Now we turn to the proof of theorem I. Without loss of generality we may suppose

(5. 1) $$w_1 = 1.$$

We consider for $|z| > 1$ the function

(5. 2) $$g(z) = \sum_{j=1}^{n} \frac{b_j}{z - w_j}.$$

Since $|w_j| \leqq 1, g(z)$ is here regular, i. e. we have from (5. 2) for $|z| > 1$

(5. 3) $$g(z) = \sum_{\nu=0}^{\infty} \frac{f(\nu)}{z^{\nu+1}}.$$

If $R > 1$ (we shall fix the value of R only later), introducing the notation

$$\max_{\substack{1 \leqq \nu \leqq M_1 \\ \nu \text{ integer}}} |f\ \nu)| = U,$$

we obtain from (5. 3), owing to the positivity of the b_j's,

$$|g(R)| \leqq \frac{f(0)}{R} + U\left(\frac{1}{R^2} + \frac{1}{R^3} + \ldots + \frac{1}{R^{M_1}}\right) + f(0)\left(\frac{1}{R^{M_1+1}} + \ldots\right) <$$

(5. 4) $$< \frac{f(0)}{R} + \frac{U}{R(R-1)} + \frac{f(0)}{R^{M_1}(R-1)}.$$

On the other hand we have, using also (5. 1) and (3. 3),[10]

$$|g(R)| \geqq \Re g(R) = \frac{b_1}{R-1} + \sum_{j=2}^{n} b_j \Re \frac{1}{R-w_j}. \tag{5.5}$$

Now we observe that

$$\Re \frac{1}{R-w_j} \geqq \frac{1}{R+1};$$

hence from (5. 5) and (5. 4) we get

$$\frac{b_1}{R-1} + \frac{b_2+\ldots+b_n}{R+1} \leqq \frac{b_1+\ldots+b_n}{R} + \frac{b_1+\ldots+b_n}{R^{M_1}(R-1)} + \frac{U}{R(R-1)}. \tag{5.6}$$

Choosing

$$R = 1 + \frac{b_1 \varepsilon}{(b_1+\ldots+b_n) - b_1\left(1+\frac{\varepsilon}{2}\right)}, \tag{5.7}$$

the condition $R>1$ is fulfilled owing to (3. 9). Since

$$\frac{b_1}{R-1} + \frac{b_2+\ldots+b_n}{R+1} - \frac{b_1+\ldots+b_n}{R} = \frac{b_1\left(1-\frac{\varepsilon}{2}\right)}{R(R-1)},$$

(5. 6) gives

$$U \geqq b_1\left(1-\frac{\varepsilon}{2}\right) - (b_1+\ldots+b_n)R^{1-M_1}. \tag{5.8}$$

Now from (5. 7) and (3. 9)

$$\log R \geqq \frac{R-1}{R} = \frac{b_1 \varepsilon}{(b_2+\ldots+b_n)+b_1\frac{\varepsilon}{2}} > \frac{b_1\varepsilon}{b_1+\ldots+b_n}$$

and using the definition of M_1 in (3. 10)

$$R^{M_1-1} \geqq \exp\left\{\left(1+\left[\frac{f(0)}{b_1\varepsilon}\log\frac{2f(0)}{b_1\varepsilon}\right]\right)\frac{b_1\varepsilon}{b_1+b_2+\ldots+b_n}\right\} \geqq$$

$$\geqq \exp\left\{\frac{f(0)}{b_1\varepsilon}\log\frac{2f(0)}{b_1\varepsilon}\cdot\frac{b_1\varepsilon}{f(0)}\right\} = \frac{2f(0)}{b_1\varepsilon};$$

putting this in (5. 8) we obtain

$$U \geqq b_1\left(1-\frac{\varepsilon}{2}\right) - \frac{b_1\varepsilon}{2} = b_1(1-\varepsilon),$$

indeed. Q. e. d.

6. The expression

$$f(t) = \sum_{j=1}^{n} b_j w_j^t$$

— in the important case, when the w_j's are of the form $w_j = e^{i\alpha_j}$ with real

[10] $\Re z$ denotes the real part of the complex number z.

α_j's and the b_j's are positive — can be brought to the form of a Fourier—Stieltjes integral $\int_{-\infty}^{\infty} e^{itx} dF(x)$ with a non-negative non-decreasing $F(x)$; if $\alpha_1 > \alpha_2 > \ldots > \alpha_n$, take simply

$$\begin{aligned} F(x) &= 0 && \text{for } -\infty < x < \alpha_n, \\ F(x) &= b_n && \text{for } \alpha_n \leqq x < \alpha_{n-1}, \\ F(x) &= b_n + b_{n-1} && \text{for } \alpha_{n-1} \leqq x < \alpha_{n-2}, \\ &\cdots\cdots\cdots\cdots \\ F(x) &= b_n + \ldots + b_2 && \text{for } \alpha_2 \leqq x < \alpha_1, \\ F(x) &= b_n + \ldots + b_1 && \text{for } \alpha_1 \leqq x < \infty. \end{aligned}$$

The extension of the whole theory, mentioned in **3,** to Fourier—Stieltjes integrals seems to be very desirable in particular with regard to possible applications to the study of characteristic functions in probability theory. As a first result in this trend we prove

THEOREM II. *If $F(x)$ is positive and non-decreasing on the real axis such that*

(6. 1) $$F(+0) - F(-0) = \Delta > 0$$

and

(6. 2) $$\int_{-\infty}^{\infty} dF(x) = 1,$$

then, if

(6. 3) $$0 < \varepsilon < \min\left(1, \frac{2(1-\Delta)}{\Delta}\right)$$

we have

(6. 4) $$\max_{\substack{1 \leqq \nu \leqq 2 + \left[\frac{1}{\Delta\varepsilon}\log\frac{2}{\Delta\varepsilon}\right] \\ \nu \text{ integer}}} \left| \int_{-\infty}^{\infty} e^{i\nu x}\, dF(x) \right| \geqq \Delta(1-\varepsilon).$$

Since the proof is very similar to that of theorem I, it will suffice only to sketch it. We denote the expression on the left of (6. 4) by V, the quantity

(6. 5) $$2 + \left[\frac{1}{\Delta\varepsilon}\log\frac{2}{\Delta\varepsilon}\right]$$

by L and choose

(6. 6) $$R = 1 + \frac{\Delta\varepsilon}{1 - \left(1 + \frac{\varepsilon}{2}\right)\Delta} \; (>1).$$

Then we have on one hand

$$\int_{-\infty}^{\infty} \frac{dF(x)}{R - e^{ix}} = \frac{1}{R} + \sum_{\nu=1}^{\infty} \frac{1}{R^{\nu+1}} \int_{-\infty}^{\infty} e^{i\nu x}\, dF(x)$$

and on the other hand

$$\int_{-\infty}^{\infty} \frac{dF(x)}{R-e^{ix}} \geqq \frac{\Delta}{R-1} + \frac{1-\Delta}{R+1}.$$

Hence as in the previous proof

$$\frac{\Delta}{R-1} + \frac{1-\Delta}{R+1} \leqq \frac{1}{R} + \frac{V}{R(R-1)} + \frac{1}{R^L(R-1)}.$$

Replacing R by its value in (6.6) we obtain

$$\frac{\Delta}{R-1} + \frac{1-\Delta}{R+1} - \frac{1}{R} = \frac{\Delta\left(1-\frac{\varepsilon}{2}\right)}{R(R-1)},$$

i. e.

(6.7) $$V \geqq \Delta\left(1-\frac{\varepsilon}{2}\right) - R^{1-L}.$$

Since

$$\log R \geqq \frac{R-1}{R} = \frac{\Delta\varepsilon}{1-\left(1-\frac{\varepsilon}{2}\right)\Delta} \geqq \Delta\varepsilon$$

$$R^{L-1} \geqq \exp\left\{\left(1+\left[\frac{1}{\Delta\varepsilon}\log\frac{2}{\Delta\varepsilon}\right]\right)\Delta\varepsilon\right\} \geqq \frac{2}{\Delta\varepsilon},$$

we obtain indeed from (6.7)

$$V \geqq \Delta(1-\varepsilon).$$

7. In all the rules mentioned above, the upper limitation is generally the better one. It would be at the same time of theoretical and practical interest to modify the procedure so as to improve the *lower* limitation at a fixed ν-value.

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О КОРНЯХ МНОГОЧЛЕНОВ

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(Резюме)

В работе доказывается следующая теорема 1: Пусть $w_1, w_2, \ldots, w_n$ комплексные числа, $|w_1| \geqq |w_2| \geqq \ldots \geqq |w_n|$ и $b_1, b_2, \ldots, b_n$ положительные числа; пусть

$$f(t) = \sum_{j=1}^{n} b_j w_j^t \qquad (t = 0, 1, 2, \ldots);$$

тогда

$$\max_{t=1,2,\ldots,M_1} |f(t)| \geqq b_1(1-\varepsilon),$$

где

$$M_1 = \left[\frac{f(0)}{b_1\varepsilon} \log \frac{2f(0)}{b_1\varepsilon}\right] + 2 \quad \text{и} \quad 0 < \varepsilon < \min\left(1, \frac{2(f(0)-b_1)}{b_1}\right).$$

Как следствие этой теоремы получается следующий результат: пусть

$$f_0(x) = \sum_{k=0}^{n} a_{k0} x^k, \quad f_\nu(x) = (-1)^n f_{\nu-1}(\sqrt{x}) f_{\nu-1}(-\sqrt{x}) \qquad (\nu = 1, 2, \ldots),$$

$$f_\nu(x) = \sum_{k=0}^{n} a_{k\nu} x^k \qquad (a_{n\nu} = 1)$$

и определим числа s_k $(k = 1, 2, \ldots, N)$ из систем линейных уравнений

$$\sum_{j=0}^{k-1} s_{k-j} a_{n-j,\nu} + k a_{n-k,\nu} = 0 \qquad (k = 1, 2, \ldots, n)$$

и

$$\sum_{j=0}^{n} s_{k-j} a_{n-j,\nu} = 0 \qquad (k = n+1, \ldots, N),$$

тогда если z_1 обозначает найбольшой по модулю корень уравнения $f_0(x) = 0$, то имеем

$$\left(\frac{1}{n}\right)^{2^{-\nu}} \leqq \frac{|z_1|}{\left(\max\limits_{j=1,2,\ldots,N} |s_j|^{\frac{1}{j}}\right)^{2^{-\nu}}} \leqq \left(\frac{1}{1-\varepsilon}\right)^{2^{-\nu}},$$

если $0 < \varepsilon < 1$ и $N = \left[\frac{n}{\varepsilon} \log \frac{2n}{\varepsilon}\right] + 2$. Таким образом получается точный вариант метода Бернулли—Данделин—Лобачевского—Грэфе.

Если вместо суммы $\sum_{j=1}^{n} b_j w_j^t$ посмотрим характеристическую функцию $f(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x)$ некоторой функции распределения $F(x)$, для которой $F(+0) - F(-0) = \Delta > 0$, тогда для $0 < \varepsilon < \min\left(1, \frac{2(1-\Delta)}{\Delta}\right)$ и $M = \left[\frac{1}{\Delta\varepsilon} \log \frac{2}{\Delta\varepsilon}\right] + 2$ имеем теорему 2:

$$\max_{t=1,2,\ldots,M} |f(t)| \geqq \Delta(1-\varepsilon).$$