ON MIXING SEQUENCES OF RANDOM VARIABLES

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Introduction

In a recent paper [1] the first named author proved the following

THEOREM 1. *Let* $[\Omega, \mathcal{A}, \mathbf{P}]$ *be a probability space (i. e.* $\mathbf{P} = \mathbf{P}(A)$ *a probability measure defined for sets* A *belonging to the* σ*-algebra* $\mathcal{A}$ *of subsets of the set* Ω*). Let* C_n *be a sequence of sets such that* $C_0 = \Omega$, $C_n \in \mathcal{A}$ *and* $\mathbf{P}(C_n) > 0$ *for* $n = 1, 2, \ldots$. *If for* $k = 0, 1, 2, \ldots$ *the condition*[1]

(1) $$\lim_{n \to \infty} \mathbf{P}(C_n | C_k) = \lambda \qquad (0 < \lambda < 1)$$

is fulfilled, then for any $B \in \mathcal{A}$ *such that* $\mathbf{P}(B) > 0$ *we have*

(2a) $$\lim_{n \to \infty} \mathbf{P}(C_n | B) = \lambda.$$

This implies that if $\mathbf{Q}$ *is an arbitrary measure on* $\mathcal{A}$ *which is absolutely continuous with respect to* $\mathbf{P}$, *we have*

(2b) $$\lim_{n \to \infty} \mathbf{Q}(C_n) = \lambda.$$

We shall say that a sequence $\{\zeta_n\}$ of random variables defined on the probability space $[\Omega, \mathcal{A}, \mathbf{P}]$ has the "mixing" property (or simply: is „mixing") if

(3) $$\mathbf{P}(\zeta_n < x | B) \Longrightarrow F(x)$$

for every $B \in \mathcal{A}$ with $\mathbf{P}(B) > 0$, where $F(x)$ is a distribution function not depending on B. Here $\Longrightarrow$ denotes, as usual, the weak convergence of a sequence of distributions to a limiting distribution (i. e. that $\mathbf{P}(\zeta_n < x)$ tends for $n \to \infty$ to $F(x)$ for every x which is a continuity point of $F(x)$).

It has been shown in [1] that in case $\{\zeta_n\}$ has the mixing property, and $\mathbf{Q}$ is a measure on $\mathcal{A}$ which is absolutely continuous with respect to $\mathbf{P}$, then

(4) $$\mathbf{Q}(\zeta_n < x) \Longrightarrow F(x).$$

In [1] it has been proved by means of Theorem 1 that if $\xi_1, \xi_2, \ldots$ are independent random variables, A_n and B_n sequences of real numbers such

[1] $\mathbf{P}(A|B)$ denotes the conditional probability of A with respect to the condition B.

that $B_n \to +\infty$ for $n \to \infty$ and putting

$$\zeta_n = \frac{\xi_1 + \xi_2 + \cdots + \xi_n - A_n}{B_n} \tag{5}$$

we have

$$\mathbf{P}(\zeta_n < x) \Longrightarrow F(x) \tag{6}$$

where $F(x)$ is a distribution function, then the sequence ζ_n is mixing, i. e. (3) holds.

This result has been proved previously only under some restrictions in [2] and [3].

As the random variables (5) form an (additive) Markov chain, it is natural to ask whether general Markov chains do possess also under suitable conditions the mixing property.

In § 1 it is shown that it is rather easy to find sufficient conditions for a sequence of random variables being mixing. The conditions in question are, especially, satisfied for certain Markov chains. This is shown in § 2 by giving some examples.

§ 1. Some conditions for a sequence of random variables being mixing

Let $\zeta_0, \zeta_1, \zeta_2, \ldots$ be a sequence of (real-valued) random variables and suppose that for any real x and for any possible value y of ζ_k we have for $n \to \infty$

$$\mathbf{P}(\zeta_n < x \mid \zeta_k = y) \Longrightarrow F(x) \tag{1.1}$$

where $F(x)$ is a distribution function.[2]

Putting $\mathbf{P}_k(x) = \mathbf{P}(\zeta_k < x)$ we have for any real z

$$\mathbf{P}(\zeta_n < x \mid \zeta_k < z) = \frac{1}{\mathbf{P}_k(z)} \int_{-\infty}^{z} \mathbf{P}(\zeta_n < x \mid \zeta_k = y)\, d\mathbf{P}_k(y). \tag{1.2}$$

Thus, by the theorem of LEBESGUE on the integration of bounded convergent sequences of functions it follows from (1.2) by virtue of (1.1) that

$$\lim_{n \to \infty} \mathbf{P}(\zeta_n < x \mid \zeta_k < z) = F(x) \tag{1.3}$$

for every point of continuity x of $F(x)$. Especially we have (for $z = +\infty$)

$$\mathbf{P}(\zeta_n < x) \Longrightarrow F(x). \tag{1.4}$$

[2] It suffices to suppose that (1.1) holds for $y \in B_k$ where $\mathbf{P}(\zeta_k \in B_k) = 1$.

Thus, applying Theorem 1 to the sets $C_0 = \Omega$, $C_n = \{\omega : \zeta_n(\omega) < x\}$ $(n = 1, 2, \ldots)$ we obtain that if $\mathbf{Q}$ is a probability measure on $\mathfrak{A}$ which is absolutely continuous with respect to $\mathbf{P}$, we have

$$\mathbf{Q}(\zeta_n < x) \Longrightarrow F(x). \tag{1.5}$$

Evidently, condition (1.3) is also necessary for the validity of (1.5). Thus we have obtained the following

THEOREM 2. *The sequence* $\{\zeta_n\}$ $(n = 0, 1, 2, \ldots)$ *of random variables defined on the probability space* $[\Omega, \mathfrak{A}, \mathbf{P}]$ *has the mixing property if and only if for every* $k = 0, 1, 2, \ldots$ *there exists a set* E_k *for which* $\mathbf{P}(\zeta_k \in E_k) = 1$ *and is such that for* $y \in E_k$ *we have for* $n \to \infty$

$$\mathbf{P}(\zeta_n < x \mid \zeta_k = y) \Longrightarrow F(x) \tag{1.6}$$

where $F(x)$ *is a distribution function not depending on* y.

Theorem 2 can be applied, especially, to verify the mixing property of Markov chains.

§ 2. Examples

EXAMPLE 1. Let $\{\zeta_n\}$ be a Markov chain. It follows from a well-known theorem of KOLMOGOROV [4] that condition (1.6) is satisfied (moreover, the convergence is uniform) if the following two conditions are fulfilled:

a) For any pair x, y of possible values of ζ_k and for any measurable set G we have

$$\mathbf{P}(\zeta_{k+1} \in G \mid \zeta_k = x) \geqq \lambda_k \mathbf{P}(\zeta_{k+1} \in G \mid \zeta_k = y) \tag{2.1}$$

for $k = 0, 1, 2, \ldots$ where the constants $\lambda_k \geqq 0$ are such that

$$\sum_{k=0}^{\infty} \lambda_k = +\infty. \tag{2.2}$$

b) $\mathbf{P}(\zeta_n < x) \Longrightarrow F(x)$ for $n \to \infty$.

EXAMPLE 2. Let $\{\zeta_n\}$ be a homogeneous Markov chain having a finite number of possible states. Suppose that each ζ_n can take on only the values $1, 2, \ldots, r$. Let us denote by p_{ij} the transition probabilities of the Markov chain $\{\zeta_n\}$, i. e. put

$$p_{ij} = \mathbf{P}(\zeta_{n+1} = j \mid \zeta_n = i) \tag{2.3}$$

and denote by π the matrix (p_{ij}). Suppose that there exists a positive integer s so that all elements of π^s are positive. Then, as well known, (1.6) is satisfied. Thus it follows from Theorem 2 that the Markov chain $\{\zeta_n\}$ is mixing.

EXAMPLE 3. Let the random variables ζ_n $(n=0, 1, 2, \dots)$ defined on the probability space $[\Omega, \mathcal{A}, \mathbf{P}]$ form a homogeneous Markov chain. Let C_n and $D_n > 0$ be two sequences of real numbers such that for any fixed $k (k=1,2,\dots)$ we have

$$\lim_{n\to\infty} \frac{D_{n-k}}{D_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{C_n - C_{n-k}}{D_n} = 0, \tag{2.4}$$

further

$$\mathbf{P}\left(\frac{\zeta_n - C_n}{D_n} < x \,\middle|\, \zeta_0 = y\right) \Longrightarrow F(x), \tag{2.5}$$

where $F(x)$ is a distribution function which is independent of the value of y where y may take on any element of a set Y such that $\mathbf{P}(\zeta_0 \in Y) = 1$. Then we have

$$\mathbf{Q}\left(\frac{\zeta_n - C_n}{D_n} < x\right) \Longrightarrow F(x) \tag{2.6}$$

for any probability measure $\mathbf{Q}$ on $\mathcal{A}$ which is absolutely continuous with respect to $\mathbf{P}$.

As a matter of fact, owing to the supposition that the Markov chain is homogeneous and the supposition (2.4), it follows from (2.5) that if $n \to \infty$, then

$$\mathbf{P}\left(\frac{\zeta_n - C_n}{D_n} < x \,\middle|\, \zeta_k = y\right) \Longrightarrow F(x) \tag{2.7}$$

for $k = 0, 1, 2, \dots$; thus our assertion follows from Theorem 2.

EXAMPLE 4. Let us consider the Engel's series of the real number t $(0 < t < 1)$

$$t = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \cdots + \frac{1}{q_1 q_2 \dots q_n} + \cdots \tag{2.8}$$

where $q_n = q_n(t) \geqq 2$ is an integer. It is known (see [5] and for another proof [6]) that $\frac{\log q_n - n}{\sqrt{n}}$ is in the limit for $n \to \infty$ normally distributed, that is

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\log q_n - n}{\sqrt{n}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \tag{2.9}$$

where $\mathbf{P}$ denotes the Lebesgue measure. It has been shown in [6] that the random variables $\log q_n(t)$ are forming a homogeneous Markov chain on the probability space $[\Omega, \mathcal{A}, \mathbf{P}]$ where Ω is the interval $(0, 1)$, $\mathcal{A}$ the set of measurable subsets of Ω and $\mathbf{P}(A)$ the Lebesgue measure of $A \in \mathcal{A}$. It is easy to see from the proof given in [6] that (2.9) remains valid also under the condition $q_1 = k$ where $k \geqq 2$ is an arbitrary integer. As for $C_n = n$,

$D_n = \sqrt{n}$ the conditions (2. 4) of Example 3 are satisfied, it follows from Theorem 2 that

$$\lim_{n\to\infty} \mathbf{Q}\left(\frac{\log q_n - n}{\sqrt{n}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du \qquad (-\infty < x < +\infty) \tag{2. 10}$$

if **Q** is any probability measure in the interval (0, 1) which is absolutely continuous with respect to the Lebesgue measure.

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