ON THE DENSITY OF CERTAIN SEQUENCES OF INTEGERS

by

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In this paper we will consider the number-theoretical function

(1) $$\Delta(n) = V(n) - U(n) \quad (n = 1, 2, \ldots.),$$

where $U(n)$ is the number of *different* prime factors and $V(n)$ is the number of *all* prime factors of n. In other words, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where $p_1, p_2, \ldots, p_r$ are different primes and $\alpha_i \geqq 1$ $(i = 1, 2, \ldots, r)$ we set

(2) $$\begin{aligned} U(n) &= r \\ V(n) &= \alpha_1 + \alpha_2 + \ldots. + \alpha_r, \\ \Delta(n) &= (\alpha_1 - 1) + (\alpha_2 - 1) + \ldots. + (\alpha_r - 1). \end{aligned}$$

We shall calculate the density of the sequence of those integers n, for which $\Delta(n) = k$, where k is an arbitrary fixed non-negative integer. The density of a sequence $n_1 < n_2 < \ldots < n_i < \ldots$ of positive integers is defined as follows:

If $N(x) = \sum_{n_i \leqq x} 1$ is the number of those elements of the sequence n_i which do not exceed x, further, if the limit $\lim_{x \to \infty} \frac{N(x)}{x} = d$, exists, we say that d is the density of the sequence n_i.

We will show that the sequence of those integers n for which $\Delta(n) = k$, has a density, which we will call d_k, and that the generating function of the sequence d_k is given by the following identity:

(3) $$\sum_{k=0}^{\infty} d_k z^k = \prod_{p=0}^{\infty} \left(1 + \frac{1}{p}\right)\left(1 + \frac{1}{p - z}\right),$$

where, in the infinite product in the right hand member p runs over the sequence of all primes; (3) is valid for $|z| < 2$.

(3) can also be written in the following equivalent forms:

$$(3') \qquad \sum_{k=0}^{\infty} d_k z^k = \frac{6}{\pi^2} \prod_{p=2}^{\infty} \left(1 + \frac{z}{(p+1)(p-z)}\right),$$

$$(3'') \qquad \sum_{k=0}^{\infty} d_k z^k = \prod_{p=2}^{\infty} \left(\frac{1 + \frac{1}{p-z}}{1 + \frac{1}{p-1}}\right),$$

or

$$(3''') \qquad \sum_{k=0}^{\infty} d_k z^k = \frac{6}{\pi^2} \prod_{p=2}^{\infty} \left(\frac{1 - \frac{z}{p+1}}{1 - \frac{z}{p}}\right).$$

Substituting $z = 0$ into (3′), we obtain the special case

$$(4) \qquad d_0 = \frac{6}{\pi^2}$$

which is well known, since d_0 is the density of square-free integers. Substituting $z = 1$ into (3″) we obtain

$$(5) \qquad \sum_{k=0}^{\infty} d_k = 1$$

which shows that the numbers d_k can be considered as elements of a probability distribution.

The values of $d_1, d_2, \ldots$ can be calculated from (3). For example, we have

$$(6) \qquad d_1 = \frac{6}{\pi^2} \sum_{p=2}^{\infty} \frac{1}{p(p+1)},$$

where p again runs over all primes. For large values of k, the following asymptotic formula can be deduced from (3):

$$(7) \qquad d_k \sim \frac{\delta}{2^k},$$

where

(8) $$\delta = \frac{1}{4} \prod_{p=3}^{\infty} \frac{(p-1)^2}{p(p-2)}.$$

(7) follows from the fact that the function $\sum_{k=0}^{\infty} d_k z^k$ is regular in every point of the circle $|z| = z$, except in $z = 2$, where it has a simple pole.

It should be mentioned that the existence of the densities d_k, follows from a general theorem on additive number-theoretical functions, stated by P. Erdös [1]. We shall give a straightforward elementary proof for the existence of the densities d_k, which gives at the same time, equation (3); the proof is essentially the same as the well known proof of (4) (see e. g. [2] p. 269.)

The idea of the proof is as follows: every positive integer n can be written in the form

(9) $$n = P \cdot Q,$$

where Q is square-free, and P is of the form $P = p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s}$, where $p_1, p_2, \dots, p_s$ are different primes and $\beta_i \geqq 2$, $(i = 1, 2, \dots, s)$ and furthermore, P and Q are relatively prime; the representation of n in the form (9) is unique. We shall call Q the *square-free part* of n and P the *first power-free* part of n. Next, we shall call the set $(\beta_1, \beta_2, \dots, \beta_s)$, the *signature* of n. We shall show that the sequence of integers which have the same given first power-free part P, has a density for every such P, and that the sequence of integers with a prescribed signature $(\beta_1, \beta_2, \dots, \beta_s) = \beta$ has a density; as the sequence of integers n for which $\Delta(n) = k$ is the union of those disjoint sequences which have such signatures $(\beta_1, \beta_2, \dots, \beta_s)$ that

$$\beta_1 + \beta_2 + \dots + \beta_s = s + k,$$

it follows that this sequence also has a density; the proof, incidentally, gives the equation (3'), i. e. (3).

We shall use the following notation:

Let $N(x, P)$ be the number of integers $n \leqq x$ with the prescribed first power-free part P; $\mathcal{N}(x, \beta)$, the number of integers $n = x$ with the prescribed signature β; $\pi(x)$, the number of primes equal to, or greater than x; $[y]$, the integral part of the positive number y, and $s(n)$ the signature of the integer n; clearly, if $n = P \cdot Q$ is the representation of n as a product of a square-free and a first power-free number, we have $s(n) = s(P)$.

Now, we prove the following relation:

(10) $$\mathcal{N}(x,\beta) = \sum_{\substack{s(P)=\beta \\ P \leq \sqrt{\lg x}}} N(x,P) + O\left(\frac{x}{(\lg x)^{1/2b}}\right),$$

where

$$\beta = (\beta_1, \beta_2, \ldots, \beta_s) \text{ and } b = \max_{1 \leqslant k \leqslant s} \beta_k.$$

As a matter of fact, if $n \leqslant x$, $s(n) = \beta$ and $n = PQ$, where

$$P = p_1^{\beta_1} p_2^{\beta_2} \ldots p_s^{\beta_s} > \sqrt{\lg x},$$

then n is divisible by $(p_1 p_2 \ldots p_s)^2$ and $p_1 p_2 \ldots p_s \geqslant P^{1/b} > (\lg x)^{1/2b}$; thus, the number of such integers does not exceed

$$\sum_{m > (\lg x)^{1/2b}} \frac{x}{m^2} = O\left(\frac{x}{(\lg x)^{1/2b}}\right),$$

which proves (10).

Let us call $N^*(x,P)$, the number of those integers $n \leqslant x$ which are of the form $n = PQ$ where $(P,Q) = 1$ and Q is not divisible by the square of any prime which is less than $\lg x$. Clearly, then, we have

(11) $$N(x,P) = N^*(x,P) + O\left(\frac{x}{\lg x}\right)$$

because the number of those integers $n \leqq x$ which are divisible by the square of a prime $> \lg x$ can not exceed

$$\sum_{p > \lg x} \frac{x}{p^2} = O\left(\frac{x}{\lg x}\right).$$

Now, $N^*(x,P)$ can be calculated by the well known sieving method

(12) $$N^*(x,P) = \left[\frac{x}{P}\right] - \sum\left[\frac{x}{Ph_i}\right] + \sum_{i \neq j}\left[\frac{x}{Ph_i h_j}\right] - \ldots\ldots$$

where $h_i, h_j, \ldots$ run over the primes $p_1, p_2, \ldots, p_s$ and over the numbers p^2 where p is a prime which is different from $p_1, p_2, \ldots, p_s$ and $p < \lg x$.

If $P < \sqrt{\lg x}$, then $p_i < \sqrt{\lg x} < \lg x$, $(i = 1, 2, \ldots, s)$ and thus the number of terms in the right hand member of (12) is $2^{\pi(\lg x)}$. Thus, if we delete all brackets in the right hand member of (12), the error committed thereby will not exceed $2^{\pi(\lg x)}$. Thus, taking into account that

$$2^{\pi(\lg x)} = O\left(e^{\frac{c_1 \lg x}{\lg \lg x}}\right),$$

where c_1 is an absolute constant, it follows that

$$N^*(x, P) = \frac{x}{P} \prod_{i=1}^{s} \left(1 - \frac{1}{p_i}\right) \prod_{\substack{p < \lg x \\ p \neq p_i \ (i = 1, \ldots, s)}} \left(1 - \frac{1}{p^2}\right) + O\left(e^{\frac{c_1 \lg x}{\lg \lg x}}\right), \tag{13}$$

where in the second product in the right hand member p runs over all primes $p < \lg x$ except $p_1, p_2, \ldots, p_s$. It follows from (11) and (13) that

$$\frac{N(x, P)}{x} = \frac{\prod_{p < \lg x} (1 - 1/p^2)}{(p_1 + 1) p_1^{\beta_1 - 1} \ldots (p_s + 1) p_s^{\beta_s - 1}} + O\left(\frac{1}{\lg x}\right). \tag{14}$$

Thus we obtain the result, that the sequence of those integers which have the given first power-free part $P = p_1 p_2 \ldots p_s$ has the density

$$d(P) = \frac{6}{\pi^2} \frac{1}{(p_1 + 1) p_1^{\beta_1 - 1} \ldots (p_s + 1) p_s^{\beta_s - 1}}. \tag{15}$$

It also follows from (10) and (14) that

$$\frac{\mathcal{N}(x, \beta)}{x} = \prod_{p < \lg x} \left(1 - \frac{1}{p^2}\right) \sum_{p_1^{\beta_1} \ldots p_s^{\beta_s} < \sqrt{\lg x}} \left(\frac{1}{(p_1 + 1) p_1^{\beta_1 - 1} \ldots (p_s + 1) p_s^{\beta_s - 1}}\right) + \tag{16}$$
$$+ O\left(1/(\lg x)^{1/2b}\right).$$

Thus, we see that the sequence of integers which have the prescribed signature $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$ has the density

$$\delta(\beta) = \frac{6}{\pi^2} \sum \frac{1}{(p_1 + 1) p_1^{\beta_1 - 1} \ldots (p_s + 1) p_s^{\beta_s - 1}}, \tag{17}$$

where the summation is to be extended over all s-tuples $(p_1, p_2, \ldots, p_s)$ of different primes.

If β is the empty set, the sum in the right hand member of (17) has to be replaced by 1.

As, clearly,

$$d_k = \sum_{\beta_1+\beta_2+\cdots+\beta_s-s=k} d(\beta) \tag{18}$$

we obtain

$$\sum_{k=0}^{\infty} d_k z^k = \frac{6}{\pi^2} \prod_{p=2}^{\infty} \left(1 + \frac{1}{p+1}\left(\frac{z}{p} + \frac{z^2}{p^2} + \cdots\right)\right) \tag{19}$$

and (19) is clearly equivalent to (3′) or (3″).

Equation (3) and (3″) can be obtained from (3′) by using the identity $\prod_{p=2}^{\infty} (1 - 1/p^2 = 6/\pi^2$. It can be seen from (3‴) that $\sum_{k=0}^{\infty} d_k z^k$ is a meromorphic function with simple poles at $z = p$, where p is a prime $\neq 3$, and simple zeros at $z = p + 1$, where p is a prime $\neq 2$.

It should be added that the existence of $d(P)$ follows from the mentioned theorem of Erdös, by applying it to the additive number-theoretical function $f(n) = \lg P$, where P is the first power-free part of n; on the other hand, the existence of $\delta(\beta)$ is *not* a consequence of the existence of $d(P)$ for any P, because the sequence of integers with a prescribed signature β is the union of an *infinite* set of sequences, each of which consists of the integers which have a prescribed first power-free part P for which $s(P) = \beta$. It would be, however, possible to state a general theorem, from which the existence of $\delta(\beta)$ follows. We hope to do this in another paper.

(Received 21 december 1055)

REFERENCES

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