GENERALIZATION OF AN INEQUALITY OF KOLMOGOROV

By

J. HÁJEK (Prague) and A. RÉNYI (Budapest), corresponding member of the Academy

In what follows $\mathbf{P}(A)$ denotes the probability of the event A, $\mathbf{M}(\xi)$ the mean value, and $\mathbf{M}(\xi|A)$ the conditional mean value, under the condition A, of the random variable ξ.

The inequality of A. N. KOLMOGOROV [1] in question states that if $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is a sequence of mutually independent random variables with mean values $\mathbf{M}(\xi_k)=0$ and finite variances $\mathbf{M}(\xi_k^2)=D_k^2$ $(k=1,2,\ldots)$, we have, for any $\varepsilon>0$,

$$\mathbf{P}(\max_{1\leq k\leq m} |\xi_1+\xi_2+\cdots+\xi_k|\geq\varepsilon)\leq\frac{1}{\varepsilon^2}\sum_{k=1}^{m}D_k^2. \tag{1}$$

This inequality is extremely useful in proving the strong law of large numbers and related theorems. In what follows the inequality (2) will be proved which contains (1) as a special case. The use of (2) instead of (1) makes it possible to simplify the proofs mentioned.

The inequality (2) has been found in 1953 by the first named author; the proof of the inequality given below is due to the second named author.

The following theorem will be proved:

THEOREM. *If $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is a sequence of mutually independent random variables with mean values $\mathbf{M}(\xi_k)=0$ and finite variances $\mathbf{M}(\xi_k^2)=D_k^2$ $(k=1,2,\ldots)$ and c_k $(k=1,2,\ldots)$ is a non-increasing sequence of positive numbers, we have for any $\varepsilon>0$ and any positive integers n and m $(n<m)$*

$$\mathbf{P}(\max_{n\leq k\leq m} c_k|\xi_1+\xi_2+\cdots+\xi_k|\geq\varepsilon)\leq\frac{1}{\varepsilon^2}\left(c_n^2\sum_{k=1}^{n}D_k^2+\sum_{k=n+1}^{m}c_k^2D_k^2\right). \tag{2}$$

PROOF. Let us put

$$\zeta=\sum_{k=n}^{m-1}(\xi_1+\xi_2+\cdots+\xi_k)^2(c_k^2-c_{k+1}^2)+c_m^2(\xi_1+\cdots+\xi_m)^2. \tag{3}$$

It follows

$$\mathbf{M}(\zeta)=c_n^2\sum_{k=1}^{n}D_k^2+\sum_{k=n+1}^{m}c_k^2D_k^2. \tag{4}$$

Denoting by A_r $(r=n, n+1, \ldots, m)$ the event consisting in the simultaneous validity of the inequalities[1]

$$c_s|\xi_1+\cdots+\xi_s|<\varepsilon \quad (n\leqq s<r) \quad \text{and} \quad c_r|\xi_1+\cdots+\xi_r|\geqq\varepsilon,$$

the inequality (2) can be written in the form

$$\sum_{r=n}^{m}\mathbf{P}(A_r)\leqq\frac{1}{\varepsilon^2}\mathbf{M}(\zeta). \tag{5}$$

The inequality (5) is the consequence of (6a), (6b) and (6c) of which the first two are evident:

$$\mathbf{M}(\zeta)\geqq\sum_{r=n}^{m}\mathbf{M}(\zeta|A_r)\,\mathbf{P}(A_r), \tag{6a}$$

$$\mathbf{M}(\zeta|A_r)\geqq\sum_{k=r}^{m-1}\mathbf{M}((\xi_1+\cdots+\xi_k)^2|A_r)\,(c_k^2-c_{k+1}^2)+c_m^2\mathbf{M}((\zeta_1+\cdots+\zeta_m)^2|A_r), \tag{6b}$$

$$\mathbf{M}((\xi_1+\cdots+\xi_k)^2|A_r)\geqq\mathbf{M}((\xi_1+\cdots+\xi_r)^2|A_r)\geqq\frac{\varepsilon^2}{c_r^2} \qquad (r\leqq k\leqq m). \tag{6c}$$

In verifying the inequality (6c), one has to use the fact that according to the definition of the event A_r the random variables ξ_k $(k>r)$ are mutually independent of each other also under the condition A_r, further that they are independent also under the condition A_r of the variables $\xi_1,\xi_2,\ldots,\xi_r$. It should be mentioned that the same fact is utilised in the proof of (1) due to KOLMOGOROV [1].

REMARK 1. If we choose $n=1$ and $c_1=c_2=\cdots=c_m=1$, we obtain from (2) as a special case the inequality (1). If we choose $c_k=\frac{1}{k}$ $(k=n,n+1,\ldots,m)$, we obtain the inequality

$$\mathbf{P}\left(\max_{n\leqq k\leqq m}\frac{|\xi_1+\xi_2+\cdots+\xi_k|}{k}\geqq\varepsilon\right)\leqq\frac{1}{\varepsilon^2}\left(\frac{\sum_{k=1}^{n}D_k^2}{n^2}+\sum_{k=n+1}^{m}\frac{D_k^2}{k^2}\right). \tag{7}$$

REMARK 2. By means of passing to the limit $m\to\infty$, it is easy to deduce from (2) the following inequality:

$$\mathbf{P}(\sup_{n\leqq k}c_k|\xi_1+\cdots+\xi_k|\geqq\varepsilon)\leqq\frac{1}{\varepsilon^2}\left(c_n^2\sum_{k=1}^{n}D_k^2+\sum_{k=n+1}^{\infty}c_k^2D_k^2\right). \tag{8}$$

Especially, if $c_k=\frac{1}{k}$ $(k=1,2,\ldots)$, we obtain the inequality

$$\mathbf{P}\left(\sup_{n\leqq k}\frac{|\xi_1+\xi_1+\cdots+\xi_k|}{k}\geqq\varepsilon\right)\leqq\frac{1}{\varepsilon^2}\left(\frac{\sum_{k=1}^{n}D_k^2}{n^2}+\sum_{k=n+1}^{\infty}\frac{D_k^2}{k^2}\right). \tag{9}$$

[1] If $r=n$, only the second inequality is supposed.

REMARK 3. It follows from (9) immediately, that the strong law of large numbers holds for the sequence of mutually independent random variables $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ if the ξ_k's have mean values 0, finite variances $D_k^2 = M(\xi_k^2)$ and

$$\sum_{k=1}^{\infty} \frac{D_k^2}{k^2} \tag{10}$$

converges ([2]).

As a matter of fact, it follows from (9) and (10) that for any $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbf{P}\left(\sup_{n \leq k} \frac{|\xi_1 + \cdots + \xi_k|}{k} \geqq \varepsilon \right) = 0 \tag{11}$$

and therefore we have

$$\mathbf{P}\left(\lim_{n \to \infty} \frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n} = 0 \right) = 1. \tag{12}$$

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References

[1] A. KOLMOGOROFF Über die Summen durch den Zufall bestimmten unabhängigen Größen, *Math Ann.*, **99** (1928), pp. 309—319. (See also the corrections, ibid., **102** (1929), pp. 484—488.)

[2] A. KOLMOGOROFF, Sur la loi forte des grands nombres, *Comptes Rendus Acad. Sci. Paris* **191** (1930), pp. 910—912. (See also A. KOLMOGOROFF, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Ergebnisse der Math. 2., no. 3 (Berlin, 1933).)

ОБОБЩЕНИЕ ОДНОГО НЕРАВЕНСТВА А. Н. КОЛМОГОРОВА

Я. Хаек (Прага) и А. Реньи (Будапешт)

(Резюме)

В работе доказывается следующее неравенство:

Пусть $\xi_1, \xi_2, \ldots, \xi_m$ независимые случайные величины, математические ожидания которых равны нулю $(\mathbf{M}(\xi_k) = 0;\ k = 1, 2, \ldots, m)$, а дисперсии конечни $(\mathbf{D}^2(\xi_k) = D_k^2;\ k = 1, 2, \ldots, m)$; пусть, далее, c_k не возрастаюшая последовательность положительных чисел. Тогда если $1 \leqq n \leqq m$ и $\varepsilon > 0$, имеем

$$\mathbf{P}(\max_{n \leqq k \leqq m} c_k |\xi_1 + \cdots + \xi_k| \geqq \varepsilon) \leqq \frac{1}{\varepsilon^2} \left(c_n^2 \sum_{k=1}^{n} D_k^2 + \sum_{k=n+1}^{m} c_k^2 D_k^2 \right). \tag{2}$$

Если $n = 1$ и $c_k \equiv 1$, то в качестве частного случая получаем хорошо известное неравенство (1) А. Н. Колмогорова. Воспользовавшись неравенством (2), можно упростить доказательство усиленных законов больших чисел.