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# ON THE DENSITY OF CERTAIN SEQUENCES OF INTEGERS

by

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In this paper we will consider the number-theoretical function

$$(1) \quad \Delta(n) = V(n) - U(n) \quad (n = 1, 2, \dots),$$

where  $U(n)$  is the number of *different* prime factors and  $V(n)$  is the number of *all* prime factors of  $n$ . In other words, if  $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ , where  $p_1, p_2, \dots, p_r$  are different primes and  $\alpha_i \geq 1$  ( $i = 1, 2, \dots, r$ ) we set

$$U(n) = r$$

$$(2) \quad V(n) = \alpha_1 + \alpha_2 + \dots + \alpha_r,$$

$$\Delta(n) = (\alpha_1 - 1) + (\alpha_2 - 1) + \dots + (\alpha_r - 1).$$

We shall calculate the density of the sequence of those integers  $n$ , for which  $\Delta(n) = k$ , where  $k$  is an arbitrary fixed non-negative integer. The density of a sequence  $n_1 < n_2 < \dots < n_i < \dots$  of positive integers is defined as follows:

If  $N(x) = \sum_{n_i \leq x} 1$  is the number of those elements of the sequence  $n_i$

which do not exceed  $x$ , further, if the limit  $\lim_{x \rightarrow \infty} \frac{N(x)}{x} = d$ , exists, we say that  $d$  is the density of the sequence  $n_i$ .

We will show that the sequence of those integers  $n$  for which  $\Delta(n) = k$ , has a density, which we will call  $d_k$ , and that the generating function of the sequence  $d_k$  is given by the following identity:

$$(3) \quad \sum_{k=0}^{\infty} d_k z^k = \prod_{p=0}^{\infty} \left(1 + \frac{1}{p}\right) \left(1 + \frac{1}{p-z}\right),$$

where, in the infinite product in the right hand member  $p$  runs over the sequence of all primes; (3) is valid for  $|z| < 2$ .

(3) can also be written in the following equivalent forms:

$$(3') \quad \sum_{k=0}^{\infty} d_k z^k = \frac{6}{\pi^2} \prod_{p=2}^{\infty} \left( 1 + \frac{z}{(p+1)(p-z)} \right),$$

$$(3'') \quad \sum_{k=0}^{\infty} d_k z^k = \prod_{p=2}^{\infty} \left( \frac{1 + \frac{1}{p-z}}{1 + \frac{1}{p-1}} \right),$$

or

$$(3''') \quad \sum_{k=0}^{\infty} d_k z^k = \frac{6}{\pi^2} \prod_{p=2}^{\infty} \left( \frac{1 - \frac{z}{p+1}}{1 - \frac{z}{p}} \right).$$

Substituting  $z = 0$  into (3'), we obtain the special case

$$(4) \quad d_0 = \frac{6}{\pi^2}$$

which is well known, since  $d_0$  is the density of square-free integers. Substituting  $z = 1$  into (3'') we obtain

$$(5) \quad \sum_{k=0}^{\infty} d_k = 1$$

which shows that the numbers  $d_k$  can be considered as elements of a probability distribution.

The values of  $d_1, d_2, \dots$  can be calculated from (3). For example, we have

$$(6) \quad d_1 = \frac{6}{\pi^2} \sum_{p=2}^{\infty} \frac{1}{p(p+1)},$$

where  $p$  again runs over all primes. For large values of  $k$ , the following asymptotic formula can be deduced from (3):

$$(7) \quad d_k \sim \frac{\delta}{2^k},$$

where

$$(8) \quad \delta = \frac{1}{4} \prod_{p=3}^{\infty} \frac{(p-1)^2}{p(p-2)}.$$

(7) follows from the fact that the function  $\sum_{k=0}^{\infty} d_k z^k$  is regular in every point of the circle  $|z| = z$ , except in  $z = 2$ , where it has a simple pole.

It should be mentioned that the existence of the densities  $d_k$ , follows from a general theorem on additive number-theoretical functions, stated by P. Erdős [1]. We shall give a straightforward elementary proof for the existence of the densities  $d_k$ , which gives at the same time, equation (3); the proof is essentially the same as the well known proof of (4) (see e. g. [2] p. 269.)

The idea of the proof is as follows: every positive integer  $n$  can be written in the form

$$(9) \quad n = P \cdot Q,$$

where  $Q$  is square-free, and  $P$  is of the form  $P = p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s}$ , where  $p_1, p_2, \dots, p_s$  are different primes and  $\beta_i \geq 2$ , ( $i = 1, 2, \dots, s$ ) and furthermore,  $P$  and  $Q$  are relatively prime; the representation of  $n$  in the form (9) is unique. We shall call  $Q$  the *square-free part* of  $n$  and  $P$  the *first power-free part* of  $n$ . Next, we shall call the set  $(\beta_1, \beta_2, \dots, \beta_s)$ , the *signature* of  $n$ . We shall show that the sequence of integers which have the same given first power-free part  $P$ , has a density for every such  $P$ , and that the sequence of integers with a prescribed signature  $(\beta_1, \beta_2, \dots, \beta_s) = \beta$  has a density; as the sequence of integers  $n$  for which  $\Delta(n) = k$  is the union of those disjoint sequences which have such signatures  $(\beta_1, \beta_2, \dots, \beta_s)$  that

$$\beta_1 + \beta_2 + \dots + \beta_s = s + k,$$

it follows that this sequence also has a density; the proof, incidentally, gives the equation (3'), i. e. (3).

We shall use the following notation:

Let  $N(x, P)$  be the number of integers  $n \leq x$  with the prescribed first power-free part  $P$ ;  $\mathcal{N}(x, \beta)$ , the number of integers  $n = x$  with the prescribed signature  $\beta$ ;  $\pi(x)$ , the number of primes equal to, or greater than  $x$ ;  $[y]$ , the integral part of the positive number  $y$ , and  $s(n)$  the signature of the integer  $n$ ; clearly, if  $n = P \cdot Q$  is the representation of  $n$  as a product of a square-free and a first power-free number, we have  $s(n) = s(P)$ .

Now, we prove the following relation:

$$(10) \quad \mathcal{N}(x, \beta) = \sum_{\substack{s(P)=\beta \\ P \leq \sqrt{\lg x}}} N(x, P) + O\left(\frac{x}{(\lg x)^{1/2b}}\right),$$

where

$$\beta = (\beta_1, \beta_2, \dots, \beta_s) \quad \text{and} \quad b = \max_{1 \leq k \leq s} \beta_k.$$

As a matter of fact, if  $n \leq x$ ,  $s(n) = \beta$  and  $n = PQ$ , where

$$P = p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s} > \sqrt{\lg x},$$

then  $n$  is divisible by  $(p_1 p_2 \dots p_s)^2$  and  $p_1 p_2 \dots p_s \geq P^{1/b} > (\lg x)^{1/2b}$ ; thus, the number of such integers does not exceed

$$m > \sum_{(\lg x)^{1/2b} < m^2} \frac{x}{m^2} = O\left(\frac{x}{(\lg x)^{1/2b}}\right),$$

which proves (10).

Let us call  $N^*(x, P)$ , the number of those integers  $n \leq x$  which are of the form  $n = PQ$  where  $(P, Q) = 1$  and  $Q$  is not divisible by the square of any prime which is less than  $\lg x$ . Clearly, then, we have

$$(11) \quad N(x, P) = N^*(x, P) + O\left(\frac{x}{\lg x}\right)$$

because the number of those integers  $n \leq x$  which are divisible by the square of a prime  $> \lg x$  can not exceed

$$\sum_{p > \lg x} \frac{x}{p^2} = O\left(\frac{x}{\lg x}\right).$$

Now,  $N^*(x, P)$  can be calculated by the well known sieving method

$$(12) \quad N^*(x, P) = \left[ \frac{x}{P} \right] - \sum \left[ \frac{x}{Ph_i} \right] + \sum_{i \neq j} \left[ \frac{x}{Ph_i h_j} \right] - \dots$$

where  $h_i, h_j, \dots$  run over the primes  $p_1, p_2, \dots, p_s$  and over the numbers  $p^2$  where  $p$  is a prime which is different from  $p_1, p_2, \dots, p_s$  and  $p < \lg x$ .

If  $P < \sqrt{\lg x}$ , then  $p_i < \sqrt{\lg x} < \lg x$ , ( $i = 1, 2, \dots, s$ ) and thus the number of terms in the right hand member of (12) is  $2^{\pi(\lg x)}$ . Thus, if we

delete all brackets in the right hand member of (12), the error committed thereby will not exceed  $2^{\pi(\lg x)}$ . Thus, taking into account that

$$2^{\pi(\lg x)} = O\left(e^{\frac{c_1 \lg x}{e^{\lg \lg x}}}\right),$$

where  $c_1$  is an absolute constant, it follows that

$$(13) \quad N^*(x, P) = \frac{x}{P} \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right) \prod_{\substack{p < \lg x \\ p \neq p_i \ (i=1, \dots, s)}} \left(1 - \frac{1}{p^2}\right) + O\left(e^{\frac{c_1 \lg x}{e^{\lg \lg x}}}\right),$$

where in the second product in the right hand member  $p$  runs over all primes  $p < \lg x$  except  $p_1, p_2, \dots, p_s$ . It follows from (11) and (13) that

$$(14) \quad \frac{N(x, P)}{x} = \frac{\prod_{p < \lg x} (1 - 1/p^2)}{(p_1 + 1) p_1^{\beta_1 - 1} \dots (p_s + 1) p_s^{\beta_s - 1}} + O\left(\frac{1}{\lg x}\right).$$

Thus we obtain the result, that the sequence of those integers which have the given first power-free part  $P = p_1 p_2 \dots p_s$  has the density

$$(15) \quad d(P) = \frac{6}{\pi^2} \frac{1}{(p_1 + 1) p_1^{\beta_1 - 1} \dots (p_s + 1) p_s^{\beta_s - 1}}.$$

It also follows from (10) and (14) that

$$(16) \quad \frac{\mathcal{N}(x, \beta)}{x} = \prod_{p < \lg x} \left(1 - \frac{1}{p^2}\right) \sum_{p_1^{\beta_1} \dots p_s^{\beta_s} < \sqrt{\lg x}} \left(\frac{1}{(p_1 + 1) p_1^{\beta_1 - 1} \dots (p_s + 1) p_s^{\beta_s - 1}}\right) + O(1/(\lg x)^{1/2b}).$$

Thus, we see that the sequence of integers which have the prescribed signature  $\beta = (\beta_1, \beta_2, \dots, \beta_s)$  has the density

$$(17) \quad \delta(\beta) = \frac{6}{\pi^2} \sum \frac{1}{(p_1 + 1) p_1^{\beta_1 - 1} \dots (p_s + 1) p_s^{\beta_s - 1}},$$

where the summation is to be extended over all  $s$ -tuples  $(p_1, p_2, \dots, p_s)$  of different primes.

If  $\beta$  is the empty set, the sum in the right hand member of (17) has to be replaced by 1.

As, clearly,

$$(18) \quad d_k = \sum_{\beta_1 + \beta_2 + \dots + \beta_s = k} d(\beta)$$

we obtain

$$(19) \quad \sum_{k=0}^{\infty} d_k z^k = \frac{6}{\pi^2} \prod_{p=2}^{\infty} \left( 1 + \frac{1}{p+1} \left( \frac{z}{p} + \frac{z^2}{p^2} + \dots \right) \right)$$

and (19) is clearly equivalent to (3') or (3'').

Equation (3) and (3'') can be obtained from (3') by using the identity  $\prod_{p=2}^{\infty} (1 - 1/p^2) = 6/\pi^2$ . It can be seen from (3''') that  $\sum_{k=0}^{\infty} d_k z^k$  is a meromorphic function with simple poles at  $z = p$ , where  $p$  is a prime  $\neq 3$ , and simple zeros at  $z = p + 1$ , where  $p$  is a prime  $\neq 2$ .

It should be added that the existence of  $d(P)$  follows from the mentioned theorem of Erdős, by applying it to the additive number-theoretical function  $f(n) = \lg P$ , where  $P$  is the first power-free part of  $n$ ; on the other hand, the existence of  $\delta(\beta)$  is *not* a consequence of the existence of  $d(P)$  for any  $P$ , because the sequence of integers with a prescribed signature  $\beta$  is the union of an *infinite* set of sequences, each of which consists of the integers which have a prescribed first power-free part  $P$  for which  $s(P) = \beta$ . It would be, however, possible to state a general theorem, from which the existence of  $\delta(\beta)$  follows. We hope to do this in another paper.

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