ON SINGULAR RADII OF POWER SERIES

by

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Let $\mathcal{R}_u$ denote the class of analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1a}$$

which are regular and unbounded in $|z| < 1$. According to D. GAIER and W. MEYER—KÖNIG [1] we call the radius R_φ defined by $z = re^{i\varphi}$, $0 \leq r < 1$ *singular* for $f(z)$, if $f(z)$ is unbounded in any sector $|z| < 1$, $\varphi - \varepsilon < \arg z < \varphi + \varepsilon$ with $\varepsilon > 0$. A radius which is not singular for $f(z)$ is called ***regular*** for $f(z)$. In [1] it has been shown that if $f(z)$ belongs to the class $\mathcal{R}_u$ and the power series of $f(z)$ has HADAMARD-gaps, i. e.

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k} \tag{1b}$$

with

$$\frac{n_{k+1}}{n_k} \geq q > 1 \qquad (k = 0, 1, \dots\ ) \tag{2a}$$

then every radius is singular for $f(z)$. Clearly for every $f(z) \in \mathcal{R}_u$ there is at least one singular radius. It is easy to see that if we suppose only that the power series (1b) has FABRY-gaps, i. e. if instead of (2a) we suppose only

$$\lim_{x \to +\infty} \frac{1}{x} \sum_{n_k < x} 1 = 0\ , \tag{2b}$$

then it is possible that there is only one singular radius for $f(z)$. A simple example is furnished by

$$f_1(z) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=0}^{k^2-1} z^{N_k + j} \tag{3a}$$

where $N_{k+1} \geq N_k + k^2$ $(k = 1, 2, \dots)$. Clearly $f_1(z)$ is regular in $|z| < 1$ and if x is real, we have

$$\lim_{x \to 1-0} f_1(x) = +\infty$$

thus $f_1(z)$ belongs to the class $\mathcal{R}_u$ and R_0 is a singular radius for $f_1(z)$. On the other hand we have by (3a)

$$|f_1(z)| \leqq \frac{\pi^2}{3|1-z|} \qquad \text{for} \quad |z| < 1 \; ; \tag{3b}$$

thus every radius R_φ with $0 < \varphi < 2\pi$ is regular for $f_1(z)$.

It is also clear from this example that to ensure that every radius should be singular for $f(z)$ it is not sufficient to prescribe the rate in which the ratio

$$\frac{1}{x} \sum_{n_k < x} 1$$

tends to 0 for $x \to +\infty$. As a matter of fact, for $f_1(z)$ defined by (3a) we have

$$\frac{1}{x} \sum_{n_k < x} 1 \leqq \frac{s^3}{N_s}$$

where s is defined by the inequality $N_s \leqq x < N_{s+1}$ and thus we can choose the sequence N_s so that

$$\frac{1}{x} \sum_{n_k < x} 1 < \varepsilon(x)$$

holds, where $\varepsilon(x)$ $(x = 1, 2, \ldots)$ is a sequence of positive numbers, tending to 0 arbitrary rapidly.

P. Erdős [2] has shown — answering a question of Gaier and Meyer—König — that to ensure that every radius should be singular for $f(z)$, it is not even sufficient to suppose that the exponents n_k of the lacunary power series (1b) of $f(z) \in \mathcal{R}_u$ satisfy the condition

$$\lim_{k\to\infty} (n_{k+1} - n_k) = +\infty \; . \tag{2c}$$

The question arises, for which sequences n_k does there exist a function $f(z)$ belonging to the class $\mathcal{R}_u$ and having the power series expansion (1b), which has only one singular radius? Clearly it is impossible to give a criterion, which depends only on the rate of growth of the sequence n_k, because the number-theoretical properties of the sequence n_k come in. As a matter of fact let the sequence n_k satisfy the following condition:

D) *for every m $(m = 1, 2, \ldots)$ there exists an integer k_m such that for $k \geqq k_m$ n_k is divisible by 2^m.*

In this case if R_φ is a singular radius for $f(z)$ then $R_{\varphi'}$, where $\varphi' = \varphi + 2\pi l/2^m$ is also singular for any pair of positive integers l and m; as a matter of fact, if z_j $(j = 1, 2, \ldots)$ is a sequence of complex numbers with $|z_j| < 1$, $\varphi - \varepsilon < \arg z_j < \varphi + \varepsilon$ and

$$\lim_{j\to+\infty} |f(z_j)| = +\infty,$$

then putting $\varphi' = \varphi + 2\pi l/2^m$ and $z_j' = z_j \exp(2\pi i\, l/2^m)$ we have $\varphi' - \varepsilon < \arg z_j' < \varphi' + \varepsilon$ and as the series for $f(z_j')$ differs from that for $f(z_j)$ only in a finite number of terms, we have also

$$\lim_{j\to+\infty} |f(z_j')| = +\infty.$$

As the set of values of φ for which R_φ is singular for $f(z)$ is clearly closed (see [1]), it follows that every radius R_φ is singular for $f(z)$. Now the divisibility condition D) implies (2c), but (except for this) is compatible with every possible order of growth of n_k; by other words if ω_k is an increasing sequence of positive integers, tending arbitrarily slowly to $+\infty$, then there exists a sequence n_k of integers having the property D) and satisfying the condition $n_{k+1} - n_k < \omega_k$. Thus our question has to be modified to some extent. We ask for which sequences n_k does there exist a sequence n'_k such that $0 \leq n'_k - n_k \leq \omega_k$ where ω_k is a sequence tending arbitrarily slowly to $+\infty$, and a function

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n'_k} \tag{1c}$$

belonging to the class $\mathcal{R}_u$, which has R_0 as its only singular radius? We shall prove, by using standard methods of probability theory, that if n_k satisfies the condition

$$\liminf_{(k-j)\to+\infty} (n_k - n_j)^{\frac{1}{k-j}} = 1 \tag{2d}$$

then there exists always such a function.

Thus we prove the following

Theorem 1. *Let n_k be an increasing sequence of natural numbers, satisfying the condition* (2d). *Then for any sequence ω_k of natural numbers for which*

$$\lim_{k\to+\infty} \omega_k = +\infty,$$

there exists a sequence n'_k of natural numbers such that $0 \leq n'_k - n_k < \omega_k$ and an analytic function $f(z)$, which is regular in the unit circle has the power series[1] (1c), *is unbounded in $|z| < 1$, but is bounded in the domain $|z| < 1$, $|\arg z| > \varepsilon$ for any $\varepsilon > 0$.*

Our proof of the above Theorem is not constructive; we prove only by using probabilistic methods, the existence of a suitable function $f(z)$, but can not give it explicitely.

The condition (2d) plays a role in other problems of a similar kind too; e. g. P. Erdős has proved [3] that if (2d) is satisfied, there exists a power series (1b) which converges uniformly but not absolutely for $|z| = 1$.

Proof of theorem 1. We shall need the following

Lemma.[2] *Let $m_1 < m_2 < \ldots < m_d$ be natural numbers, $\nu_1, \nu_2, \ldots, \nu_d$ independent random variables, each of which takes on the values $0, 1, \ldots, s-1$ with the same probability $1/s$. Let z be a complex number such that $|z| \leq 1$ and $2s|1-z| \geq 1$. Let us consider the random variable*

$$Z = \sum_{j=1}^{d} z^{m_j+\nu_j} . \tag{4a}$$

[1] $f(z)$ can be chosen so that its power series has nonnegative coefficients.

[2] A similar lemma has been used in a previous paper [4] of the authors of the present paper.

Then we have[3)]

$$\mathbf{P}\left\{|Z| \geqq \frac{4d}{s|1-z|}\right\} \leqq 4e^{-\frac{d}{32s^2}} \tag{5}$$

Proof of the Lemma. Let us put $z = re^{i\varphi}$ and denote by C resp. S the real resp. imaginary part of Z, i.e. we put

$$C = \sum_{j=1}^{d} r^{m_j+\nu_j} \cdot \cos(m_j+\nu_j)\varphi \tag{4b}$$

and

$$S = \sum_{j=1}^{d} r^{m_j+\nu_j} \cdot \sin(m_j+\nu_j)\varphi \tag{4c}$$

As

$$|Z| \leqq \sqrt{2}\max(|C|, |S|)$$

we have evidently

$$\mathbf{P}\left\{|Z| \geqq \frac{4d}{s|1-z|}\right\} \leqq \mathbf{P}\left\{|C| \geqq \frac{2\sqrt{2}\,d}{s|1-z|}\right\} + \mathbf{P}\left\{|S| \geqq \frac{2\sqrt{2}\,d}{s|1-z|}\right\} \tag{6}$$

Now let us calculate the mean value of e^{tC} where we shall choose the value of the real number t later. We have

$$\mathbf{M}\{e^{tC}\} = \prod_{j=1}^{d} \mathbf{M}\left\{e^{tr^{m_j+\nu_j}\cos(m_j+\nu_j)\varphi}\right\} =$$

$$= \prod_{j=1}^{d}\left(\sum_{N=0}^{\infty}\frac{t^N}{N!}\left(\frac{1}{s}\sum_{h=0}^{s-1} r^{N(m_j+h)}\cos^N(m_j+h)\varphi\right)\right)$$

As

$$\left|\frac{1}{s}\sum_{h=0}^{s-1} r^{m_j+h}\cos(m_j+h)\varphi\right| \leqq \left|\frac{1}{s}\sum_{h=0}^{s-1} z^{m_j+h}\right| \leqq \frac{2}{s|1-z|}$$

and

$$\left|\frac{1}{s}\sum_{k=0}^{s-1} r^{N(m_j+h)}\cos^N(m_j+h)\varphi\right| \leqq 1 \qquad (N = 2, 3, \ldots\ )$$

we have for $0 < |t| < 1/2$

$$\mathbf{M}\{e^{tc}\} \leqq \left(1 + \frac{2|t|}{s|1-z|} + t^2\right)^d \tag{7}$$

Evidently

$$\mathbf{P}\left\{|C| \geqq \frac{2\sqrt{2}\,d}{s|1-z|}\right\} = \mathbf{P}\left\{C \geqq \frac{2\sqrt{2}\,d}{s|1-z|}\right\} + \mathbf{P}\left\{C \leqq -\frac{2\sqrt{2}\,d}{s|1-z|}\right\}$$

[3)] Here and in what follows $\mathbf{P}\{\ldots\}$ denotes the probability of the event in the brackets and $\mathbf{M}\{\xi\}$ the mean value of the random variable ξ.

further if $t < 0$, then

$$\mathbf{P}\left\{C \geq \frac{2\sqrt{2}\,d}{s\,|1-z|}\right\} \leq \mathbf{M}\{e^{tC}\}\,e^{-\frac{2\sqrt{2}\,td}{s\,|1-z|}} \tag{8a}$$

and

$$\mathbf{P}\left\{C \leq -\frac{2\sqrt{2}\,d}{s\,|1-z|}\right\} \leq \mathbf{M}\{e^{-tC}\}\,e^{-\frac{2\sqrt{2}\,td}{s\,|1-z|}} \tag{8b}$$

By choosing in (7)

$$t = \frac{1}{4s\,|1-z|}$$

we obtain, taking into account that $8\sqrt{2} - 9 > 2$ and that $|1-z|^2 \leq 4$,

$$\mathbf{P}\left\{|C| \geq \frac{2\sqrt{2}\,d}{s\,|1-z|}\right\} \leq 2e^{-\frac{d}{32s^2}} \tag{9a}$$

In the same way it can be shown that

$$\mathbf{P}\left\{|S| \geq \frac{2\sqrt{2}\,d}{s\,|1-z|}\right\} \leq 2e^{-\frac{d}{32s^2}} \tag{9b}$$

Clearly (6), (9a) and (9b) imply (5). Thus our Lemma is proved.

Let us choose now a subsequence n_{k_p} of the sequence n_k such that $k_1 < k_2 < \ldots < k_p < \ldots$,

$$\lim_{p\to+\infty} (k_{2p+1} - k_{2p}) = +\infty \tag{10a}$$

and

$$\lim_{p\to+\infty} (n_{k_{2p+1}} - n_{k_{2p}})^{\frac{1}{k_{2p+1}-k_{2p}}} = 1 \tag{10b}$$

By (2d) this is possible. As a matter of fact, if $0 < \varepsilon < \frac{1}{4}$ and $(n_k - n_j)^{\frac{1}{k-j}} < 1 + \varepsilon$, then either $j > [k\varepsilon]$ or $j \leq [k\varepsilon]$; in the latter case we have

$$(n_k - n_{[k\varepsilon]})^{\frac{1}{k-[k\varepsilon]}} \leq \left[(n_k - n_j)^{\frac{1}{k-j}}\right]^{\frac{k-j}{k-[k\varepsilon]}} \leq (1+\varepsilon)^{\frac{1}{1-\varepsilon}} \leq 1 + 3\varepsilon$$

Thus we may suppose that there exists a sequence of pairs (k, j) such that $k \to +\infty$, $j \to +\infty$, $(k - j) \to +\infty$ and $(n_k - n_j)^{\frac{1}{k-j}} \to 1$. This implies the existence of a sequence k_p having the required properties.

Clearly we may rarify the sequence k_p as much as we want; thus it can be supposed that besides (10a) and (10b) the following three conditions are also satisfied:

$$(n_{k_{2p+1}} - n_{k_{2p}})^{\frac{1}{k_{2p+1}-k_{2p}}} < 1 + \frac{1}{p^9} \tag{10c}$$

$$p^4 \leq \omega_{k_{2p}} \tag{10d}$$

and

(10e) $$k_{2p+1} - k_{2p} > 64\, p^{10}$$

Now let us put

(11a) $$d_p = k_{2p+1} - k_{2p}$$

and

(11b) $$m_{pj} = n_{k_{2p}+j} - n_{k_{2p}} \qquad (j = 1, 2, \ldots, d_p)$$

further put

(11c) $$\delta_p = \frac{1}{p}$$

(11d) $$s_p = p^4$$

and

(11e) $$N_p = (m_{pd_p} + s_p)\, s_p\, \delta_p^2 \qquad (p = 1, 2, \ldots\ )$$

Let us put

(12a) $$z_{ph} = e^{\frac{2\pi i h}{N_p}} \qquad (h = 0, 1, \ldots, N_p - 1)$$

further

(12b) $$z^*_{ph} = \begin{cases} z_{ph} \text{ for } \delta_p N_p \leqq h \leqq (1-\delta_p) N_p \\ 2\cos 2\pi\delta_p - z_{ph} \text{ for } 0 \leqq h < \delta_p N_p \text{ and } (1-\delta_p) N_p < h < N_p \end{cases}$$

(clearly in the second case z^*_{ph} is obtained by reflecting z_{ph} on the line $\mathcal{Re}(z) = \cos 2\pi\delta_p$).

Evidently

(13) $$|z^*_{ph} - 1| \geqq 1 - \cos 2\pi\delta_p \geqq 8\,\delta_p^2 \quad \text{for} \quad p \geqq 4\,, \qquad h = 1, 2, \ldots, N_p$$

Let us denote by $\mathcal{L}_p$ the contour consisting of the arc $2\pi\delta_p \leqq \varphi \leqq 2\pi(1-\delta_p)$ of the unit circle $z = e^{i\varphi}$ and of the arc $|\varphi| < 2\pi\,\delta_p$ of the circle $z = 2\cos 2\pi\delta_p - e^{i\varphi}$; clearly the points z^*_{ph} $(h = 1, 2, \ldots, N_p)$ divide the line $\mathcal{L}_p$ into arcs of the length $2\pi/N_p$. By our lemma we have, denoting by ν_{pj} $(j = 1, 2, \ldots d_p)$ independent random variables, each of which takes on the values $0, 1, \ldots, s_p - 1$ with the probability $1/s_p$,

(14) $$\mathbf{P}\left\{ \max_{1 \leqq h \leqq N_p} \left| \sum_{j=1}^{d_p} z^{*\,m_{pj}+\nu_{pj}}_{ph} \right| > \frac{4\,d_p}{8\,s_p\,\delta_p^2} \right\} \leqq 4\,N_p\, e^{-\frac{d_p}{32\,s_p^2}}$$

Now putting

(15) $$Q_p(z) = \sum_{j=1}^{d_p} z^{m_{pj}+\nu_{pj}}$$

we have

(16) $$|Q'_p(z)| \leqq d_p\,(m_{pd_p} + s_p) \quad \text{for} \quad |z| \leqq 1$$

and thus for any two points z, z' of the closed unit circle

(17) $$|Q_p(z) - Q_p(z')| \leqq d_p(m_{pd_p} + s_p)\,|z - z'| \;.$$

Thus we obtain

(18) $$\max_{z \in L_p} |Q_p(z)| \leqq \max_{1 \leqq h \leqq N_p} \left| \sum_{j=1}^{c_p} z_{ph}^{*\,m_{pj}+\nu_{pj}} \right| + \frac{d_p \cdot 2\pi}{s_p \cdot \delta_p^2}$$

and therefore by (14)

(19a) $$\mathbf{P}\left\{\max_{z \in L_p} |Q_p(z)| \geqq \frac{7\,d_p}{s_p\,\delta_p^2}\right\} \leqq 4\,N_p\,e^{-\frac{c_p}{32 s_p^2}}$$

and thus with respect to (10a)—(11e) that for $p \geqq 64$

(19b) $$\mathbf{P}\left\{\max_{z \in L_p} |Q_p(z)| \geqq \frac{7\,d_p}{p^2}\right\} \leqq 8\,p^2\,e^{-p^2} \;.$$

Thus it follows that

(20) $$\sum_{p=1}^{\infty} \mathbf{P}\left\{\max_{z \in L_p} |Q_p(z)| \geqq \frac{7\,d_p}{p^2}\right\}$$

converges, and therefore, with probability 1, only a finite number of the inequalities

$$\max_{z \in L_p} |Q_p(z)| \geqq \frac{7\,d_p}{p^2}$$

is satisfied.

This implies that the values of ν_{pj} can be chosen in such a way that

(21) $$\max_{z \in L_p} |Q_p(z)| < \frac{7\,d_p}{p^2}$$

for all $p \geqq p_0$.

Let us put now

(22) $$f(z) = \sum_{p=1}^{\infty} \frac{1}{d_p} z^{n_{k_{2p}}} Q_p(z)$$

where the polynomials $Q_p(z)$ are chosen in such a way that (21) is satisfied for all $p \geqq p_0$. Clearly $f(z)$ is regular in $|z| < 1$, and also unbounded, as all its coefficients are nonnegative and $Q_p(1) = d_p$. On the other hand, for any $\varphi \not\equiv 0 \mod 2\pi$ and any $\varepsilon > 0$ with $0 < \varphi - \varepsilon < \varphi + \varepsilon < 2\pi$ we have for all values of p, for which $2\pi/p < \varphi - \varepsilon$ and $2\pi\,(1 - 1/p) > \varphi + \varepsilon$, for $\varphi - \varepsilon \leqq \arg z \leqq \varphi + \varepsilon$, $|z| < 1$ (by the maximum principle)

$$\frac{1}{d_p} |Q_p(z)| \leqq \frac{7}{p^2}$$

for $p \geqq p_0$. But this implies, that $f(z)$ is bounded in the sector $|z| < 1$, $\varphi - \varepsilon \leqq \leqq \arg z \leqq \varphi + \varepsilon$, or, by other words, R_0 is the only singular radius of $f(z)$. Taking into account that

$$\nu_{pj} \leqq s_p = p^4 \leqq \omega_{k_{2p}}$$

evidently $f(z)$ satisfies all requirements of Theorem 1., which is therewith proved.

It can be shown that the condition $n_k' - n_k = O(\omega_k)$ with ω_k tending arbitrarily slowly to $+\infty$ can not be replaced in Theorem 1. by $n_k' - n_k = O(1)$. We prove namely the following result:

Theorem 2. *Let n_k be an increasing sequence of natural numbers, such that n_k is divisible by 2^m for all $k \geqq k_m$ $(m = 1, 2, \ldots)$. Let*

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k + b_k} \tag{23}$$

be regular and unbounded in the unit circle, where the sequence b_k of integers is bounded. Then every radius R_φ is singular with respect to $f(z)$.

Proof of Theorem 2.[4] It suffices to show that $f(z)$ can not be bounded in a sector $|z| < 1$, $\alpha < \arg z < \beta$. This will be shown by proving that if $f(z)$ would be bounded in such a sector, it would be bounded in the whole unit circle. As a matter of fact, let us suppose that $f(z)$ is given by (23) and that $|b_k| \leqq B$ $(k = 1, 2, \ldots)$ and put

$$f_j(z) = \sum_{b_k = j} c_k z^{n_k} \qquad (|j| \leqq B) \tag{24}$$

Then we may write

$$f(z) = \sum_{j=-B}^{B} z^j f_j(z) \tag{23b}$$

Let us consider the values $z_l = e^{2\pi i \frac{l}{2^m}}$, where m is a fixed natural number, such that

$$2^m > \frac{4\pi(B+1)}{\beta - \alpha} \tag{25}$$

and l takes on the values $0, 1, \ldots, 2^m - 1$. Putting

$$F_{j+B}(r, \vartheta) = \left(\sum_{\substack{k \geqq k_m \\ b_k = j}} c_k r^{n_k} e^{i n_k \vartheta} \right) (re^{i\vartheta})^j \qquad (-B \leqq j \leqq +B) \tag{26}$$

we have for $0 \leqq r < 1$, $0 \leqq \vartheta < 2\pi$ and $l = 0, 1, \ldots, 2^m - 1$

$$f(re^{i\vartheta} z_l) = z_l^{-B} \sum_{l=0}^{2B} F_h(r, \vartheta) z_l^h + \Delta \tag{23c}$$

where Δ denotes a term which is bounded in the unit circle, the bound depending only on m.

As a matter of fact we have

$$|\Delta| \leqq \sum_{k < k_m} |c_k| = A \tag{27}$$

[4] It will be seen from the proof that the condition „n_k is divisible by 2^m for all $k \geqq k_m$ $(m = 1, 2, \ldots)$" could be replaced by the following more general condition: „there exists a sequence Λ_m $m = 1, 2, \ldots)$ of natural numbers, such that $\Lambda_m \to +\infty$ and n_k is divisible by Λ_m for $k \geqq k_m$ $(m = 1, 2, \ldots)$."

Now by (25) there are at least $2B+1$ terms of the sequence z_l ($l = 0, 1, \ldots, 2^m - 1$) lying on the arc $\alpha - \vartheta < \arg z < \beta - \vartheta$, $|z| = 1$.

Let us denote these numbers by $z_{l_1}, z_{l_1+1}, \ldots, z_{l_1+2B}$, let us fix the value of ϑ and put

(28a) $$Q_\vartheta(r, \zeta) = \sum_{j=0}^{2B} F_j(r, \vartheta)\, \zeta^j \,.$$

We have by the interpolation formula of Lagrange

(28b) $$Q_\vartheta(r, \zeta) = \sum_{j=0}^{2B} Q_\vartheta(r, z_{l_1+j}) \frac{\Omega(\zeta)}{\Omega'(z_{l_1+j})\,(\zeta - z_{l_1+j})}$$

where

(29) $$\Omega(\zeta) = \prod_{j=0}^{2B} (\zeta - z_{l_1+j}) \,.$$

As by supposition there exists a constant K such that $|f(z)| \leq K$ for $|z| < 1$, $\alpha < \arg z < \beta$ we have by (23c), (27) and (28a)

(30) $$|Q_\vartheta(r, z_{l_1+j})| \leqq K + A \qquad (j = 0, 1, \ldots, 2B) \,.$$

Thus it follows, that for $|\zeta| = 1$ we have

(31) $$|Q_\vartheta(r, \zeta)| \leqq \frac{(K+A)\,(2B+1)}{\left(\sin \dfrac{\pi}{2^m}\right)^{2B}} \,.$$

It follows from (23c) for $l = 0$ that

(32) $$|f(re^{i\vartheta})| \leqq \frac{(K+A)\,(2B+1)}{\left(\sin \dfrac{\pi}{2^m}\right)^{2B}} + A \quad \text{for} \quad 0 \leqq r < 1 \quad \text{and} \quad 0 \leqq \vartheta < 2\pi \,.$$

As the bound on the right hand side of (32) does not depend on r or ϑ, it follows that $f(z)$ is bounded in the whole unit circle, which contradicts our hypothesis. Thus Theorem 2. is proved.

It remains an open question, whether condition (2d) is best possible. In other words, the following problem is still unsolved:

Let

$$f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}$$

be regular and unbounded in $|z| < 1$. Suppose that

$$\liminf_{(k-j)\to+\infty} (n_k - n_j)^{\frac{1}{k-j}} = q > 1$$

Is it true that all radii R_φ ($0 \leq \varphi < 2\pi$) are singular for $f(z)$?

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HATVÁNYSOROK SZINGULÁRIS SUGARAIRÓL

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Kivonat

Legyen $f(z)$ az egységkörben reguláris és nem korlátos függvény. A $z = re^{i\varphi}$ $(0 \leqq r < 1)$ sugarat, melyet a rövidség kedvéért R_φ-vel jelölünk, D. GAIER és W. MEYER—KÖNIG nyomán (lásd [1], [2]) *szingulárisnak* nevezzük, ha $f(z)$ nem korlátos a $|z| < 1$, $\varphi - \varepsilon < \arg z < \varphi + \varepsilon$ körcikkben, akármilyen kis pozitív szám is ε. A nem-szinguláris sugarakat reguláris sugárnak nevezzük. A jelen dolgozatban a következő tételeket bizonyítjuk be:

1. tétel. *Legyen n_k természetes számok egy növekvő sorozata, amelyre*

$$\liminf_{(k-j)\to+\infty} (n_k - n_j)^{\frac{1}{k-j}} = 1 \,. \tag{1}$$

Legyen ω_k egy tetszőlegesen lassan végtelenhez tartó számsorozat. Akkor létezik olyan

$$f(z) = \sum_{k=1}^{\infty} c_k z^{n'_k} \tag{2}$$

alakú hatványsorral bíró, az egységkörben reguláris és nem korlátos $f(z)$ függvény, amelynek csak egyetlen szinguláris sugara van, és amelynek n'_k kitevői eleget tesznek a

$$0 \leqq n'_k - n_k \leqq \omega_k \tag{3}$$

feltételnek.

Az 1. tétel a dolgozatban valószínűségszámítási módszerrel van bebizonyítva.

2. tétel. *Legyen Λ_m $(m = 1, 2, \ldots)$ egy természetes számokból álló tetszőleges növekvő sorozat és n_k egy olyan természetes számokból álló sorozat, amely azzal a tulajdonsággal bír, hogy az n_k sorozat tagjai véges sok kivétellel oszthatók Λ_m-mel $(m = 1, 2, \ldots)$. Legyen b_k tetszőleges egész számokból álló korlátos sorozat. Tegyük fel, hogy*

$$f(z) = \sum_{n=1}^{\infty} c_k z^{n_k + b_k}$$

az egységkörben reguláris és nem korlátos függvény. Akkor $f(z)$-re vonatkozólag az egységkör minden sugara szinguláris.

О СИНГУЛЯРНЫХ РАДИУСАХ СТЕПЕННЫХ РЯДОВ

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Резюме

Пусть функция $f(z)$ регулярна и неограниченна в единичном круге. Радиус $z = re^{i\varphi}$ $(0 \leqq r < 1)$, обозначаемый для краткости через R_φ, следуя D. Gaier-y и W. Meyer—König-y (см. [1], [2]), называется сингулярным, если $f(z)$ неограниченна в круговом секторе $|z| < 1, \varphi - \varepsilon < \arg z < < \varphi + \varepsilon$ при любом положительном ε. Несингулярные радиусы называются регулярными. В настоящей работе доказываются следующие теоремы:

Теорема 1. *Пусть n_k есть возрастающая последовательность натуральных чисел, для которой*

$$\liminf_{(k-j)\to\infty} (n_k - n_j)^{\frac{1}{k-j}} = 1 \,. \tag{1}$$

Пусть ω_k есть как угодно медленно стремящаяся к бесконечности числовая последовательность. Тогда существует такая регулярная и неграниченная в единичном круге функция $f(z)$, разлагаемая в степенной ряд вида

$$f(z) = \sum_{k=1}^{\infty} c_k z^{n'_k} \tag{2}$$

которая имеет лишь единственный сингулярный радиус и для которой выполненно условие

$$0 \leqq n'_k - n_k \leqq \omega_k \,. \tag{3}$$

Теорема 1 доказывается в работе теоретико-вероятностным методом.

Теорема 2. *Пусть Λ_m $(m = 1, 2, \ldots)$ любая возрастающая последовательность натуральных чисел, а n_k последовтельность натуральных чисел, за исключением конечного числа делящихся на Λ_m $(m = 1, 2, \ldots)$. Пусть b_k любая ограниченная последовательность целых чисел. Предположим, что функция*

$$f(z) = \sum_{k=1}^{\infty} c_k z^{n_k + b_k}$$

регулярна и неограниченна в единичном круге. Тогда относительно $f(z)$ всякий радиус единичного круга сингулярен.