SUMMATION METHODS AND PROBABILITY THEORY

by

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§ 1. Probabilistic interpretation of methods of summation

Let $\mathbf{A} = (a_{nk})$ be an infinite matrix with *nonnegative* elements, with row-sums equal to 1 and such that the elements of each column tend to 0, i. e.

$$a_{nk} \geqq 0 \qquad (n, k = 0, 1, 2, \ldots) \tag{1.1}$$

$$\sum_{k=0}^{\infty} a_{nk} = 1 \qquad (n = 0, 1, \ldots) \tag{1.2}$$

and

$$\lim_{n \to +\infty} a_{nk} = 0 \qquad (k = 0, 1, \ldots) . \tag{1.3}$$

As well known (see [1]), the summation method which consists in forming from a given sequence s_k $(k = 0,1,\ldots)$ the transformed sequence $t_n = \mathbf{A}\, s_n$ defined by

$$t_n = \sum_{k=0}^{\infty} a_{nk}\, s_k \qquad (n = 0, 1, \ldots) \tag{1.4}$$

and considering the limit of t_n (if it exists), is *permanent,* i. e. if $\lim\limits_{n\to+\infty} s_n = s$ then $\lim\limits_{n\to+\infty} t_n = s$ too. Such a method can be interpreted probabilistically as follows: let ν_n $(n = 0,1,\ldots)$ be a sequence of random variables, taking on only nonnegative integral values, with the corresponding probabilities

$$\mathbf{P}(\nu_n = k) = a_{nk} \qquad (n, k = 0, 1, \ldots) . \tag{1.5}$$

(Here and in what follows we denote by $\mathbf{P}(\ldots)$ the probability of the event in the brackets.) The conditions (1.1) and (1.2) clearly express only that the sequence a_{nk} $(k = 0,1,\ldots)$ is for each value of n the probability distribution of such a random variable. Condition (1.3) expresses that ν_n tends in probability to $+\infty$ (which we denote by $\nu_n \Rightarrow +\infty$); as a matter of fact by (1.5) and (1.3) we have

$$\lim_{n\to+\infty} \mathbf{P}(\nu_n \geq N) = 1 \tag{1.6}$$

for all positive values of N, which is equivalent with $\nu_n \Rightarrow +\infty$. Now (1.4) can be interpreted as follows: t_n is the mean value of the random variable s_{ν_n} that is

$$t_n = \mathbf{M}(s_{\nu_n}) \,. \tag{1.7}$$

(Here and in what follows $\mathbf{M}(\dots)$ denotes the mean value of the random variable in the brackets.) Thus the summation procedure defined by the matrix $\mathbf{A} = (a_{nk})$ satisfying (1.1) — (1.2) — (1.3), can be interpreted as follows: we consider the mean value of the random term s_{ν_n} of the sequence s_k and consider the limit of this mean value if $n \to +\infty$.

§ 2. Hausdorff-methods

For HAUSDORFF-summation methods, still more can be said. The summation corresponding to the matrix **A** is called a HAUSDORFF-method of summation if

$$a_{nk} = \binom{n}{k} \int_0^1 x^k (1-x)^{n-k} \, dF(x) \tag{2.1}$$

where $F(x)$ is a probability distribution function in the interval [0, 1]; the process is as well known (see [1]) permanent if $F(x)$ is continuous in the point $x = 0$.

In case of a *Hausdorff*-method the random variables ν_n introduced by (1.5) can be characterized as follows: $\nu_n = \beta(n, \xi)$, where ξ is a random variable which has the distribution function $F(x)$, and $\beta(n, x)$ is a random variable for each fixed value of x $(0 < x < 1)$ which has a binomial distribution of order n with parameter x, i. e.

$$\mathbf{P}\big(\beta(n, x) = k\big) = \binom{n}{k} x^k (1-x)^{n-k} \qquad (k = 0, 1, \dots, n) \,. \tag{2.2}$$

Thus, we first choose a value of the random variable ξ; if this value is x we draw with replacement n balls from an urn containing red and white balls in the proportion x to $1 - x$, and if the number of red balls among the balls chosen is k, we put $\nu_n = k$. Thus we have

$$t_n = \mathbf{M}(s_{\beta(n,\xi)}) \,. \tag{2.3}$$

This interpretation enables us to prove a number of known facts about HAUSDORFF-summation methods in a surprisingly simple manner. For instance let us consider two HAUSDORFF-methods corresponding to the matrices $\mathbf{A} = (a_{nk})$ and $\mathbf{B} = (b_{nk})$ where a_{nk} is defined by (2.1) and b_{nk} by

$$b_{nk} = \binom{n}{k} \int_0^1 x^k (1-x)^{n-k} \, dG(x) \tag{2.4}$$

where $G(x)$ is an other distribution function in [0, 1]. Let us now perform the two transformations $t_n = \mathbf{A}\, s_n$ and $v_n = \mathbf{B}\, t_n$ after another, i. e. consider

$v_n = \mathbf{BA}\, s_n$. It is well known that the resulting method is again one of the Hausdorff type. This can be shown as follows: If η is a random variable having $G(x)$ for its distribution function, and independent of ξ, we have by (2.4)

$$v_n = \mathbf{M}(\mathbf{M}(s_{\beta(\beta(n,\eta),\xi)})) . \tag{2.5}$$

Now the double mean value can be replaced according to a known theorem of probability theory (see [2]) by a simple one, i. e.

$$v_n = \mathbf{M}(s_{\beta(\beta(n,\eta),\xi)}) . \tag{2.6}$$

Thus we have to proceed as follows: first observe the values of the random variables ξ and η which are independent and have the distribution functions $F(x)$ and $G(x)$; if these values are $\xi = x$ and $\eta = y$, then take at random (with replacement) n balls from an urn which contains red and white balls in the proportion y to $1 - y$. If among the balls there are k red ones, take at random (with replacement) k balls form an urn composed of red and white balls in the proportion x to $1 - x$; if among the balls chosen there are l red ones, put $v_n = l$. Now by a well known theorem of probability, the distribution of v_n is the same as if we would have used only one urn which is composed of red and white balls in the proportion xy to $1 - xy$ and choose at random (with replacement) n balls.

This fact can be expressed by saying that the mixture of binomial distributions with the same parameter x and of different orders, with weights forming also a binomial distribution with parameter y and order n, is itself a binomial distribution of order n and parameter xy. This can be proved e. g. by the symbolical calculus of distributions (see [3], p. 129—133) as follows:

$$\sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} \big(xE_1 + (1-x)\big)^k = \big(xy\, E_1 + (1-xy)\big)^n \tag{2.7}$$

where E_1 denotes the distribution attributing the probability 1 to the value 1. Thus it follows that

$$v_n = \mathbf{M}(s_{\beta(n,\xi\eta)}) . \tag{2.8}$$

Thus $v_n = \mathbf{C}\, s_n$ where $\mathbf{C}$ is the HAUSDORFF-matrix with the elements

$$c_{nk} = \binom{n}{k} \int_0^1 x^k (1-x)^{n-k}\, dH(x) \tag{2.9}$$

where $H(x)$ is the distribution function of the random variable $\xi\,\eta$, that is

$$H(x) = \int_0^1 F\left(\frac{x}{u}\right) dG(u) = \int_0^1 G\left(\frac{x}{u}\right) dF(u) . \tag{2.10}$$

Clearly the continuity of either $F(x)$ or $G(x)$ in $x = 0$ implies the same for $H(x)$.

§ 3. Henriksson-methods

If we replace in the definition of the HAUSDORFF-method the binomial distribution $\left\{\binom{n}{k} x^k(1-x)^{n-k}\right\}$ by the Poisson-distribution $\left\{\frac{(\lambda x)^k e^{-\lambda x}}{k!}\right\}$ where $\lambda > 0$,we obtain an other important class of summation methods: *Henriksson's class of methods of summation.* The definition of this class of summation methods given below is slightly different from that given by HENRIKSSON (see [5]).

Let $F(x)$ be a distribution function in the interval $[0, +\infty)$; there corresponds to every $F(x)$ a method of summation which is defined as follows: we put

$$a_k(\lambda) = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} dF(x) \qquad (k = 0, 1, \ldots) \tag{3.1}$$

form the transformed values

$$t(\lambda) = \sum_{k=0}^\infty a_k(\lambda) s_k \tag{3.2}$$

and consider the limit

$$\lim_{\lambda \to +\infty} t(\lambda) = s . \tag{3.3}$$

If the limit (3.3) exists we shall say that the sequence $\{s_n\}$ *is summable to s by the Henriksson-method of summation corresponding to the distribution function $F(x)$.* If $\gamma(\lambda)$ denotes a random variable having Poisson-distribution with mean value λ, then clearly

$$t(\lambda) = \mathbf{M}(s_{\gamma(\lambda\xi)}) \tag{3.4}$$

where ξ is a random variable having the distribution function $F(x)$. It is easy to see that the Henriksson method corresponding to the distribution function $F(x)$ is permanent if and only if $F(x)$ is continuous for $x = 0$.

The class of Henriksson methods includes of course Borel's methods, which is obtained if $F(x)$ is the distribution function of the constant 1 in which case

$$t(\lambda) = \sum_{k=0}^\infty \frac{\lambda^k e^{-\lambda}}{k!} s_k . \tag{3.5}$$

The class of Henriksson's summation methods includes further Abel's method, which is obtained if $F(x)$ is an exponential distribution function, $F(x) = 1 - e^{-x}$ for $x \geq 0$ in which case, putting $u = \frac{\lambda}{1+\lambda}$ we have

$$t(\lambda) = (1-u) \sum_{k=0}^\infty s_k u^k . \tag{3.6}$$

If $\lambda \to +\infty$ then clearly $u \to 1-0$, thus $t(\lambda) \to s$ means nothing else than the Abel-summability of s_n to s.

If we choose for $F(x)$ the gamma-distribution of order r

$$F(x) = \int_0^x \frac{t^{r-1} e^{-t}}{(r-1)!} dt \qquad (r > 0) \tag{3.7}$$

we obtain putting again $U = \frac{\lambda}{1+\lambda}$

$$t(\lambda) = (1-u)^r \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} u^k s_k \tag{3.8}$$

which is a generalization of the Abel-method and may be called the Abel-method of order $r(r = 1$ corresponds to the ordinary Abel-method of summation).

O. HENRIKSSON defined his class of summation methods in a different way, by putting

$$t(\lambda) = \sum_{k=0}^{\infty} s_k \lambda^k \frac{q^{(k)}(-\lambda)}{k!} \tag{3.9}$$

where he supposed that $q(z)$ is a power series

$$q(z) = \sum_{u=0}^{\infty} q_n z^n . \tag{3.10}$$

Clearly, putting

$$q(-z) = \int_0^{\infty} e^{-zx} dF(x) \qquad \text{for } z \geqq 0 \tag{3.11}$$

we have

$$\frac{\lambda^k q^{(k)}(-\lambda)}{k!} = a_k(\lambda) . \tag{3.12}$$

Thus the of summations-methods defined by (3.9) is essentially the same as the class defined by (3.2). Our treatment is somewhat more general then that of HENRIKSSON, as we do not suppose that $q(z)$ is regular in the point $z = 0$, only that it is of the form (3.11) where $F(x)$ is a distribution function, i. e. that $q(-z)$ is the Laplace-transform of a distribution function in the interval $(0, +\infty)$. By other words we suppose that $q(-z)$ is completely monotonic for $z \geqq 0$ and $q(0) = 1$ as by a well known theorem of HAUSDORFF (see e. g. [6], p. 89, Theorem 3.5.) it follows that $q(-z) = \int_0^{\infty} e^{-zx} dF(x)$ where $F(x)$ is a distribution function, further that $\lim_{z \to +\infty} q(-z) = 0$, which implies that $F(x)$ is continuous at $x = 0$. On the other hand Henriksson did not suppose that the coefficients q_n in (3.10) are non-negative.

It has been already remarked by HENRIKSSON (see [5]) that if one performs on $\{s_n\}$ a Hausdorff-transformation and after this a Henriksson-transformation, this is equivalent to a single Henriksson-transformation. As a matter of fact if ξ has the distribution function $F(x)$ $(0 \leq x \leq 1)$ and η the distribution function $G(y)$ $(0 \leq y \leq +\infty)$ and ξ and η are independent then putting $t_n = \mathbf{M}(s_{\beta(n,\xi)})$ and $v(\lambda) = \mathbf{M}(t_{\gamma(\lambda\eta)})$ one has

$$v(\lambda) = \mathbf{M}(\mathbf{M}(s_{\beta(\gamma(\lambda\eta),\xi)})) = \mathbf{M}(s_{\beta(\gamma(\lambda\eta),\xi)}) . \tag{3.13}$$

Now as well known from probability theory, the mixture of binomial distributions by Poissonian weights is again a Poisson distribution. This can be shown by the symbolical calculus as follows:

$$\sum_{n=0}^{\infty} \frac{\mu^n}{n!} e^{-\mu} (p E_1 + (1-p))^n = e^{\mu p (E_1 - 1)} . \tag{3.14}$$

Thus we have

$$\mathbf{M}(s_{\beta(\gamma(\lambda\eta),\xi)}) = \mathbf{M}(s_{\gamma(\lambda\xi\eta)}) . \tag{3.15}$$

Thus putting

$$b_k(\lambda) = \int_0^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} \, dH(x) \tag{3.16}$$

where

$$H(z) = \int_0^1 G\left(\frac{z}{x}\right) dF(x) = \int_0^{\infty} F\left(\frac{z}{x}\right) dG(x) \tag{3.17}$$

we have

$$v(\lambda) = \sum_{k=0}^{\infty} b_k(\lambda) \, s_k . \tag{3.18}$$

Thus $v(\lambda)$ is obtained by using the Henriksson-transformation corresponding to the distribution function $H(x)$ given by (3.17).

§ 4. Limit-distribution methods of summation

It is a natural idea to try to characterize the „sum" of a divergent series by a distribution instead of by a number. This can be done in different ways. For instance if $\{s_k\}$ is a sequence of real numbers, let us put

$$S(y) = \lim_{n \to +\infty} \frac{1}{n+1} \sum_{\substack{k \leq n \\ s_k < y}} 1 \tag{4.1}$$

provided that the limit on the right of (4.1) exists. If $S(y)$ is a distribution function and convergence in (4.1) takes place for all points of continuity y of $S(y)$, we call $S(y)$ *the* $(C, 1)$*-limiting distribution of the sequence* $\{s_n\}$. The relation (4.1) can be put also in the following form:

$$S(y) = \lim_{n \to +\infty} \frac{1}{n+1} \sum_{k=0}^{n} H(y - s_k) \tag{4.2}$$

where $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x \leq 0 . \end{cases}$$

Clearly if (4.1) exists for any real y, and $f(x)$ is an arbitrary step-function, we have

$$(4.3) \qquad \lim_{n\to+\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(s_k) = \int_{-\infty}^{+\infty} f(y)\, dS(y) .$$

Formula (4.3) shows the real meaning of the limit distribution function $S(y)$, further shows the way of its natural generalization. If A is an arbitrary linear method of summation, we may consider the limit (if it exists)

$$(4.4) \qquad S(y) = A - \lim_{n\to+\infty} H(y - s_n)$$

and if $S(y)$ is a probability distribution function, call it the A-limiting distribution function of the sequence $\{s_n\}$. (4.4) implies that for any step function f we have

$$(4.5) \qquad A - \lim_{n\to+\infty} f(s_n) = \int_{-\infty}^{+\infty} f(y)\, d\, S(y) .$$

Of course (4.3) holds also for any Riemann-integrable function $f(x)$ for which $\int_{-\infty}^{+\infty} f(x)\, d\, S(x)$ exists, as for such a function and for any $\varepsilon > 0$ two step functions $f_1(x)$ and $f_2(x)$ can be found so that

$$f_1(x) \leq f(x) \leq f_2(x) \quad \text{and} \quad \int_{-\infty}^{+\infty} \big(f_2(x) - f_1(x)\big)\, d\, S(x) < \varepsilon .$$

Concerning (4.5) the same holds if A is a linear method having a matrix with nonnegative elements.

Conversely, if the limit in (4.3) resp. (4.5) exists for $f(x) = e^{ixt}$ for all real t and if it is a continuous function of t for $t = 0$, then by the well known theorem on characteristic functions (4.2) resp. (4.4) holds.

If (4.2) holds and $\int_{-\infty}^{+\infty} x\, d\, S(x)$ is finite, we may consider $s = \int_{-\infty}^{+\infty} x\, d\, S(x)$ as the generalized limit of the sequence $\{s_n\}$. If moreover s_n is bounded, then clearly $\lim_{n\to+\infty} \frac{1}{n+1} \sum_{k=0}^{n} s_k = \int_{-\infty}^{+\infty} x dS(x)$. If s_n is unbounded, this is evidently not true, as $S(y)$ remains unchanged if s_n is arbitrarily changed for a sequence of values of n which have 0 density, while by such an operation the $(C, 1)$-summability can be clearly destroyed.

Note that $S(y)$ is the distribution function of a constant s, i. e.

$$(4.6) \qquad S(y) = \begin{cases} 0 & \text{for} \quad y \leq s \\ 1 & \text{for} \quad y > s \end{cases}$$

if and only if s_n is almost convergent (see [7]. Vol. II. p. 181), i. e. the sequence of positive integers can be split into two disjoint sequences m_k and l_k ($k = 1,2, \ldots$) such that $\lim_{k\to+\infty} s_{m_k} = s$ and the sequence l_k has zero density, i. e. $\lim_{N\to+\infty} \frac{1}{N} \sum_{l_k<N} 1 = 0$. Thus the existence of a limiting distribution of a sequence may be considered as the generalization of almost convergence, not of the ordinary convergence.

§ 5. On the sequence of generalized partial sums of a sequence

In a recent paper [3] I considered the following problem. Let $\sum_{k=0}^{\infty} a_k$ be a series, and denote by A_n the sum

$$A_n = a_{k_1} + a_{k_2} + \ldots + a_{k_r} \tag{5.1}$$

provided that

$$n = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_r} \qquad (k_1 > k_2 > \ldots > k_r) \tag{5.2}$$

is the representation of n in the binary number system.

The sequence $\{A_n\}$ contains clearly all finite sums formed from the terms of the series Σa_k, arranged in lexicographic order. It has been shown in [3] that the sequence $\{A_n\}$ has a $(C, 1)$-limiting distribution in the sense of § 4. if and only if both Σa_k and Σa_k^2 are convergent. It has been proved in [3] also that the sequence $\{A_n\}$ is $(C, 2)$-summable if and only if the series Σa_k is convergent.

I conjectured that the same holds for Abel-summability, i. e. we have

$$\lim_{x \to 1-0} (1 - x) \sum_{n=0}^{\infty} A_n x^n = s \tag{5.3}$$

if and only if the series Σa_k is convergent and

$$\sum_{k=0}^{\infty} a_k = 2s. \tag{5.4}$$

That (5.4) implies (5.3) is easily shown. However I did not succeed up to now to prove that (5.3) implies (5.4). This would follow from the following equivalent statement:

If

$$\lim_{x \to 1-0} \sum_{k=0}^{\infty} a_k \frac{x^{2^k}}{1 + x^{2^k}} = s \tag{5.5}$$

then the series $\sum_{k=0}^{\infty} a_k$ *is convergent and its sum is* $2s$.

This „high-indices"-type conjecture (5.5) is up to now not proved. It is easy to prove however (5.5) if some additional Tauberian condition is supposed, e. g. if we suppose

$$\lim_{k \to +\infty} a_k = 0. \tag{5.6}$$

As a matter of fact, if (5.6) holds then (5.5) follows from the „high-indices" theorem (see [1], Theorem 114.).

As a matter of fact, if (5.5) and (5.6) hold, we have

$$\left| \sum_{k=0}^{\infty} \frac{2 a_k x^{2^k}}{1 + x^{2^k}} - \sum_{k=0}^{\infty} a_k x^{2^k} \right| \leq \sum_{k=0}^{\infty} |a_k| \, x^{2^k}(1 - x^{2^k})$$

and thus, if $\operatorname{Max}_{k \geq n} |a_k| = B_n$, then

$$\left| \sum_{k=0}^{\infty} \frac{2\, a_k x^{2^k}}{1 + x^{2^k}} - \sum_{k=0}^{\infty} a_k x^{2^k} \right| \leq B_0(1 - x^{2^n}) + B_n . \tag{5.7}$$

Now let $x \to 1 - 0$ and $n \to +\infty$ but so that $x^{2^n} \to 1$. Then the right-hand side of (5.7) tends to 0. Thus it follows from (5.5) that

$$\lim_{x \to 1-0} \sum_{k=0}^{\infty} a_k x^{2^k} = 2\, s \tag{5.8}$$

and by the high-indices theorem it follows that $\sum_{k=0}^{\infty} a_k$ is convergent with the sum $2\, s$.

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The following reference should be mentioned: D. D. KOSAMBI in his paper "Classical Tauberian theorems" (Journal of the Indian Society of Agricultural Statistics 10 (1958) 141—149) makes use of the same probabilistic interpretation of methods of summation as given in § 1.

SZUMMÁCIÓS ELJÁRÁSOK VALÓSZÍNŰSÉGSZÁMÍTÁSI INTERPRETÁCIÓJA

RÉNYI A.

Legyen $\mathbf{A} = (a_{nk})$ $(n, k = 0, 1, \ldots)$ egy végtelen mátrix, amelyre teljesülnek a következő feltételek:

$$a_{nk} \geq 0 \qquad (n, k = 0, 1, \ldots), \tag{1.1}$$

$$\sum_{k=0}^{\infty} a_{nk} = 1 \qquad (n = 0, 1, \ldots). \tag{1.2}$$

és

(1.3) $$\lim_{n\to+\infty} a_{nk} = 0 \qquad (k = 0, 1, \ldots),$$

legyen s_n a szummálandó sorozat és

(1.4) $$t_n = \sum_{k=0}^{\infty} a_{nk} s_k$$

a transzformált sorozat, amelynek konvergenciája esetén az s_n sorozatot az **A** mátrix által definiált szummációs eljárással szummálhatónak nevezzük. A $\{t_n\}$ sorozat elemei a következőképpen interpretálhatók: legyen ν_n egy nemnegatív egész értékű valószínűségi változó, amely a k értéket a_{nk} valószínűséggel veszi fel, mely esetben az (1.3) feltétel azt fejezi ki, hogy ν_n sztochasztikusan $+\infty$-hez tart, ha $n \to +\infty$. t_n felfogható, mint s_{ν_n} várható értéke. Ez az interpretáció különösen a HAUSDORFF-féle és HENRIKSSON-féle szummációs eljárások esetében hasznos, és lehetővé teszi ismert összefüggések egyszerű bizonyítását oly módon, hogy azokat jólismert binomális, ill. Poisson-eloszlások keverésére vonatkozó tételekre vezethetjük vissza. A 4. §. egy divergens sorozat határeloszlásával foglalkozik, és az ennek segítségével értelmezhető szummációs eljárásokkal. Az 5. §. a szerző egy sor összes véges rész-összegeinek sorozataira vonatkozó sejtésével foglalkozik, amely a következő alakra hozható: ha

$$\lim_{x\to 1-0} \sum a_k \frac{x^{2^k}}{1+x^{2^k}} = s$$

létezik, akkor a Σa_k sor konvergens, és összege $2s$. Ezt az állítást a szerzőnek ezideig csak a $\lim a_k = 0$ kiegészítő feltevés mellett sikerült bebizonyítania.

МЕТОДЫ СУММИРОВАНИЯ И ИХ ТЕОРЕТИКО-ВЕРОЯТНОСТНАЯ ИНТЕРПРЕТАЦИЯ

А. RÉNYI

Резюме

Пусть $\mathbf{A} = (a_{nk})$ бесконечная матрица, для элементов которой имеют место следующие соотношение

(1.1) $$a_{nk} \geqq 0 \qquad (n, k = 0, 1, \ldots),$$

(1.2) $$\sum_{k=0}^{\infty} a_{nk} = 1 \qquad (n = 0, 1, \ldots),$$

(1.3) $$\lim_{n\to+\infty} a_{nk} = 0 .$$

Пусть s_n — последовательность числа, которую хотим суммировать и посмотрим последовательность

(1.4) $$t_n = \sum a_{nk} s_k .$$

Если t_n сходится к пределу s, то мы скажем, что последовательность s_n суммируемый к s с методом суммирования определённым матрицой **A**.

Эту методу можно истолковать на языке теории вероятностей следующим образом: пусть ν_n неотрицательная целочисленная случайная величина которая принимает значение k с вероятностей a_{nk}. Тогда t_n равно математическому ожидании от s_{ν_n}. Это истолкование особенно полезно при рассмотрении методов суммирования типа HAUSDORFF и HENRIKSSON и делает возможным сводить доказательство нескольких теорем об этих методов суммирования на хорошо известных элементарных фактов относительно смесы биномиальных и Пуассонских распределении вероятностей. § 4 занимается методами суммирования определенных с помощью предельных распределении расходящихся последовательностей.

В § 5 формулируется недоказанная гипотеза о том что если имеет место

$$\lim_{x\to 1-0} \sum_{k=0}^{\infty} a_k \frac{x^{2^k}}{1+x^{2^k}} = s\,, \tag{5.5}$$

то ряд $\sum_{k=0}^{\infty} a_k$ сходится и имеет сумму $2\,s$; доказывается, что это верно если кроме (5.5) предполагается что a_k стремится к нулю.