LEGENDRE POLYNOMIALS AND PROBABILITY THEORY

By

ALFRÉD RÉNYI

Mathematical Institut of the Eötvös Loránd University

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Dedicated to the memory of Professor L. FEJÉR

The theory of Legendre polynomials is one of the many topics to which FEJÉR contributed beautiful, nowadays classical results. Besides ordinary Legendre polynomials, he also considered [1] generalized Legendre polynomials corresponding to an arbitrary power series with real coefficients. If

(1) $$G(z) = \sum_{k=0}^{\infty} g_k z^k$$

is a power series with real coefficients, then we have

(2) $$|G(re^{i\vartheta})|^2 = \sum_{n=0}^{\infty} Q_n(\cos\vartheta)\, r^n$$

where

(3) $$Q_n(\cos\vartheta) = \sum_{k=0}^{n} g_k g_{n-k} \cos(n-2k)\,\vartheta.$$

If we choose

(4) $$G(z) = \frac{1}{\sqrt{1-z}}$$

that is, if

(5) $$g_0 = 1, \quad g_k = \frac{1.3\ldots(2k-1)}{2.4\ldots 2k} = \frac{\binom{2k}{k}}{2^{2k}} \quad \text{for} \quad k = 1, 2, \ldots,$$

the polynomials (3) will be the ordinary Legendre polynomials. Thus if $G(z)$ is given by (4), we have $Q_n(\cos\vartheta) = P_n(\cos\vartheta)$ with

(6) $$P_n(\cos\vartheta) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \cos(2k-n)\,\vartheta$$

where $P_n(x)$ is the ordinary Legendre polynomial of order n, i. e.

(7) $$P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

The aim of the present note is to point out an evident probabilistic interpretation of Legendre polynomials, which seems to have escaped attention up to now, though it has some interesting consequences.

Let us consider the tossing of a (symmetric) coin. Let ε_n be equal to $+1$ or -1 according to whether at the n-th tossing we obtained head or tail. We denote by $P(\cdot)$ the probability of the event in the brackets and by $M(\cdot)$ the mean value of the random variable in the brackets. According to our supposition the random variables $\varepsilon_1, \varepsilon_2 \ldots, \varepsilon_n, \ldots$ are independent and $P(\varepsilon_n=+1)=$ $=P(\varepsilon_n=-1)=\frac{1}{2}$ $(\eta=1, 2, \ldots)$. We put $s_n=\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n$. Let us call s_n „positive" if either $s_n>0$ or $s_n=0$ and at the same time $s_{n-1}>0$. Let N_n denote the number of those amoung the sums s_k $(k=1, 2, \ldots, n)$ which are positive in the mentioned sense. It has been proved by K. L. CHUNG and W. FELLER [2] that

$$P(N_{2n}=2r)=\frac{\binom{2r}{r}\binom{2n-2r}{n-r}}{2^{2n}} \qquad (r=0, 1, \ldots, n). \tag{8}$$

(Clearly N_{2n} can not be odd.)

Let us denote by $\varphi_{2n}(t)$ the characteristic function of the random variable $N_{2n}-n$, i. e. let us put

$$\varphi_{2n}(t)=M(e^{it(N_{2n}-n)})=\sum_{r=0}^{n} P(N_{2n}=2r)\, e^{it(2r-n)}. \tag{9}$$

Now it can be seen from (8) that the distribution of $N_{2n}-n$ is symmetric and thus we have, taking (6) into account, that

$$\varphi_{2n}(t)=P_n(\cos t). \tag{10}$$

Thus $P_n(\cos t)$ is the characteristic function of the random variable $N_{2n}-n$. Clearly $N_{2n}-n$ is the excess of the number of winning positions of a player tossing a coin $2n$ times over the average value of this number (provided that a neutral position following a winning position is counted also as a winning position). Now it is well known, (see [3], p. 186) that

$$\lim_{n\to+\infty} P_n\left(\cos\frac{t}{n}\right)=J_0(t) \tag{11}$$

uniformly in t, where $J_0(t)$ denotes the Bessel function of order 0, that is

$$J_0(t)=\sum_{k=0}^{+\infty}\frac{(-1)^k\left(\frac{t}{2}\right)^{2k}}{k!^2}. \tag{12}$$

On the other hand it is known (see [3], p. 15) that

$$J_0(t)=\frac{1}{\pi}\int_{1}^{+1}\frac{e^{itx}\,dx}{\sqrt{1-x^2}}. \tag{13}$$

Thus $J_0(t)$ is the characteristic function of the distribution which has the density function

$$(14)\qquad f(x)=\begin{cases}\dfrac{1}{\pi\sqrt{1-x^2}} & \text{for } -1<x<+1\\ 0 & \text{for } |x|>1.\end{cases}$$

By means of (10), (11), (13) and the continuity theorem for characteristic functions it follows that

$$(15)\qquad \lim_{n\to+\infty} P\left(\frac{N_{2n}-n}{n}<x\right)=\frac{1}{\pi}\int_{-1}^{x}\frac{du}{\sqrt{1-u^2}}$$

which implies

$$(16)\qquad \lim_{n\to+\infty} P\left(\frac{N_{2n}}{2n}<y\right)=\frac{2}{\pi}\arc\sin\sqrt{y} \qquad \text{for } 0\leq y\leq 1.$$

As clearly $N_{2n}\leq N_{2n+1}\leq N_{2n}+1$, it follows from (16) that

$$(17)\qquad \lim_{n\to+\infty} P\left(\frac{N_n}{n}<y\right)=\frac{2}{\pi}\arc\sin\sqrt{y} \qquad (0\leq y\leq 1)$$

(17) is the celebrated *arc-sine law*. Obviously using instead of (11) a corresponding formula with a remainder term (see [3]) one could obtain a remainder term in (17) too.

In his mentioned paper [1] **Fejér** makes the remark that the sequence g_k defined by (5) is the sequence of moments of the distribution having the density function

$$(18)\qquad g(x)=\begin{cases}\dfrac{1}{\pi\sqrt{x(1-x)}} & \text{for } 0<x<1,\\ 0 & \text{for } x<0 \text{ and } x>1,\end{cases}$$

that is, we have

$$(19)\qquad g_k=\int_0^1 g(x)\,x^k\,dx \qquad (k=0,1,\ldots).$$

Now clearly $g(x)$ defined by (18) is nothing else than the density function of the arc-sine distribution figuring as a limiting distribution in (17), that is

$$(20)\qquad \int_0^y g(x)\,dx=\frac{2}{\pi}\arc\sin\sqrt{y} \qquad \text{for } 0\leq y\leq 1.$$

Thus if the sequence g_k is defined by (19) where $g(x)$ is defined by (18), we have

$$\lim_{n\to+\infty} \sum_{0\leq k\leq ny} g_k g_{n-k} = \int_0^y g(x)\,dx. \tag{21}$$

The question arises whether the fact that the function $g(x)$ is connected with the sequence $\{g_k\}$ both by (19) and (21) can be generalized for other functions $g(x)$? It is easy to see that this is *not* the case, and the fact that the same function $g(x)$ occurs both in (21) and (19) is a mere coincidence. As a matter of fact it is easy to see that if $g(x)$ is an arbitrary density function which vanishes outside the interval $(0,1)$ and whose asymptotic behaviour for $x\to 1-0$ is given by

$$g(x) \sim \frac{c}{(1-x)^\alpha} \quad \text{where } c>0 \text{ and } 0<\alpha<1, \tag{22}$$

then defining g_k as the k-th moment of the distribution having the density function $g(t)$ we obtain

$$g_k \sim \frac{c\,\Gamma(1-\alpha)}{k^{1-\alpha}} \quad \text{for } k\to+\infty. \tag{23}$$

Thus especially if in (22) $\alpha = {}^1/_2$, we obtain

$$\lim_{n\to+\infty} \sum_{0\leq k\leq ny} g_k g_{n-k} = c^2\pi \int_0^y \frac{dx}{\sqrt{x(1-x)}} \quad \text{for } 0\leq y\leq 1. \tag{24}$$

Thus at the right hand side of (24) we find always the same function $\frac{1}{\sqrt{x(1-x)}}$ if only $g(x)$ satisfies (22) with $\alpha = {}^1/_2$. It is also easy to see that if we choose $g(x)$ so that (22) holds with some α $(0<\alpha<1)$ and choose another density function $h(x)$ which vanishes outside the interval $(0,1)$ and for which for $x\to 1-0$

$$h(x) \sim \frac{d}{(1-x)^{1-\alpha}} \quad \text{with } d>0 \tag{25}$$

and define h_k as the k-th moment of the distribution having the density function $h(x)$ we obtain

$$\lim_{n\to+\infty} \sum_{0\leq k\leq ny} g_k h_{n-k} = c\,d\,\Gamma(\alpha)\,\Gamma(1-\alpha) \int_0^y \frac{dx}{x^{1-\alpha}(1-x)^\alpha}. \tag{26}$$

The limit relation (26) contain some further special cases which are connencted with probability theory. Let us put

$$g(\alpha, x) = \frac{\sin\alpha\pi}{\pi\, x^{1-\alpha}(1-x)^\alpha} \quad \text{for } 0<x<1. \tag{27}$$

Then, as mentioned in [1], we have

$$(28)\qquad g_k(\alpha) = \int_0^1 g(\alpha, x)\, x^k\, dx = \binom{-\alpha}{k}(-1)^k \qquad (k = 0, 1, \ldots).$$

Thus choosing $g(x) = g(\alpha, x)$ and $h(x) = g(1-\alpha, x)$ it follows from (26), taking into account that $\Gamma(\alpha)\,\Gamma(1-\alpha) = \dfrac{\pi}{\sin \alpha\pi}$,

$$(29)\qquad \lim_{n\to+\infty} \sum_{k\le yn} \binom{-\alpha}{k}\binom{\alpha-1}{n-k}(-1)^n = \frac{\sin\alpha\pi}{\pi}\int_0^y \frac{dx}{x^{1-\alpha}(1-x)^{\alpha}}.$$

Now it has been shown by E. Sparre Andersen [4] that if $x_1, x_2, \ldots, x_n, \ldots$ are independent random variables with the same continuous distribution and putting $s_n = x_1 + x_2 + \ldots + x_n$, we have $P(s_n > 0) = \alpha$ $(n = 1, 2, \ldots)$ $0 < \alpha < 1$, then denoting by N_n the number of positive terms of the sequence where $s_1, s_2, \ldots, s_n$ we have

$$(30)\qquad P(N_n = k) = \binom{-\alpha}{k}\binom{\alpha-1}{n-k}(-1)^n$$

and

$$(31)\qquad \lim_{n\to+\infty} P\left(\frac{N_n}{n} < y\right) = \frac{\sin\alpha\pi}{\pi}\int_0^y \frac{dx}{x^{1-\alpha}(1-x)^{\alpha}}.$$

Now clearly (29) expresses that (31) follows from (30).

References

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[4] E. Sparre Andersen, On the frequency of positive partial sums of a series of random variables, *Matematisk Tidsskrift B*, 1950, 33—35.