A NEW APPROACH TO THE THEORY OF ENGEL'S SERIES

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Introduction

In a previous joint paper with P. ERDŐS and P. SZÜSZ in this journal ([1]) we have given detailed proofs of certain results on Engel's series for the representation of real numbers, due originally to É. BOREL [2] and P. LÉVY [3]. Recently I have found a new approach to these theorems which leads to a much simpler proof of these results. The new approach consists in introducing a certain sequence of independent random variables, by means of which all the quantities in question can be expressed in a simple way. The mentioned theorems are obtained as direct consequences of well known general theorems on the sum of independent random variables. This is an essential simplification compared with [1] where the same results have been obtained by the rather difficult technique of dealing with sums of almost independent random variables.

In **§ 1** we give the definition of Engel's series, and collect certain simple formulas, all of which have already been given in [1], which will be needed in the sequel. In **§ 2** we introduce the mentioned independent random variables. In **§ 3** we show how Theorems 2, 3 and 4 of [1] can be obtained as special cases of well known limit theorems of probability theory. In **§ 4** we prove by the same method some new results on Engel's series. These are obtained by combining the method of the present paper by some previous results of the author (see [4]). Finally in **§ 5** we discuss a modification of Engel's representation of real numbers, which we call *modified Engel's series*. For these series similar results hold as for ordinary Engel's series. A similar simple approach to the corresponding theory of Sylvester's series, given in [1] is still lacking.

§ 1. Engel's representation of real numbers

Let x be an arbitrary real number in the interval $(0, 1]$. Then x can be represented in the form of the infinite series

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \dots + \frac{1}{q_1 q_2 \cdots q_n} + \dots \tag{1.1}$$

where $q_1, q_2, \dots, q_n, \dots$ is a nondecreasing sequence of positive integers $\geqq 2$, which can be obtained by the following algorithm: we choose for q_1 the least positive integer for which $\frac{1}{q_1} < x$; for q_2 the least positive integer for which $\frac{1}{q_1}+\frac{1}{q_1 q_2}<x$, and in general (for $n=2, 3, \ldots$) if $q_1, q_2, \ldots, q_n$ have already been determined we choose for q_{n+1} the least positive integer for which $\frac{1}{q_1}+\frac{1}{q_1 q_2}+\ldots+\frac{1}{q_1 q_2 \cdots q_n q_{n+1}}<x$. This algorithm ensures that q_n is a non-decreasing sequence, further that

$$\left| x-\sum_{n=1}^{N}\frac{1}{q_1 q_2 \cdots q_n}\right| < \frac{1}{q_1 q_2 \cdots q_N(q_N-1)}, \tag{1.2}$$

which implies that (1.1) holds.

Clearly the n-th denominator q_n is a well defined function of x, that is $q_n=q_n(x)$. If we suppose now that x is a random variable, uniformly distributed in the interval (0,1], then $q_n=q_n(x)$ is a random variable too ($n=1, 2, \ldots$). It was shown in [1] that the sequence $\{q_n\}$ is a homogeneous Markov chain with transition probabilities

$$\mathbf{P}(q_{n+1}=k \mid q_n=j)=\frac{j-1}{k(k-1)} \qquad (n=1,2,\ldots) \tag{1.3}$$

where k and j are arbitrary positive integers satisfying $k \geqq j \geqq 2$, and the (unconditional) distribution of q_1 is given by

$$\mathbf{P}(q_1=k)=\frac{1}{k(k-1)} \qquad (k=2,3,\ldots). \tag{1.4}$$

(Here and in what follows $\mathbf{P}(A)$ denotes the probability of the event A and $P(A \mid B)$ the conditional probability of the event A under condition B).

§ 2. The basic independent variables

Let $\varepsilon_k=\varepsilon_k(x)$ ($k=2, 3, \ldots$) denote the number of times the integer k occurs in the sequence $q_n=q_n(x)$ ($n=1, 2, \ldots$). As the sequence q_n is non-decreasing, it follows that $\varepsilon_k=r$ means that there exists a nonnegative integer j such that

$$\begin{aligned} &q_n<k && \text{for*} && n \leqq j, \\ &q_n=k && \text{for} && n=j+1,\ j+2, \ldots, j+r, \\ &q_n>k && \text{for} && n>j+r. \end{aligned} \tag{2.1}$$

The probability distribution of ε_k can be easily obtained. As a matter of fact this has already been done in [1] (Theorem 1). We obtain by (1.3)

$$\mathbf{P}(\varepsilon_k \geqq r)=\frac{1}{k^r(k-1)}\prod_{l=2}^{k-1}\frac{1}{1-\frac{1}{l}}=\frac{1}{k^r}, \tag{2.2}$$

* if $j=0$ then there is of course no $n \leqq j$ for which q_n is defined.

and as evidently

$$P(\varepsilon_k = r) = P(\varepsilon_k \geqq r) - P(\varepsilon_k \geqq r+1) \tag{2.3}$$

it follows

$$P(\varepsilon_k = r) = \frac{k-1}{k^{r+1}} \qquad (r = 0, 1, 2, \ldots; k = 2, 3, \ldots). \tag{2.4}$$

Now what escaped our attention when writing the paper [1] is the fact that the random variables $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_k, \ldots$ are independent.

This can be shown as follows: Let $r_2, r_3, \ldots, r_n$ be arbitrary nonnegative integers, then again by (1.3)

$$P(\varepsilon_2 = r_2, \varepsilon_3 = r_3, \ldots, \varepsilon_{n-1} = r_{n-1}, \varepsilon_n \geqq r_n) = \frac{1}{2^{r_2} \cdot 3^{r_3} \ldots (n-1)^{r_{n-1}} n^{r_n}} \tag{2.5}$$

and thus

$$P(\varepsilon_2 = r_2, \varepsilon_3 = r_3, \ldots, \varepsilon_n = r_n) = \frac{n-1}{n} \prod_{k=1}^{n} \frac{1}{k^{r_k}} = \prod_{k=1}^{n} \frac{(k-1)}{k^{r_k+1}}. \tag{2.6}$$

Compared with (2.4) it follows that

$$P(\varepsilon_2 = r_2, \ldots, \varepsilon_n = r_n) = \prod_{k=2}^{n} P(\varepsilon_k = r_k) \tag{2.7}$$

which makes the independence of the variables $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_k, \ldots$ obvious.

Thus we have proved the following

THEOREM 1. *If ε_k $(k=2, 3, \ldots)$ denotes how many times the number k occurs in the sequence of denominators q_n $(n=1, 2, \ldots)$ of Engel's series of a random real number x, uniformly distributed in the interval* $(0, 1]$, *then the random variables ε_k are independent and the distribution of ε_k is given by* (2.4).

Let us calculate the mean value, the variance and the third absolute central moment of the random variable ε_k. (Here and in what follows we denote by $M(\zeta)$ the mean value and by $D^2(\zeta)$ the variance of the random variable ζ.)

We clearly have for $k=2, 3, \ldots$

$$M(\varepsilon_k) = \frac{1}{k-1}, \tag{2.8}$$

$$D^2(\varepsilon_k) = \frac{1}{k-1} + \frac{1}{(k-1)^2}, \tag{2.9}$$

$$M[|\varepsilon_k - M(\varepsilon_k)|^3] \leqq \frac{C}{k-1}, \tag{2.10}$$

where C is a positive constant, not depending on k.

§ 3. Limit theorems on the denominators of Engel's series

Let $\mu_N=\mu_N(x)$ denote the number of terms of the sequence $q_n=q_n(x)$ which are $\leqq N$. By other words, let us put

$$\mu_N = \sum_{k=2}^{N} \varepsilon_k. \tag{3.1}$$

Then clearly in view of (2.8), (2.9) and (2.10) and Theorem 1

$$\mathsf{M}(\mu_N) = \sum_{j=1}^{N-1} \frac{1}{j} \sim \log N, \tag{3.2}$$

$$\mathsf{D}^2(\mu_N) = \sum_{j=1}^{N-1} \left(\frac{1}{j} + \frac{1}{j^2}\right) \sim \log N \tag{3.3}$$

and

$$\sqrt[3]{\sum_{k=2}^{N} \mathsf{M}[|\varepsilon_k - \mathsf{M}(\varepsilon_k)|^3]} = o\left(\sqrt[3]{\log N}\right) \tag{3.4}$$

Thus in view of Theorem 1. Liapounoff's form of the central limit law is applicable, and we get

THEOREM 2a. *For every real y we have*

$$\lim_{N\to+\infty} \mathsf{P}\left(\frac{\mu_N - \log N}{\sqrt{\log N}} < y\right) = \Phi(y) \tag{3.5}$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2} du. \tag{3.6}$$

As clearly

$$\mathsf{P}(\mu_N < n) = \mathsf{P}(q_n > N). \tag{3.7}$$

Theorem 2a can be written in the following equivalent form

THEOREM 2c. *For every real y*

$$\lim_{n\to+\infty} \mathsf{P}\left(\frac{\log q_n - n}{\sqrt{n}} < y\right) = \Phi(y). \tag{3.8}$$

Theorem 2c is identical with Theorem 2 of [12] and is originally due to P. LÉVY.

To prove the strong law of large numbers for the random variables ε_k we need the following general form of this law: *If $\xi_1, \xi_2, \ldots, \xi_N, \ldots$ are inde-*

pendent, nonnegative random variables with finite mean values $M_k = \mathbf{M}(\xi_k)$ *and variances* $D_k^2 = \mathbf{D}^2(\xi_k)$, *and if putting* $A_N = \sum_{k=1}^{N} M_k$ *one has*

$$\lim_{N \to +\infty} A_N = +\infty, \tag{3.9}$$

and further

$$\sum_{N=1}^{\infty} \frac{D_N^2}{A_N^2} < +\infty \tag{3.10}$$

then with probability 1 *one has*

$$\lim_{N \to +\infty} \frac{\sum_{k=1}^{N} \xi_k}{A_N} = 1. \tag{3.11}$$

This theorem follows easily from the three-series theorem of A. N. KOLMOGOROFF and KRONECKER'S lemma.
(See [5], [6], [7]). The conditions of this theorem are clearly fulfilled for $\xi_k = \varepsilon_k$, in view of the fact that the series $\sum_{N=3}^{\infty} \frac{1}{N \log^2 N}$ is convergent. Thus we obtain

THEOREM 3a. *We have with probability* 1 (*i. e. for almost all* x)

$$\lim_{N \to +\infty} \frac{\mu_N}{\log N} = 1. \tag{3.12}$$

In view of $\mu_{q_n} = n$ Theorem 3*a* is equivalent with

THEOREM 3b. *We have with probability* 1

$$\lim_{n \to +\infty} \sqrt[n]{q_n} = e. \tag{3.13}$$

Theorem 3b is identical with Theorem 3 of [1] and is originally due to E. BOREL.

Similarly, as the conditions given by A. N. KOLMOGOROFF [8] for the law of the iterated logarithm are evidently fulfilled for the variables ε_k, we obtain

THEOREM 4a. *We have with probability* 1 (*i. e. for almost all* x)

$$\limsup_{N \to +\infty} \frac{\mu_N - \log N}{\sqrt{2 \log N \cdot \log \log \log N}} = +1 \quad \text{and} \quad \liminf_{N \to +\infty} \frac{\mu_N - \log N}{\sqrt{2 \log N \cdot \log \log \log N}} = -1 \tag{3.14}$$

or equivalently

THEOREM 4b. *We have with probability* 1 (*i. e. for almost all* x)

$$\limsup_{n \to +\infty} \frac{\log q_n - n}{\sqrt{2n \cdot \log \log n}} = +1 \quad \text{and} \quad \liminf_{n \to +\infty} \frac{\log q_n - n}{\sqrt{2n \cdot \log \log n}} = -1. \tag{3.15}$$

Theorem 4b is identical with Theorem 4 of [1], due also to P. LÉVY, which was proved in [1] in a rather cumbersone way; here we obtain this result without any effort as a special case of a well-known general theorem.

§ 4. Some further results

We first prove the following theorem, which is a simple consequence of the Borel-Cantelli lemma.

THEOREM 5. *Let* $2 \leqq k_1 < k_2 < \ldots < k_j < \ldots$ *be an increasing sequence of positive integers. Then the sequence* $q_n = q_n(x)$ *contains for almost all* x *infinitely many or only a finite number of terms of the sequence* $k_j (j = 1, 2, \ldots)$ *according to whether the series* $\sum_{j=1}^{\infty} \frac{1}{k_j}$ *is divergent or convergent.*

PROOF OF THEOREM 5. Let A_k denote the event that the number k is contained (at least once) in the sequence q_n. Then by Theorem 1 the events A_k are independent and

$$\mathsf{P}(A_k) = \mathsf{P}(\varepsilon_k \geqq 1) = \frac{1}{k}. \tag{4.1}$$

Thus Theorem 5 follows immediately by the Borel-Cantelli lemma.

Especially it follows that for almost all x there are infinitely many primes in the sequence $q_n(x)$.

Let us consider now the following problem: what is the probability that the sequence $q_n(x)$ is strictly increasing for $n \geqq n_0(x)$.

This question is answered by

THEOREM 6. *For almost all* x *the sequence* $q_n(x)$ *is strictly increasing for* $n \geqq n_0(x)$ *where* $n_0(x)$ *depends on* x.

PROOF OF THEOREM 6. Let B_k denote the event that the number k occurs more than once in the sequence $q_n(x)$. Clearly the events B_k are independent and

$$\mathsf{P}(B_k) = \mathsf{P}(\varepsilon_k \geqq 2) = \frac{1}{k^2}. \tag{4.2}$$

Thus the series

$$\sum_{k=2}^{\infty} \mathsf{P}(B_k) \tag{4.3}$$

is convergent, and Theorem 6 follows also from the Borel-Cantelli lemma.

The least number $\varkappa(x)$ having the property that no number $k > \varkappa(x)$ occurs more than once in the sequence $q_n(x)$ is a random variable, and it is not difficult to obtain its distribution. As a matter of fact we have

$$\mathsf{P}[\varkappa(x) = k] = \frac{1}{k^2} \prod_{j=k+1}^{\infty} \left(1 - \frac{1}{j^2}\right). \tag{4.4}$$

As by an easy calculation

$$\sum_{k=2}^{\infty} \mathbf{P}[\varkappa(x)=k] = \sum_{k=2}^{\infty} \frac{1}{k^2} \prod_{j=k+1}^{\infty} \left(1-\frac{1}{j^2}\right) = \frac{1}{2} \tag{4.5}$$

it follows that the probability of $\varkappa(x)=1$ is $^1/_2$. By other words, $^1/_2$ is the probability that q_n is strictly increasing from the very beginning.

Now let us drop our assumption that x is uniformly distributed in the interval $(0,1]$ and replace this assumption by the supposition that x has an arbitrary absolutely continuous distribution in $(0, 1]$. As the random variables ε_k are independent, we can apply the theorem (see [4]) that if for a sequence of independent random variables a limit distribution theorem holds, then the assertion of this theorem remains valid if we replace the probability measure on the underlying probability space by a new measure which is absolutely continuous with respect to the original measure.

Thus it follows that the assertions of Theorems 2b and 3b and thus also those of Theorems 2a and 3a remain valid if x is a random variable having an arbitrary absolutely continuous distribution in the interval $(0, 1]$. Especially x may be uniformly distributed in an arbitrary subinterval (α, β) of $(0, 1]$.

§ 5. Modified Engel's series

In this § we consider the representation of a real number $x (0 < x \leqq 1)$ in the form

$$x = \frac{1}{m_1} + \frac{1}{(m_1-1)m_2} + \dots + \frac{1}{(m_1-1)(m_2-1)\dots(m_{n-1}-1)m_n} + \dots \tag{5.1}$$

where m_n is a *strictly increasing* sequence of positive integers, $\geqq 2$ which is defined as follows: we choose for m_1 the least positive integer for which $\frac{1}{m_1} < x$; evidently $m_1 \geqq 2$; if m_1 is already chosen, we choose for m_2 the least positive integer for which $\frac{1}{m_1} + \frac{1}{(m_1-1)m_2} < x$. Then clearly $m_2 > m_1$ as $\frac{1}{m_1} + \frac{1}{(m_1-1)m_1} = \frac{1}{m_1-1} \geqq x$ because of the definition of m_1. If $m_1, m_2, \dots, m_n$ are already determined, we choose for m_{n+1} the least positive integer for which $\frac{1}{m_1} + \frac{1}{(m_1-1)m_2} + \dots + \frac{1}{(m_1-1)\dots(m_n-1)m_{n+1}} < x$

It is easy to see that $m_{n+1} > m_n$ and that

$$0 < x - \frac{1}{m_1} - \sum_{k=1}^{n} \frac{1}{(m_1-1)\dots(m_{k-1}-1)m_k} \leqq \frac{1}{(m_1-1)(m_2-1)\dots(m_n-1)m_n} \tag{5.2}$$

and thus as $m_n \geqq n+1$, (5.1) holds. We shall call (5.1) the *modified Engel's series* of x.

The denominators m_n have the same probable asymptotic behaviour as the denominators q_n of the ordinary Engel's series.

In fact the variables m_n $(n=1, 2, \ldots)$ form also a homogeneous Markov chain, with the transition probabilities

$$\mathsf{P}(m_n = k \mid m_{n-1} = l) = \frac{l}{k(k-1)} \qquad (k \geqq l+1). \tag{5.3}$$

The basic independent random variables are defined here analogously. In fact if δ_k is 1 or 0 according to whether the number k occurs in the sequence m_n or not, then it can be shown — analogously as in **§ 1** we proved the corresponding fact for ordinary Engel's series — that the random variables $\delta_2, \delta_3, \ldots, \delta_k, \ldots$ are independent and $\mathsf{P}(\delta_k = 1) = \frac{1}{k}$. Starting from this fact it follows that Theorems 2b, 3b and 4b remain valid for m_n instead of q_n.

These results are connected with the theory of order statistics (see [9]).

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