ON THE NUMBER OF ZEROS OF SUCCESSIVE DERIVATIVES OF ANALYTIC FUNCTIONS

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Introduction

Let $f(z)$ be regular in the circle $|z|<R$. Let us denote by $N_k(f(z), r)$ the number of zeros of the k-th derivative $f^{(k)}(z)$ of $f(z)$ in the closed circle $|z|\leqq r<R$. In the present paper we shall investigate the asymptotic properties of the sequence $N_k(f(z), r)$ $(k=1,2,\ldots)$.

In this direction several results have been obtained by G. PÓLYA (see [1]). One of the results of PÓLYA is the following: If $f(z)$ is an entire function of finite order $\lambda \geqq 1$, then for any $r>0$ we have

$$\liminf_{k\to\infty} \frac{\log N_k(f(z), r)}{\log k} \leqq \frac{\lambda-1}{\lambda}. \tag{1}$$

Let us denote by $\mathfrak{N}_k(f(z), I)$ the number of zeros of $f^{(k)}(z)$ in the real closed interval I. Further results of PÓLYA are as follows: If $f(z)$ is real on the real axis, and it is analytic in the closed interval I, we have

$$\liminf_{k\to\infty} \frac{\mathfrak{N}_k(f(z), I)}{k} < +\infty; \tag{2}$$

if $f(z)$ is an entire function, we have

$$\liminf_{k\to\infty} \frac{\mathfrak{N}_k(f(z), I)}{k} = 0; \tag{3}$$

finally, that if $f(z)$ is an entire function of exponential type, we have

$$\liminf_{k\to\infty} \mathfrak{N}_k(f(z), I) < +\infty. \tag{4}$$

Recently, M. A. YEVGRAFOV [2] proved the following general result:[1] *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function, the coefficients of which satisfy the inequality*

$$|a_n| \leqq \frac{M A^n}{q(1)\,q(2)\ldots q(n)} \qquad (n=1,2,\ldots)$$

[1] The authors are indebted to R. P. BOAS, Jr. who kindly called their attention to this result.

where $q(x)$ *is positive and increasing for* $x \geqq 1$, *further* $q'(x)$ *exists and* $\lim_{x\to\infty} x\frac{q'(x)}{q(x)} = \varrho$ *where* $0 \leqq \varrho \leqq 1$. *Then we have*

$$\liminf_{k\to\infty} \frac{N_k(f(z), r)q(k)}{k} < +\infty. \tag{5}$$

In § 2 of the present paper we shall prove that the theorem of YEVGRAFOV is a consequence of the following simpler and more general theorem:

If $\operatorname{Max}_{|z|=r} |f(z)| = M(r) = e^{G(r)}$, *further if* $x = H(y)$ *denotes the inverse function of* $y = G(x)$, *we have*

$$\liminf_{k\to\infty} \frac{N_k(f(z), r)H(k)}{k} < +\infty. \tag{6}$$

We shall show also that (6) can be replaced by

$$\liminf_{k\to\infty} \frac{N_k(f(z), 1)H(k)}{k} \leqq e^2 \tag{7}$$

(Theorem 2'). As a matter of fact, we shall prove more, namely we obtain a theorem (Theorem 2) which is much stronger than YEVGRAFOV's theorem. Our theorem states that *if* $f(z)$ *is an entire function,* $M(r) = \operatorname{Max}_{|z|=r} |f(z)|$ *and if we suppose only*

$$\liminf_{r\to\infty} \frac{\log M(r)}{g(r)} < 1 \tag{8}$$

where $g(r)$ *is an arbitrary continuous and monotonically increasing function for which* $\lim_{r\to\infty} g(r) = +\infty$, *then we have*

$$\liminf_{k\to\infty} \frac{N_k(f(z), 1)\, h(k)}{k} \leqq e^2 \tag{9}$$

where $x = h(y)$ *denotes the inverse function of* $y = g(x)$.

The results (1), (3) and (4) are included in YEVGRAFOV's theorem and in our Theorem 2, respectively. In § 1 we prove a theorem on functions analytic in a circle. In § 3 we prove some results on the sequence $r_k = |z_k|$ $(k = 1, 2, \ldots)$ where z_k denotes that root of $f^{(k)}(z)$ which is nearest to the origin; we generalize thereby some previous results, e. g. theorems of ÅLANDER [3] and ERWE [8].

§ 1. Functions regular in a circle

We begin by proving

THEOREM 1. *If $f(z)$ is regular in the circle $|z|<1$ and $0<r<1$, we have*

$$\liminf_{k\to\infty} \frac{N_k(f(z), r)}{k} \leqq K(r) \tag{10}$$

where $K=K(r)$ is the only positive root of the transcendental equation

$$r=\frac{K}{(1+K)^{1+\frac{1}{K}}}. \tag{11}$$

Theorem 1 can also be written in the following equivalent form:

THEOREM 1'. *If $f(z)$ is regular in the circle $|z|<\dfrac{(1+K)^{1+\frac{1}{K}}}{K}$ $(K>0)$, we have*

$$\liminf_{k\to\infty} \frac{N_k(f(z), 1)}{k} \leqq K. \tag{12}$$

Let us mention the following special case of Theorem 1': (12) is valid with $K=1$ if $f(z)$ is regular in the circle $|z|<4$.

Theorem 1' implies that if $f(z)$ is an entire function, we have

$$\liminf_{k\to\infty} \frac{N_k(f(z), r)}{k}=0$$

for any $r>0$.

The proofs of the above theorems are based on the well-known theorem of JENSEN (see e.g. [4]): *If $g(z)$ is regular in a circle $|z|<R$, $g(0)\neq 0$ and $z_1, z_2, \ldots, z_n$ are the zeros of $g(z)$ in the circle $|z|\leqq \varrho<R$, then we have*

$$\log \frac{\varrho^n}{|z_1|\cdot|z_2|\cdots|z_n|}=\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{g(\varrho e^{i\varphi})}{g(0)}\right| d\varphi.$$

If $N_0(g(z), r)$ denotes the number of zeros of $g(z)$ in the circle $|z|\leqq r<\varrho$, it follows from (12) that

$$N_0(g(z), r)\log\frac{\varrho}{r} \leqq \operatorname*{Max}_{|z|=\varrho} \log\left|\frac{g(z)}{g(0)}\right|. \tag{13}$$

We shall always use JENSEN*'s theorem in the form* (13).

Some simple inequalities, which will be frequently used in this paper, are collected in the following

LEMMA. *If* $f(z)=\sum_{k=0}^{\infty} a_k z^k$ *is regular in* $|z|<R$ *and for some value of* $A\geqq 1$ *and* $B>0$ *we have*

$$|a_{k+j}|<\frac{A|a_k|}{B^j} \qquad (j=1,2,\ldots), \tag{14}$$

then for $|z|=\varrho<R$

$$\left|\frac{f^{(k)}(z)}{f^{(k)}(0)}-1\right|\leqq A\left(\frac{1}{\left(1-\frac{\varrho}{B}\right)^{k+1}}-1\right) \tag{15}$$

and thus

$$\left|\frac{f^{(k)}(z)}{f^{(k)}(0)}\right|\leqq \frac{A}{\left(1-\frac{\varrho}{B}\right)^{k+1}}. \tag{16}$$

PROOF. (14) implies $|a_k|>0$ and

$$\frac{f^{(k)}(z)}{f^{(k)}(0)}=1+\sum_{j=1}^{\infty}\frac{a_{k+j}}{a_k}\cdot\frac{(k+1)(k+2)\cdots(k+j)}{j!}z^j. \tag{17}$$

Taking into account that

$$\frac{1}{(1-x)^{k+1}}=1+\sum_{j=1}^{\infty}\frac{(k+1)(k+2)\cdots(k+j)}{j!}x^j$$

for $|x|<1$, (15) and from this (16) follows.

PROOF OF THEOREM 1′. Let us suppose that the radius of convergence of the power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is finite and equal to $R>1$. In this case we have $\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}=\frac{1}{R}$. Thus if $1<B<R<C$, we can find an infinity of values of k for which $\sqrt[k]{|a_k|}>\frac{1}{C}$ and $\sqrt[k+j]{|a_{k+j}|}\leqq\frac{1}{B}$ $(j=1,2,\ldots)$ and thus

$$|a_{k+j}|\leqq\frac{\left(\frac{C}{B}\right)^k|a_k|}{B^j} \qquad (j=1,2,\ldots). \tag{18}$$

On the other hand, if $R=\infty$, then $\sqrt[n]{|a_n|}\to 0$ and thus we can find for any $B>0$ an infinity of values of k for which

$$|a_{k+j}|\leqq\frac{|a_k|}{B^j} \qquad (j=1,2,\ldots). \tag{19}$$

As a matter of fact, if $\max_{n \geqq N} \sqrt[n]{|a_n|} = \sqrt[k_N]{|a_{k_N}|} < \frac{1}{B}$ (which will be true for all sufficiently large values of N), then $k = k_N$ satisfies (19).

The inequalities (18) and (19) can be combined, and it follows that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is regular in the circle $|z| < R$ $(R > 1)$ (but may be regular also in a larger circle or in the whole plane), then for any $q > 1$ and $B < R$ we can find an infinity of values of k such that

$$|a_{k+j}| \leqq \frac{q^k |a_k|}{B^j} \qquad (j = 1, 2, \ldots). \tag{20}$$

It follows from our Lemma that for $|z| = \varrho$ $(1 < \varrho < R)$

$$\left| \frac{f^{(k)}(z)}{f^{(k)}(0)} \right| \leqq \frac{q^k}{\left(1 - \frac{\varrho}{B}\right)^{k+1}} \tag{21}$$

and thus, applying (13) with $r = 1$ and $g(z) = f^{(k)}(z)$, we obtain

$$N_k(f(z), 1) \leqq \frac{k \log q + (k+1) \log \left(1 - \frac{\varrho}{B}\right)^{-1}}{\log \varrho} \tag{22}$$

which implies, as q may be chosen arbitrarily near to 1 and B to R, that

$$\liminf_{k \to \infty} \frac{N_k(f(z), 1)}{k} \leqq \frac{\log \left(1 - \frac{\varrho}{R}\right)^{-1}}{\log \varrho} \qquad \text{for} \qquad 1 < \varrho < R. \tag{23}$$

Now let us choose the value of ϱ so as to minimize the right hand side of (23), that is, let ϱ be equal to $(1+K)^{\frac{1}{K}}$ where K is the positive root of the equation $R = \frac{(1+K)^{1+\frac{1}{K}}}{K}$ which has a unique solution for any $R > 1$. Thus we have proved Theorem 1′, and therefore Theorem 1, too.

We do not know whether the bound in (10) is best possible or not. The estimation (10) is, however, best possible in the following sense: it is clear from the proof of Theorem 1′ that we considered only such values of k for which $f^{(k)}(0) \neq 0$; thus we have obtained slightly more than is expressed by (10), namely we proved

$$\liminf_{\substack{k \to \infty \\ f^{(k)}(0) \neq 0}} \frac{N_k(f(z), r)}{k} \leqq K(r). \tag{10′}$$

Now (10′) is a best possible estimation; this can be shown by considering the function

$$(24)\qquad g(z,K)=\sum_{n=0}^{\infty} z^{[(1+K)^n]}$$

where K is the only positive root of the equation (10) and $[x]$ denotes the integer part of x. Let us put $k_n=[(1+K)^n]$ and consider $g^{(k_n)}(z,K)$. We have clearly

$$\frac{g^{(k_n)}(z,K)}{g^{(k_n)}(0,K)}=P_n(z)+Q_n(z)$$

where

$$P_n(z)=1+\frac{(k_n+1)\cdots k_{n+1}}{(k_{n+1}-k_n)!}z^{k_{n+1}-k_n}$$

and

$$Q_n(z)=\sum_{j=2}^{\infty}\frac{(k_n+1)\cdots k_{n+j}}{(k_{n+j}-k_n)!}z^{k_{n+j}-k_n}.$$

The roots of the equation $P_n(z)=0$ are all lying on the circle

$$|z|=\varrho_n=\left(\frac{(k_{n+1}-k_n)!}{(k_n+1)\cdots k_{n+1}}\right)^{\frac{1}{k_{n+1}-k_n}}$$

and by Stirling's formula we obtain

$$\lim_{n\to\infty}\varrho_n=r=\frac{K}{(1+K)^{1+\frac{1}{K}}}.$$

If $\varepsilon>0$, we have on the circle $|z|=r(1+\varepsilon)$

$$|P_n(z)|\geqq\left(1+\frac{\varepsilon}{2}\right)^{k_{n+1}-k_n}$$

for $n\geqq n_0(\varepsilon)$. On the same circle we have

$$\left|\frac{(k_n+1)\cdots(k_{n+j})}{(k_{n+j}-k_n)!}z^{k_{n+j}-k_n}\right|\leqq\left[\frac{(1+K)^{\frac{j}{(1+K)^j-1}}}{1-\frac{1}{(1+K)^j}}\cdot\frac{K(1+2\varepsilon)}{(1+K)^{1+\frac{1}{K}}}\right]^{k_{n+j}-k_n}\qquad (j=2,3,\ldots).$$

As $\dfrac{j}{(1+K)^j-1}\leqq\dfrac{1}{K}$, we have

$$|Q_n(z)|\leqq 4+2K\quad\text{for}\quad |z|=r(1+\varepsilon)$$

if $0<\varepsilon<\frac{1}{4(K+1)}$. It follows by ROUCHÉ's theorem, that

$$\lim_{n\to\infty}\frac{N_{(1+k)^n}(g(z,k),r(1+\varepsilon))}{[(1+K)^n]}=K \tag{25}$$

if $0<\varepsilon<\frac{1}{4(K+1)}$.

Let us mention that $g^{(n)}(z,K)$ has more than cn zeros $(c>0)$ in $|z|<r_0$ for some $r_0>r$ and *every* $n=1,2,\ldots$.

§ 2. Entire functions

As it has been mentioned in § 1, it follows from Theorem 1 that if $f(z)$ is an *entire* function, we have

$$\liminf_{k\to\infty}\frac{N_k(f(z),1)}{k}=0. \tag{26}$$

(26) can not be improved, i. e. no relation of the form

$$\liminf_{k\to\infty}\frac{N_k(f(z),1)}{k\varepsilon(k)}=0$$

holds with $\lim_{k\to\infty}\varepsilon(k)=0$ $(\varepsilon(k)>0)$ for *all* entire functions. (26) can, however, be strengthened if we put some restriction on the rate of growth of $f(z)$. This is expressed by the following

THEOREM 2. *Let $g(r)$ denote an arbitrary function, monotonically increasing in $0<r<+\infty$, for which $\lim_{r\to+\infty}g(r)=+\infty$. Let $x=h(y)$ denote the inverse function of $y=g(x)$. Let us suppose that $f(z)$ is an entire function for which, putting $M(r)=\operatorname{Max}_{|z|=r}|f(z)|$, we have*

$$\liminf_{r\to+\infty}\frac{\log M(r)}{g(r)}<1.$$

Then we have

$$\liminf_{k\to\infty}\frac{N_k(f(z),1)h(k)}{k}\leqq e^2. \tag{27}$$

PROOF OF THEOREM 2. Let $\varepsilon>0$ denote an arbitrary small positive number. Let us denote by $\nu(r)$ $(0<r<+\infty)$ the central index of the series

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

for $|z|=r$, i. e. suppose

$$|a_n|r^n \leqq |a_{\nu(r)}|r^{\nu(r)} \qquad (n=0,1,2,\ldots)$$

and thus

(28)
$$|a_{\nu(r)+j}| \leqq \frac{|a_{\nu(r)}|}{r^j} \qquad (j=1,2,\ldots).$$

Let us consider such a value $r>0$ for which

(29)
$$\log M(re) \leqq g(re).$$

By our supposition we can find arbitrarily large values of r satisfying (29).

Applying our Lemma with $A=1$, $B=r$, $k=\nu(r)$, $R>r$, $\varrho=e$, we obtain

$$\left|\frac{f^{\nu(r)}(z)}{f^{\nu(r)}(0)}\right| \leqq \frac{1}{\left(1-\frac{e}{r}\right)^{\nu(r)+1}} \quad \text{for} \quad |z|=e,$$

and thus by JENSEN's theorem

(30)
$$N_{\nu(r)}(f(z),1) \leqq (\nu(r)+1)\log\frac{1}{1-\frac{e}{r}} \leqq \frac{\nu(r)e(1+\varepsilon)}{r}$$

if $r \geqq r_0(\varepsilon)$

Now, taking into account that for every $n=1,2,\ldots$ and every $R>0$ we have $|a_N|R^N \leqq M(R)$, and using (29), we have

$$|a_n|(re)^n \leqq M(re) < e^{g(re)}$$

and thus

$$|a_n|r^n \leqq e^{g(re)-n} \qquad (n=1,2,\ldots).$$

Therefore

$$|a_n|r^n \leqq 1 \quad \text{if} \quad n \geqq g(re).$$

But it is known,[2] that the absolute value of the maximal term on $|z|=r$ of the power series of an entire function is tending to $+\infty$ for $r\to\infty$; thus it follows that if r is a sufficiently large value, satisfying (29), we have $\nu(r) \leqq g(re)$, and thus $h(\nu(r)) \leqq re$. It follows from (30) that

(31)
$$N_{\nu(r)}(f(z),1) \leqq \frac{\nu(r)e^2(1+\varepsilon)}{h(\nu(r))}.$$

Thus, taking into account that $\nu(r)\to\infty$ for $r\to\infty$ and that $\varepsilon>0$ is arbitrary, (27) follows.

We can prove quite similarly also the following

[2] See e. g. [7], p. 2, Problem No. 9.

THEOREM 2′. *If $f(z)$ is an arbitrary entire function, $M(r)=\operatorname{Max}_{|z|=r}|f(z)|$, and $x=H(y)$ denotes the inverse function of $y=\log M(r)$, then we have*

$$\liminf_{k\to\infty}\frac{N_k(f(z),1)H(k)}{k}\leqq e^2.$$

PROOF. Clearly, the condition $\liminf_{n\to\infty}\frac{\log M(r)}{G(r)}<1$ is needed in the proof of Theorem 2 only to ensure the existence of arbitrary large values of r for which (29) is valid. Now for $g(r)=G(r)$ (29) is valid for *all* values of r, thus Theorem 2′ follows.

Theorem 2 is best possible in the following case: if $g(r)$ is a monotonically increasing and convex function for which $g(0)=0$, $g'(0)=0$, $g(1)=1$ and $\lim_{r\to\infty}\frac{g(r)}{r}=+\infty$, then there can be found an entire function $f(z)$ such that putting $M(r)=\operatorname{Max}_{|z|=r}|f(z)|$ we have $\liminf_{r\to\infty}\frac{\log M(r)}{g(r)}<+\infty$ and nevertheless

$$\liminf_{k\to\infty}\frac{N_k(f(z),2)\,h(k)}{k}>0$$

where $x=h(y)$ is the inverse of $y=g(x)$. As a matter of fact, if the sequence n_k is defined by $n_0=0$, $n_1=1$ and by the recursion formula $n_{k+1}=\left[n_k\left(1+\frac{e}{h(n_k)}\right)\right]$, the function

$$f(z)=\sum_{k=1}^{\infty}\frac{z^{n_k}}{\prod_{j=1}^{k}[h(n_j)]^{n_j-n_{j-1}}}$$

has all the properties required. This can be shown again by using ROUCHÉ's theorem as follows:

Let us consider first $f^{(n_k)}(z)$. We have clearly

$$\frac{f^{(n_k)}(z)}{f^{(n_k)}(0)}=P_k(z)+Q_k(z)$$

where

$$P_k(z)=1+\frac{(n_k+1)\dots n_{k+1}}{(n_{k+1}-n_k)!}\left(\frac{z}{h(n_{k+1})}\right)^{n_{k+1}-n_k}$$

and

$$Q_k(z)=\sum_{j=2}^{\infty}\frac{(n_k+1)\dots n_{k+j}}{(n_{k+j}-n_k)!}\,\frac{z^{n_{k+j}-n_k}}{\prod_{s=k+1}^{k+j}h(n_s)^{n_s-n_{s-1}}}\,.$$

Clearly, all roots of $P_k(z)=0$ are lying on the circle

$$|z|=\varrho_k=h(n_{k+1})\left[\frac{(n_{k+1}-n_k)!}{(n_k+1)\ldots n_{k+1}}\right]^{\frac{1}{n_{k+1}-n_k}}$$

and we have for $k\to\infty$ $\varrho_k\sim\dfrac{h(n_{k+1})}{h(n_k)}$. But as $h'(y)=\dfrac{1}{g'(x)}$ is decreasing, we have

$$1\leqq\frac{h(n_{k+1})}{h(n_k)}\leqq 1+\frac{n_k\,h'(n_k)e}{h^2(n_k)}.$$

But

$$\frac{yh'(y)}{h(y)}=\frac{g(x)}{xg'(x)}$$

and as $g''(x)\geqq 0$ we have

$$\frac{g(x)}{xg'(x)}=\frac{\int\limits_0^x g''(t)\,(x-t)\,dt}{\int\limits_0^x g''(t)\,x\,dt}\leqq 1.$$

Thus it follows $\lim\limits_{k\to\infty}\varrho_k=1$.

Clearly, on the circle $|z|=1+\varepsilon$ we have

$$|P_k(z)|\geqq\left(1+\frac{\varepsilon}{2}\right)^{n_{k+1}-n_k}\quad\text{for}\quad k\geqq k_0(\varepsilon).$$

On the other hand, on the same circle we have

$$\left|\frac{(n_k+1)\ldots n_{k+j}}{(n_{k+j}-n_k)!}\cdot\frac{z^{n_{k+j}-n_k}}{\prod\limits_{s=k+1}^{k+j}h(n_s)^{n_s-n_{s-1}}}\right|\leqq\left[\frac{\left(1+\dfrac{n_{k+j}-n_k}{n_k}\right)^{\frac{n_k}{n_{k+j}-n_k}}\left(1+\dfrac{3\varepsilon}{2}\right)}{\left(1-\dfrac{n_k}{n_{k+2}}\right)h(n_{k+1})}\right]^{n_{k+j}-n_k}$$

for sufficiently large values of k. As $(1+x)^{\frac{1}{x}}\leqq e$ and $\left(1-\dfrac{n_k}{n_{k+2}}\right)h(n_{k+1})\to 2e$ for $k\to\infty$, it follows that for $|z|=1+\varepsilon$ $\left(0<\varepsilon<\dfrac{1}{2}\right)$

$$|Q_k(z)|\leqq\frac{2}{1-2\varepsilon}\quad\text{for}\quad k\geqq k_1(\varepsilon).$$

Thus $f^{(n_k)}(z)$ has $n_{k+1}-n_k$ roots in the circle $|z|=1+\varepsilon$ for $0<\varepsilon<1/2$ and $k\geqq k_1(\varepsilon)$.

Let us consider now a number N, $n_{k-1} < N < n_k$. If $N \leqq \dfrac{n_k}{1+\dfrac{e}{4h(n_k)}}$, then $f^{(N)}(z)$ has more than $\dfrac{N}{2h(N)}$ roots in the point $z=0$. On the other hand, if $N > \dfrac{n_k}{1+\dfrac{e}{4h(n_k)}}$, let us have $N \sim \dfrac{n_k}{1+\dfrac{\lambda e}{h(n_k)}}$ $\left(0<\lambda<\dfrac{1}{4}\right)$.

We have clearly

$$\frac{f^{(N)}(z)}{z^{n_k-N} n_k(n_k-1)\dots(n_k-N+1)} = p_N(z)+q_N(z)$$

where

$$q_N(z) = \sum_{j=2}^{\infty} \frac{n_{k+j}(n_{k+j}-1)\cdots(n_{k+j}-N+1)}{n_k(n_k-1)\cdots(n_k-N+1)} \cdot \frac{z^{n_{k+j}-n_k}}{\prod_{s=k+1}^{k+j} h(n_s)^{n_s-n_{s-1}}}$$

and

$$p_N(z) = 1 + \frac{n_{k+1}(n_{k+1}-1)\cdots(n_{k+1}-N+1)}{n_k(n_k-1)\cdots(n_k-N+1)} \left(\frac{z}{h(n_{k+1})}\right)^{n_{k+1}-n_k}.$$

The roots of $p_N(z)=0$ are all lying in the circle $|z|=R_N$ where $R_N \sim \sim (1+\lambda)\left(1+\dfrac{1}{\lambda}\right)^{\lambda} \leqq \dfrac{5}{4}\sqrt[4]{5}$ as $0<\lambda<\dfrac{1}{4}$. But on the circle $|z|=\left(1+\dfrac{\delta}{2}\right)\dfrac{5}{4}\sqrt[4]{5}$ we have for any $\delta>0$, if K is sufficiently large,

$$\left|\frac{n_{k+j}(n_{k+j}-1)\cdots(n_{k+j}-N+1)}{n_k(n_k-1)\cdots(n_k-N+1)} \cdot \frac{z^{n_{k+j}-n_k}}{\prod_{s=k+1}^{k+j} h(n_s)^{n_s-n_{s-1}}}\right| \leqq$$

$$\leqq \left(\frac{(1+2\delta)\frac{5}{4}\sqrt[4]{5}}{2}\right)^{n_{k+j}-n_k} = \beta^{n_{k+j}-n_k} \qquad (j=2,3,\dots)$$

where $\beta<1$ if $0<\delta<\dfrac{1}{2}\left(\dfrac{8}{5\sqrt[4]{5}}-1\right)$.

Thus it follows by ROUCHÉ's theorem that $f^{(N)}(z)$ has $n_{k+1}-n_k$ roots in the circle $|z|=\left(1+\dfrac{\delta}{2}\right)\dfrac{5}{4}\sqrt[4]{5}$. As $n_{k+1}-n_k \geqq \dfrac{N}{2h(N)}$, combining the cases $n_{k-1}<N\leqq \dfrac{n_k}{1+\dfrac{e}{4h(n_k)}}$ and $n_k \geqq N > \dfrac{n_k}{1+\dfrac{e}{4h(n_k)}}$, it follows that $f^{(N)}(z)$ has

$\geqq \frac{N}{2h(N)}$ roots in the circle $|z| < 2$. Thus we have

$$\liminf_{k\to\infty} \frac{N_k(f(z), 2)h(k)}{k} \geqq \frac{1}{2},$$

what was to be proved.

It remains to show that $\liminf_{r\to\infty} \frac{\log M(r)}{g(r)} < +\infty$. This can be done as follows: let us put $r_k = h(n_k)$ and

$$\mu(r_k) = \frac{r_k^{n_k}}{\prod_{j=1}^{k} h(n_j)^{n_j - n_{j-1}}}.$$

First we show that

$$\limsup_{k\to\infty} \frac{\log \mu(r_k)}{n_k} < +\infty.$$

This can be proved by starting from the evident formula

$$\frac{\log \mu(r_k)}{n_k} = \frac{1}{n_k} \sum_{j=1}^{k} (n_j - n_{j-1}) \log \frac{h(n_k)}{h(n_j)}.$$

Let us denote by S_r $(r = 0, 1, \ldots)$ the set of those values of j for which

$$\frac{h(n_k)}{2^{r+1}} \leqq h(n_j) < \frac{h(n_k)}{2^r}.$$

Let I_r denote the greatest element of the set S_r. Then we have clearly

$$\frac{\log \mu(r_k)}{n_k} \leqq \frac{1}{n_k} \sum_{r=0}^{\infty} (r+1) n_{I_r}.$$

Now $n_{I_r} \leqq g\left(\frac{h(n_k)}{2^r}\right)$ and $g(x)$ is convex, therefore

$$n_{I_r} \leqq \frac{g(h(n_k))}{2^r} = \frac{n_k}{2^r}$$

and thus

$$\frac{\log \mu(r_k)}{n_k} \leqq \sum_{r=0}^{\infty} \frac{r+1}{2^r} = 4.$$

Now $\mu(r_k)$ is the maximal term of the series

$$M(r_k) = \sum_{S=1}^{\infty} \frac{r_k^{n_S}}{\prod_{j=1}^{S} h(n_j)^{n_j - n_{j-1}}}$$

and it is easy to show that

$$\lim_{k\to\infty} \frac{\log M(r_k)}{\log \mu(r_k)} = 1.$$

Taking into account that $n_k = g(r_k)$, we obtain

$$\liminf_{r\to\infty} \frac{\log M(r)}{g(r)} \leqq 4.$$

By the same method it can be shown that $\liminf_{r\to\infty} \frac{\log M(r)}{g(r)} \leqq 1$, but for our purpose this is not necessary.

The theorem of YEVGRAFOV can be deduced from Theorem 2 as follows: Let us suppose that $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is an entire function and

$$|a_n| \leqq \frac{MA^n}{q(1)q(2)\cdots q(n)} \qquad (n=1, 2, \ldots)$$

where $q(x)$ is positive and monotonically increasing for $x \geqq 1$, $\lim_{x\to\infty} q(x) = +\infty$ and $\lim_{x\to+\infty} \frac{xq'(x)}{q(x)} = \varrho$ where $0 \leqq \varrho \leqq 1$; clearly it can be supposed that $q(1) > 1$; let us denote by $x = \gamma(y)$ the inverse of $y = q(x)$, and let us for a given $r > 0$ determine the integer N by

$$N = [\gamma(2Ar)], \quad \text{i. e.} \quad N \leqq \gamma(2Ar) < N+1.$$

Then

$$q(N) \leqq 2Ar \leqq q(N+1). \tag{32}$$

It follows that for $|z| = r$

$$|f(z)| \leqq M \frac{(Ar)^N}{q(1)q(2)\cdots q(N)} (S_1 + S_2),$$

where

$$S_1 = \frac{q(N)}{Ar} + \frac{q(N)q(N-1)}{(Ar)^2} + \cdots + \frac{q(N)q(N-1)\cdots q(2)}{(Ar)^{N-1}}$$

and

$$S_2 = 1 + \frac{Ar}{q(N+1)} + \frac{(Ar)^2}{q(N+1)q(N+2)} + \cdots.$$

Clearly we have

$$|S_1| \leqq 2 + 2^2 + \cdots + 2^{N-1} \leqq 2^N \quad \text{and} \quad |S_2| \leqq 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2.$$

Thus it follows that

$$M(r) \leqq 2M \exp\left[N \log 2Ar - \sum_{k=1}^{N} \log q(k)\right]. \tag{33}$$

As $\log q(k)$ is positive and increasing,

$$\sum_{k=1}^{N} \log q(k) \geqq \int_{1}^{N} \log q(x)\,dx$$

and therefore, by (32),

$$\log M(r) \leqq \log 2M + N \log q(N+1) - \int_{1}^{N} \log q(x)\,dx.$$

According to our supposition $\log q(x)$ is of the form

$$\log q(x) = \varrho \log x + \int_{1}^{x} \frac{\varepsilon(t)}{t}\,dt$$

where $\lim_{t\to\infty} \varepsilon(t) = 0$; it follows that if $\varrho > 0$, $\log M(r) \leqq \varrho N + o(N)$, i. e. for an arbitrary $\varepsilon > 0$ we have

$$\log M(r) \leqq \varrho\gamma(2Ar)(1+\varepsilon) \tag{34}$$

if r is sufficiently large, and thus if $g(r) = 2\varrho\cdot\gamma(2Ar)$ and $x = h(y)$ is the inverse of $y = g(x)$, we obtain by Theorem 2

$$\liminf_{k\to\infty} \frac{N_k(f(z), 1)\,h(k)}{k} \leqq e^2.$$

As

$$h(k) = \frac{1}{2A} q\left(\frac{k}{2\varrho}\right)$$

and

$$\lim_{k\to\infty} \frac{q\left(\frac{k}{2\varrho}\right)}{q(k)} = \left(\frac{1}{2\varrho}\right)^{\varrho},$$

it follows that

$$\liminf_{k\to\infty} \frac{N_k(f(2), 1)\,q(k)}{k} \leqq 2Ae^2\left(\frac{1}{2\varrho}\right)^{\varrho}. \tag{35}$$

Thus we have proved YEVGRAFOV's theorem for $\varrho > 0$.

If $\varrho = 0$, we have $\log M(r) = o(\gamma(2Ar))$ and thus it follows in this case also that

$$\liminf_{k\to\infty} \frac{N_k(f(2), 1)\,q(k)}{k} < +\infty. \tag{36}$$

Now we shall suppose that $f(z)$ is an entire function of order $\geqq 1$ for which, putting $M(r) = \operatorname{Max}_{|z|=r} |f(z)|$, further $\log M(r) = G(r)$, the limit

$$\lim_{n\to\infty} \frac{d \log G(r)}{d \log r} = \alpha \geqq 1 \tag{37}$$

exists; we shall show that in this case, denoting by $x = H(y)$ the inverse function of $y = G(x)$, we have

$$\liminf \frac{N_k(f(z), 1) H(k)}{k} < +\infty, \tag{38}$$

and thus for entire functions of order $\geqq 1$ and satisfying the condition (37) the assertion of Theorem 2′ follows[3] from YEVGRAFOV's theorem. Substituting $r = H(n)$ in the inequality $|a_n| \leqq \frac{e^{G(r)}}{r^n}$ $(n = 1, 2, \ldots)$, we obtain

$$|a_n| \leqq \frac{e^n}{(H(n))^n} \tag{39}$$

and thus

$$|a_n| \leqq \frac{e^n}{H(1) H(2) \cdots H(n)}. \tag{40}$$

Now let us suppose that $f(z)$ is such an entire function for which the finite or infinite limit (37) exists. As

$$\frac{y H'(y)}{H(y)} = \frac{1}{\left(\frac{d \log G(x)}{d \log x}\right)},$$

it follows from the existence of $\lim_{x\to\infty} \frac{d \log G(x)}{d \log x} = \alpha$ that $\lim_{y\to\infty} \frac{y H'(y)}{H(y)} = \varrho = \frac{1}{\alpha}$ exists. As we have supposed that $G(r)$ is of order $\geqq 1$, it follows that $0 \leqq \varrho \leqq 1$.

Thus we have shown that YEVGRAFOV's theorem is equivalent to the special case of Theorem 2′ for entire functions satisfying (37). Thus Theorem 2′ is slightly stronger but, of course, Theorem 2 is essentially stronger than YEVGRAFOV's theorem.

§ 3. Remarks on the zero z_k of $f^{(k)}(z)$ which is nearest to the origin

It follows from our Theorem 2 that especially if

$$\liminf_{r\to\infty} \frac{\log M(r)}{r} < A,$$

[3] Except the numerical estimation of the left hand side of (38).

we obtain[4]

$$\liminf_{k\to\infty} N_k(f(z), 1) < e^2 A.$$

This can be formulated as follows: If r_k denotes the absolute value of the zero z_k of $f^{(k)}(z)$ which is nearest to the origin, we have for an entire function for which $\liminf_{r\to\infty}\frac{\log M(r)}{r} < A$,

$$\limsup_{k\to\infty} r_k \geqq \frac{1}{Ae^2}.$$

For entire functions of finite order $\lambda \geqq 1$, the behaviour of r_k has been investigated by ÅLANDER [3] who proved that

$$\liminf_{k\to\infty} \frac{\log \frac{1}{r_k}}{\log k} \leqq \frac{\lambda-1}{\lambda}.$$

Now we shall prove a general theorem which includes this result of ÅLANDER as a special case.

THEOREM 3. *If $f(z)$ is an entire function, $M(r) = \operatorname{Max}_{|z|=r} |f(z)|$ and r_k denotes the absolute value of the zero z_k of $f^{(k)}(z)$ which is nearest to the origin $(k = 1, 2, \ldots)$, then denoting by $x = H(y)$ the inverse function of $y = \log M(x)$ we have*

$$\liminf_{k\to\infty} \frac{H(k)}{kr_k} \leqq \frac{e}{\log 2}. \tag{41}$$

PROOF. Let us start from the inequality (38). This implies that for any $\varepsilon > 0$

$$\lim_{n\to\infty} \left(\frac{H(n)}{e^{1+\varepsilon}}\right)^n |a_n| = 0. \tag{42}$$

Thus we can find arbitrary large values of k for which

$$|a_{k+j}| \leqq \left(\frac{e^{1+\varepsilon}}{H(k)}\right)^j |a_k| \qquad (j = 1, 2, \ldots). \tag{43}$$

[4] This implies that for $A < \frac{1}{e^2}$

$$\liminf_{k\to\infty} N_k(f(z), 1) < 1,$$

i. e. an infinity of derivatives of $f(z)$ have no zeros in the unit circle. It is known that if $f(z)$ is of exponential type and $\limsup_{r\to\infty}\frac{\log M(r)}{r} < A$, the same assertion holds for $A \leqq 0{,}7199$. (See [5])

It follows from inequality (15) that for such values of k for which (43) holds and for $|z| \leqq \varrho$ we have

$$\left|\frac{f^{(k)}(z)}{f^{(k)}(0)}-1\right| \leqq \frac{1}{\left(1-\frac{\varrho e^{1+\varepsilon}}{H(k)}\right)^{k+1}}-1, \tag{44}$$

and thus $f^{(n)}(z) \neq 0$ for $|z| \leqq \varrho$ if

$$\left(1-\frac{\varrho e^{1+\varepsilon}}{H(k)}\right)^{k+1} > \frac{1}{2},$$

i. e. for a sufficiently large k if

$$\varrho < \frac{H(k) \log 2}{k e^{1+2\varepsilon}}. \tag{45}$$

But (45) implies that

$$\liminf_{k\to\infty} \frac{H(k)}{k r_k} \leqq \frac{e^{1+2\varepsilon}}{\log 2}. \tag{46}$$

As $\varepsilon > 0$ is arbitrary, Theorem 3 is proved.

Clearly, (41) implies

$$\limsup_{k\to\infty} k r_k = +\infty \tag{47}$$

for every entire function.

For functions, which are regular in a circle $|z| < R$, instead of (47) we can prove only

THEOREM 4. *If $f(z)$ is regular in the circle $|z| < R$ and is not a polynomial, further z_k is the root of $f^{(k)}(z)$ which is nearest to the origin, then putting $r_k = |z_k|$ we have*

$$\limsup_{k\to\infty} k r_k \geqq R \log 2. \tag{48}$$

PROOF. The proof is very similar to that of Theorem 3. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leqq \frac{1}{R}$ and thus $\frac{R^n |a_n|}{(1+\varepsilon)^n} \to 0$ for any $\varepsilon > 0$. This implies that putting $\max_{n \geqq N} \frac{R^n |a_n|}{(1+\varepsilon)^n} = \frac{R^{k_N} |a_{k_N}|}{(1+\varepsilon)^{k_N}}$ we have for $k = k_N$ $(N = 1, 2, \ldots)$

$$|a_{k+j}| \leqq \frac{|a_k|}{\left(\frac{R}{1+\varepsilon}\right)^j} \qquad (j = 1, 2, \ldots). \tag{49}$$

Thus by inequality (15) we have for $|z| \leqq \varrho$ and the mentioned values of k

$$\left|\frac{f^{(k)}(z)}{f^{(k)}(0)}-1\right| \leqq \frac{1}{\left(1-\frac{\varrho(1+\varepsilon)}{R}\right)^{k+1}}-1, \tag{50}$$

therefore $f^{(k)}(z) \neq 0$ for $|z| \leqq \varrho$ if

$$\left(1-\frac{\varrho(1+\varepsilon)}{R}\right)^{k+1} > \frac{1}{2}$$

and thus if

$$\varrho \leqq \frac{R \log 2}{(k+1)(1+2\varepsilon)} \tag{51}$$

for sufficiently large k.

The assertion of Theorem 4 follows immediately.

It should be mentioned that there exist functions $f(z)$ regular in the unit circle for which $\limsup_{k\to\infty} k r_k < +\infty$, for example if $f(z)=\frac{1}{1-z^2}$, we have $\limsup_{k\to\infty} k r_k = \frac{\pi}{4}$. This example is due to ERWE [8].

It would be interesting to determine the greatest constant by which $\log 2$ can be replaced in (48).

The question may be raised: what can be said about the series

$$\sum_{k=1}^{\infty} r_k. \tag{52}$$

It can be shown that the series (52) is divergent not only for every entire function but also for every function which is regular in some circle $|z|<R$ (except for polynomials) with $R>0$. As a matter of fact, this follows easily from the results of W. GONTCHAROFF ([6], p. 34).

The following conjecture[5] of ERWE is a simple consequence of this remark: If $f(z)$ is regular in $|z|<R$, $|z_1|<R$, $|z_{n+1}| \leqq \frac{1}{2}|z_n|$ and $f^{(n)}(z_n)=0$ $(n=1,2,\ldots)$, then $f(z)$ is a polynomial. As a matter of fact, we have $r_n \leqq |z_n|$ and thus our suppositions imply $\sum_{n=1}^{\infty} r_n < +\infty$. More can be said about the sequence r_k if the power series of $f(z)$ has Hadamard gaps. If $f(z)=\sum_{k=0}^{\infty} a_n z^{n_k}$ where $\frac{n_{k+1}}{n_k} \geqq q > 1$ and $f(z)$ is an entire function, then

[5] ERWE proved that if $f(z)$ is regular in a circle around $z=0$ containing the points z_n for which $|z_{n+1}| \leqq \frac{1}{2}|z_n|^2$, further $f^{(n)}(z_n)=0$ $(n=1,2,\ldots)$, then $f(z)$ is a polynomial.

$\limsup_{k\to\infty} r_k = +\infty$; if it is supposed only that $f(z)$ is regular in the circle $|z| < R$ and $f(z) = \sum_{k=0}^{\infty} a_n z^{n_k}$ with $\frac{n_{k+1}}{n_k} \geqq q > 1$, then $\limsup_{k\to\infty} r_k \geqq \frac{R\left(1-\frac{1}{q}\right)}{2e}$.

It seems that the following conjecture is true: If $f(z)$ is an entire function, we have

$$\limsup_{k\to\infty} \frac{r_1+r_2+\cdots+r_k}{\log k} = +\infty.$$

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О ЧИСЛЕ КОРНЕЙ ПОСЛЕДОВАТЕЛЬНЫХ ПРОИЗВОДНЫХ АНАЛИТИЧЕСКИХ ФУНКЦИЙ

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(Резюме)

Пусть $f(z)$ регулярна в некоторой области плоскости комплексной переменной, содержащей внутри себя круг $|z| \leq r$ $(r > 0)$, и пусть $N_k(f(z), r)$ означает число корней $f^{(k)}(z)$ в круге $|z| \leq r$ $(k = 1, 2, \ldots)$. Обозначим через z_k наиболее близкий к точке $z=0$ корень от $f^{(k)}(z)$ и пусть $r_k = |z_k|$.

Работа изучает асимптотические свойства последовательностей $N_k(f(z), r)$ и r_k $(k = 1, 2, \ldots)$. В частности, в работе доказываются следующие теоремы:

Теорема 1. Пусть $f(z)$ регулярна в единичном круге и пусть $0 < r < 1$. Тогда

$$\liminf_{k\to\infty} \frac{N_k(f(z), r)}{k} \leq K(r),$$

где $K = K(r)$ есть единственный положительный корень трансцендентного уравнения

$$r = \frac{K}{(1+K)^{1+\frac{1}{K}}}$$

Теорема 2. Пусть $g(r)$ есть любая непрерывная и монотонно возрастающая в интервале $(0 < r < \infty)$ функция и пусть $\lim g(r) = +\infty$. Обозначим через $x = h(y)$ функцию, обратную функции $y = f(x)$. Пусть $f(z)$ есть целая функция, $M(r) = \max_{|z|=r} |f(z)|$ и предположим, что

$$\liminf_{r \to \infty} \frac{\log M(r)}{g(r)} < 1.$$

Тогда

$$\liminf_{k \to \infty} \frac{N_k(f(z), 1)\, h(k)}{k} \leqq e^2.$$

Теорема 3. Пусть $f(z)$ есть целая функция, $M(r) = \max_{|z|=r} |f(z)|$, $x = H(y)$ обозначает функцию, обратную функции $y = \log M(x)$. Тогда

$$\liminf_{k \to \infty} \frac{H(k)}{k r_k} \leqq \frac{e}{\log 2}.$$

Теорема 4. Если $f(z)$ регулярна в единичном круге и не многочлен, то

$$\limsup_{k \to \infty} k r_k \geqq \log 2.$$

Перечисленные теоремы являются обобщениями результатов Пойа [1], Евграфова [2] и Аландера [3]. Работа содержит также доказательство одной гипотезы Эрве [8].