ASYMMETRIC GRAPHS

By

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Dedicated to T. GALLAI, *at the occasion of his* 50^{th} *birthday*

Introduction

We consider in this paper only non-directed graphs without multiple edges and without loops. The number of vertices of a graph G will be called its order, and will be denoted by $N(G)$. We shall call such a graph symmetric, if there exists a non-identical permutation of its vertices, which leaves the graph invariant. By other words a graph is called symmetric if the group of its automorphisms has degree greater than 1. A graph which is not symmetric will be called *asymmetric*. The degree of symmetry of a symmetric graph is evidently measured by the degree of its group of automorphisms. The question which led us to the results contained in the present paper is the following: how can we measure the degree of asymmetry of an asymmetric graph?

Evidently any asymmetric graph can be made symmetric by deleting certain of its edges and by adding certain new edges connecting its vertices. We shall call such a transformation of the graph its symmetrization. For each symmetrization of the graph let us take the sum of the number of deleted edges — say r — and the number of new edges — say s; it is reasonable to define the degree of asymmetry $A[G]$ of a graph G, as the minimum of $r+s$ where the minimum is taken over all possible symmetrizations of the graph G. (In what follows if in order to make a graph symmetric we delete r of its edges and add s new edges, we shall say that we *changed* $r+s$ edges.) Clearly the asymmetry of a symmetric graph is according to this definition equal to 0, while the asymmetry of any asymmetric graph is a positive integer.

The question arises: how large can be the degree of asymmetry of a graph of order n (i. e. a graph which has n vertices)? We shall denote by $A(n)$ the maximum of $A[G]$ for all graphs G of order n ($n=2, 3, \ldots$). We put further $A(1) = +\infty$. It is evident that $A(2)=A(3)=0$.

Now let $\bar{G}$ denote the complementary graph of G, that is the graph which consists of the same vertices as G and of those and only those edges which do not belong to G; then we have evidently

LEMMA 1.

$$A[G]=A[\bar{G}]. \tag{1}$$

As a matter of fact the complementary graph of a symmetric graph is evidently also symmetric (i. e. (1) holds if $A[G]=0$) and if a transformation T, consisting in deleting r edges and adding s new edges, makes G symmetric, then the transformation $\bar{T}$, consisting in adding those s edges which are deleted by T and deleting those r edges which are added by T, is clearly a symmetrization of $\bar{G}$, and thus Lemma 1 follows.

We shall need also the following evident fact:

LEMMA 2. *If a graph G is not connected and its components are $G_1, G_2, \ldots, G_c$ then we have*

$$A[G] \leqq \min_{1 \leqq i \leqq c} A[G_i]. \tag{2}$$

Let us mention further that a graph containing more than one isolated point is symmetric.

Now we can prove that $A(4)=0$ and $A(5)=0$. Let us first consider $A(4)$. Clearly any not connected graph of order 4 is symmetric by Lemma 2, further by Lemma 1 we may restrict ourselves to graphs fo order 4 having not more than $\frac{1}{2}\binom{4}{2}=3$ edges, because if the graph has more than 3 edges, the complementary graph has less than 3 edges. But the only connected graphs of order 4 with not more than 3 edges are the path and the star shown on Fig. 1

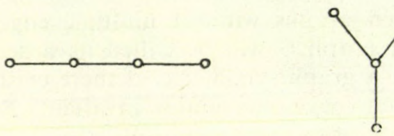

Fig. 1

which are clearly symmetric. Thus $A(4)=0$. Now we show $A(5)=0$. Again we can restrict ourselves to graphs of order 5 which are connected and which contain not more than $\frac{1}{2}\binom{5}{2}=5$ edges. These belong however all to one of the 8 types shown on Fig. 2 which are evidently all symmetric.

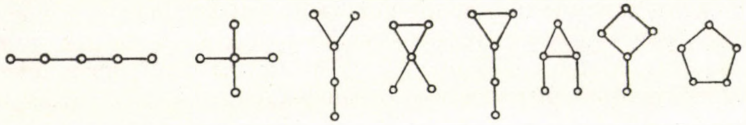

Fig. 2

(We have drawn the graphs so that each is symmetric with respect to its vertical axis.)

Now we shall show that $A(6)=1$. Here again we may restrict ourselves to consider connected graphs having not more than $\left[\frac{1}{2}\binom{6}{2}\right]=7$ edges.* Among these we find four asymmetric types, shown by Fig. 3.

All have their degree of asymmetry equal to 1. As a matter of fact, each can be made symmetric by deleting the edge which is indicated by a thick line. It is easy to see that any of these graphs can also be made symmetric

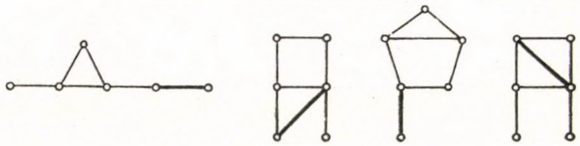

Fig. 3

* Here and in what follows $[x]$ denotes the integral part of the real number x.

by adding a suitably chosen edge, as shown on Fig. 4, where the edge to be added is indicated by a dotted line.

However it is not true in general that if a graph can be made symmetric by omitting one edge, it can also be made symmetric by adding one edge. For instance Fig. 5 shows a graph of order 10 which can be made symmetric by omitting one edge (that which is drawn by a thick line) but can not be made symmetric by adding one new edge. (Of course if by omitting an edge an involutory symmetry is produced, then the same symmetry can be produced by adding (instead of omitting) a suitably chosen edge.)

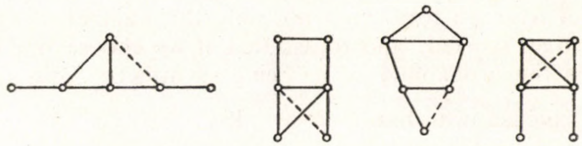

Fig. 4

In § 1 we shall show (Theorem 1) by a simple argument that the asymmetry of a graph of order n can not exceed $\frac{n-1}{2}$ if n is odd, while if n is even the asymmetry can not exceed $\frac{n}{2}-1$; in § 2 we prove (Theorem 2) that this estimate is asymptotically best possible, that is for any $\varepsilon>0$ there can be found an integer $n_0(\varepsilon)$ such that for any $n>n_0(\varepsilon)$ there exists a graph G_n of order n for which $A[G_n]>\frac{n}{2}(1-\varepsilon)$.

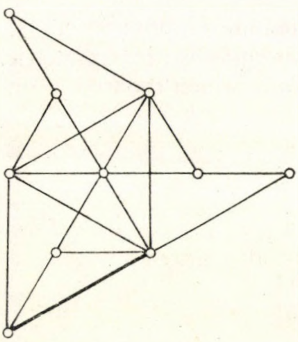

Fig. 5

In other words we have

$$\lim_{n\to+\infty}\frac{A(n)}{n}=\frac{1}{2}. \tag{3}$$

We can prove still more, namely that there exists a positive constant C such that

$$A(n)>\frac{n}{2}-C\sqrt{n\log n}.$$

However we do not know whether there exist graphs G_n of even resp. odd order n for which $A[G_n]=\frac{n}{2}-1$ resp. $A[G_n]=\frac{n-1}{2}$; we can prove that this is impossible if $n\equiv 3 \bmod 4$ and we guess that this is impossible for all n.

In view of (3) it is reasonable to introduce the quantity

$$a[G]=\frac{A[G]}{\left[\frac{N(G)-1}{2}\right]} \tag{4}$$

for any graph G with $N(G)\geqq 3$, and call it the relative asymmetry of G. It follows from our results that for any graph G with $N(G)\geqq 3$ one has

$$0\leqq a[G]\leqq 1. \tag{5}$$

The proof of Theorem 2 is not constructive, only a proof of existence. It uses probabilistic considerations. This method gives however more than stated above: it shows that for large values of n most graphs of order n are asymmetric, the degree of asymmetry of most of them being near to $\frac{n}{2}$.

An other interesting question is to investigate the asymmetry or symmetry of a graph for which not only the number of vertices but also the number N of edges is fixed, and to ask that if we choose one of these graphs at random, what is the probability of its being asymmetric. We have solved this question too, and have shown that if $N=\frac{n}{2}\log n+\omega(n)n$ where $\omega(n)$ tends arbitrarily slowly to $+\infty$ for $n\to+\infty$, then the probability that a graph with n vertices and N edges chosen at random (so that any such graph has the same probability $\binom{\binom{n}{2}}{N}^{-1}$ to be chosen) should be asymmetric, tends to 1 for $n\to+\infty$. This and some further related results will be published in an other forthcoming paper.

In § 3 we deal with (denumerably) infinite graphs, more exactly with random infinite graphs Γ defined as follows. Let $P_1, P_2, \ldots, P_n, \ldots$ be an infinite sequence of vertices. Let us suppose that for each j and k ($j\neq k$) if E_{jk} denotes the event that P_j and P_k are connected by an edge, then the events E_{jk} are independent and each has the probability $\frac{1}{2}$. We prove the simple but surprising fact that Γ is symmetric with probability 1 (Theorem 3).

Thus there is a striking contrast between finite and infinite graphs: while „almost all" finite graphs are asymmetric, „almost all" infinite graphs are symmetric.

In § 4 we deal with the asymmetry of graphs of order n in which the total number N of edges is fixed.

In § 5 we deal with some related unsolved problems.

Our thanks are due to T. Gallai for his valuable remarks.

§ 1. Proof of the theorem that the asymmetry of a graph of order n can not exceed $\left[\frac{n-1}{2}\right]$

In this § we prove

Theorem 1.

$$A(n)\leqq\left[\frac{n-1}{2}\right].$$

Remark. Of course Theorem 1 implies that if n is odd, then $A(n)\leqq\frac{n-1}{2}$ and if n is even, we have $A(n)\leqq\frac{n}{2}-1$.

Proof. Let G be an arbitrary graph of order n. We may suppose $n\geqq 6$. Let $P_1, P_2, \ldots, P_n$ be the vertices of G and let us denote by v_k the valency of P_k in G (i. e. v_k is the number of edges having P_k as one of their endpoints.)

Let further v_{jk} $(j \neq k)$ denote the number of vertices P_h of G $(h \neq j, h \neq k)$ which are connected in G both with P_j and with P_k. Let us put further $v_{jj} = 0$. Clearly $v_{kj} = v_{jk}$ and

$$\sum_{j=1}^{n} \sum_{k=1}^{n} v_{jk} = \sum_{h=1}^{n} v_h(v_h - 1). \tag{1. 1}$$

As a matter of fact, both the left hand side and the right hand side of (1. 1)are equal to the number of (ordered) pairs of edges of G which have one common endpoint. Let us choose now two distinct vertices P_j and P_k of G $(j \neq k)$ and let us put

$$\Delta_{jk} = v_j + v_k - 2v_{jk} - 2\delta_{jk} \tag{1. 2}$$

where $\delta_{jk} = 1$ or 0 according to whether P_j and P_k are connected by an edge in G or not. Let us put further $\Delta_{jj} = 0$. Evidently Δ_{jk} is the number of vertices of G which are in different relation with P_j and P_k (i. e. which are either connected with P_j and not connected with P_k or connected with P_k and not connected with P_j). Clearly by omitting all edges connecting P_j (resp. P_k) with some point of G which is not connected with P_k (resp. P_j) we obtain a graph G' in which P_j and P_k are connected with the same points. Thus G' has the symmetry consisting in the interchange of P_j and P_k and leaving all other points unchanged. But G' is obtained from G by deleting Δ_{jk} edges. Thus G can be made symmetric by deleting Δ_{jk} edges. It follows that

$$A[G] \leqq \min_{j \neq k} \Delta_{jk} \leqq \frac{\sum_{j=1}^{n} \sum_{k=1}^{n} \Delta_{jk}}{n(n-1)}. \tag{1. 3}$$

On the other hand we have

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \Delta_{jk} = 2 \sum_{l=1}^{n} v_l(n-1-v_l). \tag{1. 4}$$

As a matter of fact, the left hand side of (1. 4) is equal to the number of ordered triplets (P_j, P_k, P_l) of vertices such that G contains exactly one of the two possible edges $\widehat{P_j P_l}$ and $\widehat{P_k P_l}$; if we fix P_l then that among P_j and P_k which is connected with P_l can be chosen in v_l ways and the other in $n-1-v_l$ ways; this proves (1. 4). ((1. 4) could also be deduced from (1. 1) and (1. 2)).

As clearly

$$v_l(n-1-v_l) = \left(\frac{n-1}{2}\right)^2 - \left(v_l - \frac{n-1}{2}\right)^2 \tag{1. 5}$$

we obtain

$$2 \sum_{l=1}^{n} v_l(n-1-v_l) \leqq \begin{cases} \dfrac{n(n-1)^2}{2} & \text{if } n \text{ is odd} \\[2ex] n\dfrac{[(n-1)^2-1]}{2} & \text{if } n \text{ is even.} \end{cases} \tag{1. 6}$$

It follows from (1. 3), (1. 4) and (1. 6)

$$A(G) \leqq \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n(n-2)}{2(n-1)} & \text{if } n \text{ is even.} \end{cases} \tag{1. 7}$$

Now we have evidently

$$\frac{n(n-2)}{2(n-1)} < \frac{n}{2} \quad \text{if} \quad n > 1.$$

Thus it follows from (1. 7) that

$$A[G] \leqq \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2} - 1 & \text{if } n \text{ is even} \end{cases} \tag{1. 8}$$

and thus for every n

$$A[G] \leqq \left[\frac{n-1}{2}\right]. \tag{1. 9}$$

As (1. 9) holds for every graph G of order n, Theorem 1 is proved.

The problem arises, for which odd values of n does there exist a graph G of order n such that

$$\min_{j \neq k} \Delta_{jk} = \frac{n-1}{2}. \tag{1. 10}$$

As by (1. 3) and (1. 6) we have for odd n

$$\min_{j \neq k} \Delta_{jk} \leqq \frac{\sum\sum_{j \neq k} \Delta_{jk}}{n(n-1)} \leqq \frac{n-1}{2} \tag{1. 11}$$

it follows that (1. 10) can hold only if $\Delta_{jk} = \dfrac{n-1}{2}$ for all $j \neq k$. It follows from (1. 5) that in this case we have also $v_l = \dfrac{n-1}{2}$ for $l = 1, 2, \ldots, n$. Now if $n \equiv 3 \bmod 4$ then $\dfrac{n-1}{2}$ is odd, and as in any graph the number of vertices having an odd valency is even we obtain a contradiction. Thus (1. 10) can hold for an odd n only if $n \equiv 1 \bmod 4$.

We shall call a graph G of order $n \equiv 1 \bmod 4$ for which (1. 10) holds a Δ-graph.

For $n = 5$ the cycle of order 5 is a Δ-graph. For $n = 9$ a Δ-graph is shown by Fig. 6.

A simple way to describe the Δ-graph shown by Fig. 6 is as follows: let the 9 vertices be labelled by ordered pairs of numbers (a, b) where a and b may take on independently the values 0, 1, 2. Let us connect the vertices labelled by (a, b) and (a', b') if and only if either $a = a'$ or $b = b'$.

We can construct a Δ-graph of order p, if p is an arbitrary prime for which $p \equiv 1 \bmod 4$, as follows: Let $P_1, P_2, \ldots, P_p$ be the vertices of G and let us connect the vertices P_j and P_k if and only if $k-j$ is a quadratic residue mod p. In this case clearly each vertex P_j has valency $\frac{p-1}{2}$. We show that for each $j \neq k$ we have $\Delta_{jk} = \frac{p-1}{2}$. This follows immediately from the following well-known property of quadratic residues observed first by Lagrange (see [1] and [2]): If $r_1, r_2, \ldots, r_{\frac{p-1}{2}}$ are all quadratic residues among the numbers $1, 2, \ldots, p-1$, then among the num-

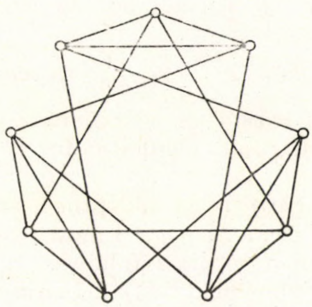

Fig. 6

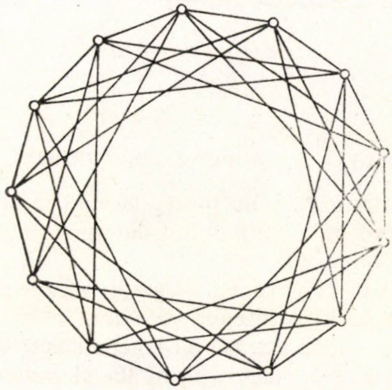

Fig. 7

bers $r_l + d \left(l = 1, 2, \ldots, \frac{p-1}{2}\right)$ where d is any of the numbers $1, 2, \ldots, p-1$, there are exactly $\frac{p-1}{4}$ which are congruent to a quadratic non-residue mod p. As a matter of fact Δ_{jk} is equal to the number of those integers h $(h = 1, 2, \ldots, p)$ for which $h-j$ is a quadratic residue and $h-k$ a non-residue, or $h-j$ a quadratic nonresidue and $h-k$ a residue. Putting $d = k-j$ this means that Δ_{jk} is equal to te sum of the number of non-residues among the numbers $r_l + d$ $\left(l = 1, 2, \ldots, \frac{p-1}{2}\right)$ and the number of non-residues among the numbers $r_l - d$ $\left(l = 1, 2, \ldots, \frac{p-1}{2}\right)$ and thus $\Delta_{jk} = 2\left(\frac{p-1}{4}\right) = \frac{p-1}{2}$.

Thus there exists a Δ-graph of every order n which is a prime of the form $4k+1$. Clearly the Δ-graph of order 5 mentioned above is the same as that obtained by the above general construction for $p=5$. For $p=13$ the Δ-graph obtained by our construction is shown by Fig. 7.

We can construct also a Δ-graph of order $n = p^2$ if p is a prime of the form $p = 4k+3$. The construction is as follows: let us label the vertices by the pairs of numbers (a, b) where $0 \leqq a \leqq p-1$, $0 \leqq b \leqq p-1$. Let us connect the vertices labelled with (a, b) and (a', b') if either $a=a'$, or $(a-a')(b-b')$ is a quadratic residue mod p. In this case each vertex (a, b) has the valency $\frac{p^2-1}{2}$, because it is connected with the $p-1$ vertices (a, b') where $b' \neq b$ and with the $\left(\frac{p-1}{2}\right)^2$ vertices (a', b') such that $a-a'$ and $b-b'$ are both quadratic residues mod p and the $\left(\frac{p-1}{2}\right)^2$ vertices (a', b') such that $a-a'$ and $b-b'$ are both quadratic non-residues mod p and $2\left(\frac{p-1}{2}\right)^2+p-1=\frac{p^2-1}{2}$. Further denoting by v the number of vertices which are connected with one of the vertices (a, b) and (a', b') but not with the other, we always have $v=\frac{p^2-1}{2}$. This follows from the theorem (due to Lagrange) according to which if $r_1, r_2 \ldots, r_{\frac{p-1}{2}}$ is a complete set of quadratic residues mod p then exactly $\frac{p-3}{2}$ among the numbers $r_j+d\left(j=1, 2, \ldots, \frac{p-1}{2}\right)$ are congruent to a quadratic residue mod p (see [3]). Mr. A. HEPPES (oral communication) has constructed by a similar but different method a Δ-graph of order p^2 for every odd prime p.

We can construct a Δ-graph of order p^r where p is an odd prime, and r an arbitrary positive integer such that $p^r \equiv 1 \bmod 4$ (that is, if $p \equiv 1 \bmod 4$ then r is arbitrary, while if $p \equiv 3 \bmod 4$ then r has to be an even number), as follows. Let us label the p^r vertices of the graph by the elements of a Galois-field $GF(p^r)$. Let us connect two vertices labelled by U and $V(U \in GF(p^r), V \in GF(p^r))$ if and only if $U-V = C^2$ where C is some element of $GF(p^r)$. Now J. B. KELLY [2] has proved that for any $GF(p^r)$ with $p^r \equiv 1 \bmod 4$ by denoting by A the subset of those non-zero elements which are squares, and by B the subset of those elements which are not squares, it follows that any non-zero element d can be represented in exactly $\frac{p^r-1}{2}$ ways in the form $d = a-b$ where $a \in A$ and $b \in B$. Thus it follows (exactly as in the case $r=1$) that our graph G is a Δ- graph. Thus there exists a Δ-graph of order n if $n=p^r \equiv 1 \bmod 4$ where p is a prime. We do not know whether there exists a Δ-graph of order n if $n \equiv 1 \bmod 4$ and n is not a prime-power.

Let us mention that all Δ-graphs which we have constructed are symmetric; for instance the Δ-graph of order 9 shown by Fig. 6. has the automorphism which carries over the vertex labelled with (a, b) into the vertex labelled with (a', b') where $a' \equiv a+1 \pmod 3$ and $b' \equiv b+1 \pmod 3$. The Δ-graph of order p where p is a prime of the form $4k+1$ constructed above has the symmetry which carries p_l into $p_{l'}$ where $l' = l+1 \bmod p$.

Thus while there exist at least for certain odd values of n graphs for which $\min \Delta_{jk}=\frac{n-1}{2}$, we do not know any graph of (odd) order n for which $A[G] = \frac{n-1}{2}$. We guess that this is impossible.

It is possible that the following stronger conjecture holds also: all Δ graphs are symmetric.

Finally we should add the following remark: Let $C_3(G)$ denote the number of triangles contained in a graph G. A. GOODMAN [4] (see also [5] and [6]) has determined the minimum of $C_3(G_n)+C_3(\overline{G}_n)$ for all graphs of order n. For $n\equiv 1$ mod 4 his result is as follows:

$$\min\left(C_3(G_n)+C_3(\overline{G}_n)\right) = \frac{n(n-1)(n-5)}{24}. \tag{1.12}$$

Let us call a graph of order n for which the minimum in (1. 12) is attained, a Goodman-graph. Now it is easy to see that any Δ-graph is at the same time a Goodman-graph (but not conversely). This can be proved as follows: If G_n is a Δ-graph of order n, then the number of triangles contained in G_n and containing the edge PQ is equal to the number of vertices connected with both P and Q, and thus is equal to $\frac{n-1}{2}-\frac{n-1}{4}-1=\frac{n-5}{4}$. As the total number of edges of G_n is $\frac{n(n-1)}{4}$ and each triangle is counted in this way three times, $C_3(G_n)=\frac{n(n-1)(n-5)}{48}$. Clearly if G_n is a Δ-graph then $\overline{G}_n$ is a Δ-graph too; thus it follows that $C_3(G_n)+C_3(\overline{G}_n)=\frac{n(n-1)(n-5)}{24}$, i. e. that (1. 12) holds for G_n.

§ 2. The asymmetry of a random graph of order n

In this § we prove the following

THEOREM 2. *Let us choose at random a graph Γ having n given vertices so that all possible $2^{\binom{n}{2}}$ graphs should have the same probability to be chosen. Let $\varepsilon>0$ be arbitrary. Let $\mathbf{P}_n(\varepsilon)$ denote the probability that by changing not more than $\frac{n(1-\varepsilon)}{2}$ edges of Γ it can be transformed into a symmetric graph. Then we have*

$$\lim_{n\to+\infty}\mathbf{P}_n(\varepsilon)=0. \tag{2.2}$$

COROLLARY. *For any ε with $0<\varepsilon<1$ there exists an integer $n_0(\varepsilon)$ depending only on ε, such that for $n>n_0(\varepsilon)$ there exist graphs G of order n with $A[G]>\frac{n(1-\varepsilon)}{2}$.*

REMARK. Clearly it follows from Theorem 1 and the corollary of Theorem 2 that

$$\lim_{n\to+\infty}\frac{A(n)}{n}=\frac{1}{2}. \tag{2.3}$$

The same method yields also $\frac{n}{2}-A(n)=O(\sqrt{n\log n})$ but we shall not prove this in detail.

PROOF of Theorem 2. As the proof is not simple, we first give a sketch of the proof.

Let us denote by $\mathbf{P}_n(\varepsilon, q)$ $(q=2, 3, \ldots)$ the probability that a random graph of order n can be transformed by changing $m < \frac{n}{2}(1-\varepsilon)$ of its edges into a graph admitting a permutation Π as an automorphism, where Π is a permutation which leaves exactly $l = n-q$ of the n vertices of the graph unchanged, but which can not be transformed into a symmetric graph by changing less than m of its edges. Then we have

$$\mathbf{P}_n(\varepsilon) \leqq \sum_{q=2}^{n} \mathbf{P}_n(\varepsilon, q). \tag{2. 4}$$

We shall estimate $\mathbf{P}_n(\varepsilon, q)$ as follows

$$\mathbf{P}_n(\varepsilon, q) \leqq \frac{A_{n,q}\cdot B_{n,q}\cdot C_{n,q}}{2^{\binom{n}{2}}} \tag{2. 5}$$

where $A_{n,q}$ is the number of ways a permutation Π_q, leaving exactly $l = n-q$ of the n vertices of the graph invariant, can be chosen; $B_{n,q}$ is an upper bound for the number of graphs which are invariant under such a permutation Π_q and $C_{n,q}$ is an upper bound for the number of graphs which can be transformed into a graph admitting a given permutation Π_q by changing $m \leqq \frac{n}{2}(1-\varepsilon)$ of its edges, and can not be transformed into a symmetric graph by changing less than m of its edges.

We shall deal first with the terms for which q lies in the range

$$\sqrt[4]{n} \leqq q \leqq n \tag{2. 6}$$

then with the terms for which q lies in the range

$$5 \leqq q < \sqrt[4]{n} \tag{2. 7}$$

and finally with the terms corresponding to $q=2$, 3 and 4 separately. We shall show that the sum figuring on the right hand side of (2. 4) tends to 0 for $n \to +\infty$; this clearly implies the assertion of Theorem 2.

Let us go now into the details.

Let Π be an arbitrary permutation of order n having the cycle-representation

$$\Pi = (a_{1,1}, \ldots, a_{1,c_1})(a_{2,1}, \ldots, a_{2,c_2})\ldots(a_{r,1}, \ldots, a_{r,c_r}) \tag{2. 7}$$

where $a_{i,j}$ $(1 \leqq j \leqq c_i;\ 1 \leqq i \leqq r)$ are the numbers $1, 2, \ldots, n$ in some order. Thus $c_1, c_2, \ldots, c_r$ are the cycle-lengths of Π. The permutation Π can also be interpreted as a one-to-one mapping of the set $\{1, 2, \ldots, n\}$ onto itself. Let Π, interpreted this way, map k into Πk $(k=1, 2, \ldots, n)$. We shall denote by Π^s the mapping obtained by applying the mapping Π s times. Clearly $\Pi a_{i,j} = a_{i,j+1}$ for $j=1, 2, \ldots, c_i$ where a_{i,c_i+1} stands for $a_{i,1}$. Let us calculate first the probability that a graph Γ of order n chosen at random should admit Π as its automorphism. By choosing a graph Γ at random we mean that n vertices $P_1, \ldots, P_n$ are prescribed and we choose some set of edges connecting these vertices at random, so that each of the $2^{\binom{n}{2}}$ possible choices has the same probability $2^{-\binom{n}{2}}$.

Thus the random choice of Γ is equivalent with a sequence of $\binom{n}{2}$ independent random decisions concerning all possible $\binom{n}{2}$ edges, so that with respect to any possible edge the probability of including it into Γ is equal to $\frac{1}{2}$. An equivalent way of characterizing the random choice of Γ is as follows: let us put $\varepsilon_{j,k}=1$ if the edge $\widehat{P_jP_k}$ is contained in Γ and $\varepsilon_{j,k}=0$ if not $(1\leqq j<k\leqq n)$. Then the random choice of Γ means that the $\varepsilon_{j,k}$ with $j<k$ are independent random variables each taking on the values 1 and 0 with probability $\frac{1}{2}$. Let us put $\varepsilon_{k,j}=\varepsilon_{j,k}$ for $j<k$. Now Γ admits the automorphism Π if and only if for any pair j, k $(j\neq k)$ one has $\varepsilon_{\Pi j,\Pi k}=\varepsilon_{j,k}$. Let us calculate now how many of the values $\varepsilon_{j,k}$ can still be chosen arbitrarily. An easy argument shows that if j belongs to the a-th cycle of Π (of length c_a) and k to the b-th cycle of Π (of length c_b) (where $a\neq b$) then the sequence of equations

$$\varepsilon_{j,k}=\varepsilon_{\Pi j,\Pi k}=\varepsilon_{\Pi^2 j,\Pi^2 k}=\ldots=\varepsilon_{\Pi^s j,\Pi^s k}=\ldots$$

contains $[c_a, c_b]$ different terms where $[A, B]$ denotes the least common multiple of A and B. Thus among the $c_a\cdot c_b$ values $\varepsilon_{j,k}$ where j belongs to the a-th cycle and k to the b-th cycle of Π we can choose only $\frac{c_a\cdot c_b}{[c_a, c_b]}=(c_a, c_b)$ values independently, where (A, B) stands for the greatest common divisor of A and B; all other such $\varepsilon_{j,k}$ are then determined $(a\neq b;\ 1\leqq a\leqq r;\ 1\leqq b\leqq r)$.

By a similar argument we get that among the $\varepsilon_{j,k}$ with both j and k belonging to the a-th cycle of Π we can choose $\left[\frac{c_a}{2}\right]$ independently, where $[x]$ denotes the integral part of x. Thus there are exactly

$$2^{\sum_{1\leqq a<b\leqq r}(c_a, c_b)+\sum_{a=1}^{r}\left[\frac{c_a}{2}\right]} \tag{2.8}$$

different graphs of order n which admit the automorphism Π, having the cycle representation (2.7), and the probability of Γ admitting the automorphism Π, i. e. of $\Pi\Gamma=\Gamma$ is

$$\mathbf{P}(\Pi\Gamma=\Gamma)=2^{\sum_{1\leqq a<b\leqq r}(c_a, c_b)+\sum_{a=1}^{r}\left[\frac{c_a}{2}\right]-\binom{n}{2}}. \tag{2.9}$$

Now let us fix a graph G for which $\Pi G=G$ and count the number of such graphs which can be transformed into G by changing m of its edges.

Clearly the m edges to be changed can be chosen in $\binom{\binom{n}{2}}{m}$ ways. Thus the number of graphs which can be transformed into one admitting the automorphism Π by changing m edges can not exceed

$$\binom{\binom{n}{2}}{m}2^{\sum_{1\leqq a<b\leqq r}(c_a, c_b)+\sum_{a=1}^{r}\left[\frac{c_a}{2}\right]}. \tag{2.10}$$

Now let us suppose that among the cycle-lengths c_a of Π there are exactly $l = n-q$ which are equal to 1; we may suppose $c_1=c_2=\ldots=c_l=1$ and $c_{l+i}\geqq 2$ for $i=1, 2, \ldots\ r-l$. Then we have

$$q = \sum_{i=1}^{r-l} c_{l+i} \geqq 2(r-l)$$

and thus

$$(r-l)\leqq \frac{q}{2}=\frac{n-l}{2} \quad \text{and} \quad r\leqq \frac{n+l}{2}. \tag{2.11}$$

As $(c_a, c_b)\leqq \min(c_a, c_b)\leqq \frac{c_a+c_b}{2}$, it follows

$$\sum_{1\leqq a<b<r} (c_a, c_b)+\sum_{a=1}^{r}\left[\frac{c_a}{2}\right]\leqq \binom{l}{2}+(r-l)\left(\frac{n+l}{2}\right). \tag{2.12}$$

Thus

$$\sum_{1\leqq a<b\leqq r} (c_a, c_b)+\sum_{a=1}^{r}\left[\frac{c_a}{2}\right]\leqq \binom{l}{2}+\frac{n^2-l^2}{4}. \tag{2.13}$$

Thus the probability of choosing at random such a graph which can be transformed by less than $(1-\varepsilon)\frac{n}{2}$ changes into one admitting Π as an automorphism does not exceed

$$2^{\frac{l^2-n^2}{4}+O(n\log n)}. \tag{2.14}$$

As the number of permutations Π with a fixed l is less than $\binom{n}{l}(n-l)! = 2^{O(n\log n)}$ we have

$$\sum_{\sqrt[4]{n}\leqq q\leqq n} \mathbf{P}_n(\varepsilon, q)\leqq 2^{-\frac{n^{5/4}}{2}+O(n\log n)}. \tag{2.15}$$

Now we consider the permutations with $5\leqq q\leqq \sqrt[4]{n}$. Concerning these we have to use much careful estimations.

Let us fix first the value of q $(5\leqq q<\sqrt[4]{n})$. The number of permutations which leave $n-q = l$ elements unchanged is clearly less than $\binom{n}{q}q!\leqq n^{\sqrt[4]{n}}$. In estimating the number of ways in which the m edges to be changed can be chosen, we may restrict ourselves to those edges, which connect either the q points which do not remain unchanged by Π among themselves, or edges connecting such points with the invariant ones. Thus an upper bound for the number of choices of the

$m < \frac{n}{2}(1-\varepsilon)$ edges is given by

$$\sum_{m<\frac{n}{2}(1-\varepsilon)} \binom{\binom{q}{2}+q(n-q)}{m} \leqq n \binom{nq}{\frac{n}{2}(1-\varepsilon)} = 2^{H(\alpha)nq+O(\log n)} \tag{2.16}$$

where $\alpha = \frac{1-\varepsilon}{2q}$ and $H(\alpha) = \alpha \log_2 \frac{1}{\alpha} + (1-\alpha)\log_2 \frac{1}{1-\alpha}$.

Now if $q \geqq 5$ then $\alpha \leqq \frac{1-\varepsilon}{10} < \frac{1}{10}$ and thus (as $H(x)$ is increasing for $0 < x < \frac{1}{2}$)

$$H(\alpha) < H\left(\frac{1}{10}\right) < 0{,}47. \tag{2.17}$$

It follows that for $5 \leqq q < \sqrt[4]{n}$

$$\mathbf{P}_n(\varepsilon, q) = 2^{O(\sqrt[4]{n}\log n) + \frac{n^2-(n-q)^2}{4} + \binom{n-q}{2} - \binom{n}{2} + 0{,}47qn}$$

and thus

$$\sum_{q=5}^{\sqrt[4]{n}} \mathbf{P}_n(\varepsilon, q) \leqq 2^{-0{,}03qn + O(\sqrt[4]{n})}. \tag{2.18}$$

Thus it remains only to consider permutations Π for which $q=2$, 3 or 4, i. e. which interchange not more than 4 points and leave all others untouched. Let us start with the case $q=2$. The number of such permutations is clearly $\binom{n}{2}$. The number of graphs of order n admitting such a permutation as an automorphism is (as in this case $c_1=c_2=\ldots=c_{n-2}=1$, $c_{n-1}=2$) $2^{\binom{n-1}{2}+1}$, and thus the probability that a random graph admits an automorphism interchanging two points is $\leqq 2^{-n+O(\log n)}$. Now, if a graph G^* can be transformed by changing m of its edges into a graph G admitting the permutation Π interchanging P_j and P_k and leaving all other points invariant, we may suppose that all edges changed have either P_j or P_k (but not both) as one of their endpoints.

It is clear that we may restrict ourselves to count those graphs, which can be transformed into G by *deleting* edges, because any graph G^* which can be transformed in a graph, which is invariant with respect to the permutation interchanging P_j and P_k, by changing (i. e. deleting or adding) m edges, can also be transformed in such a graph by deleting m edges. Thus the number of such graphs G^* belonging to a fixed G does not exceed

$$\sum_{m<\frac{n}{2}(1-\varepsilon)} \binom{n-2}{m} = 2^{H\left(\frac{1-\varepsilon}{2}\right)n + O(\log n)}.$$

As however $H(x) < 1$ for $x \neq \frac{1}{2}$ it follows that

$$P_n(\varepsilon, 2) \leqq 2^{-c(\varepsilon)n} \tag{2.19}$$

where $c(\varepsilon)$ is a positive constant depending only on ε. Let us consider now the case $q=3$. In this case clearly $c_1 = c_2 = \dots = c_{n-3} = 1$ and $c_{n-2} = 3$ and therefore the number of graphs admitting such an automorphism is

$$2^{\binom{n-2}{2}+1}.$$

As further we can select the 3 points which are moved by the permutation in $\binom{n}{3}$ ways, and the permutation itself in two ways, the probability that a graph admits as an automorphism a permutation cyclically interchanging 3 points and leaving all others unmoved, does not exceed

$$2\binom{n}{3}2^{\binom{n-2}{2}+1-\binom{n}{2}} = 2^{-2n+O(\log n)}.$$

Now if such a graph G is fixed, all graphs which can be transformed into G by changing $m < \frac{n}{2}(1-\varepsilon)$ edges (and can not be transformed into any other graph admitting the same automorphism by changing a smaller number of edges) are obtained if we choose m among the $n-3$ points left unchanged by the permutation, and select one of the three edges connecting this point with the 3 points moved by the permutation and delete or add this edge according to whether it is or is not contained in G. Thus the number of graphs which can in this way be transformed into G is $\sum_{m<\frac{n}{2}(1-\varepsilon)} \binom{n-3}{m} 3^m = O\left(2^n 3^{\frac{n}{2}(1-\varepsilon)}\right)$. Besides this, we may change some of the 3 edges between the 3 points moved by the permutation. As the number of ways doing this is 8 we obtain that

$$\mathbf{P}_n(\varepsilon, 3) = O\left(\left(\frac{\sqrt{3}}{2}\right)^n\right). \tag{2.20}$$

Let us consider finally the case $q=4$. Here two cases have to be distinguished: either $c_1 = c_2 = \dots = c_{n-4} = 1$ and $c_{n-3} = 4$ or $c_1 = c_2 = \dots = c_{n-4} = 1$ and $c_{n-3} = c_{n-2} = 2$. For the first case we obtain

$$\sum_{1\le a<b\le r} (c_a, c_b) + \sum_{a=1}^{r}\left[\frac{c_a}{2}\right] = \binom{n-3}{2} + 2$$

for the second

$$\sum_{1\le a<b\le r} (c_a, c_b) + \sum_{a=1}^{r}\left[\frac{c_a}{2}\right] = \binom{n-2}{2} + 3.$$

Thus the probabilities of a graph admitting such an automorphism are $\le 6\binom{n}{4}2^{\binom{n-3}{2}+2-\binom{n}{2}} = 2^{-3n+O(\log n)}$ and $\le 6\binom{n}{4}2^{\binom{n-2}{2}+3-\binom{n}{2}} = 2^{-2n+O(\log n)}$, respectively. As regards the number of graphs which can be transformed into a given graph G invariant with respect to a fixed permutation of the mentioned types by changing not more than $\frac{n}{2}(1-\varepsilon)$ edges, we obtain in the first case an upper bound of order

$$\sum_{m\leqq\frac{n(1-\varepsilon)}{2}}\sum_{l\leqq\frac{m}{2}}\binom{n-4}{l}\binom{n-4-l}{m-2l}6^l4^{m-2l}\leqq 2^{1,82n+6(\log n)}$$

and in the second case an upper bound of order

$$\sum_{l\leqq m\leqq\frac{n(1-\varepsilon)}{2}}\sum\binom{n-4}{l}\binom{n-4}{m-l}2^m\leqq 2^{1,32n+O(\log n)}.$$

It follows that

$$\mathbf{P}_n(\varepsilon,4)\leqq 2^{-0,68n+O(\log n)}. \tag{2.21}$$

Collecting the estimates (2. 15), (2. 18), (2. 19), (2. 20) and (2. 21) in view of (2. 4) Theorem 2 follows.

§ 3. Symmetries of infinite graphs

Let Γ_∞ denote a random infinite graph which has the vertices P_n $(n=1,2,\ldots)$ and which is such that denoting by $E_{j,k}$ the event that P_j and P_k are connected by an edge $(j\neq k)$ the events $E_{j,k}$ $(j,k=1,2,\ldots;\ j<k)$ are independent and $P(E_{j,k})=\frac{1}{2}$. We shall prove that with probability one Γ_∞ admits non-trivial automorphisms. We can construct such an automorphism as follows.

Let us denote by $A(k)$ the index of the vertex into which the automorphism carries over the vertex P_k. We put $A(1)=2$ and $A(2)=1$.

Now let us consider P_3. This vertex can be in 4 possible relations with P_1 and P_2 (connected with both; connected with P_1 but not with P_2; connected with P_2 but not with P_1; connected neither with P_1 nor with P_2). Let $A(3)$ be the least integer (if there exists any) for which $P_{A(3)}$ is in the same relation with P_2 and P_1 as P_3 with P_1 and P_2 and put $A(A(3))=3$. If $A(n)$ is already defined for any finite number of values of n, for instance if $A(n_j)=n'_j$ and $A(n'_j)=A(n_j)$ $(j=1,2,\ldots,s)$ where $n_1,n_2,\ldots,n_s,n'_1,n'_2,\ldots,n'_s$ are different integers, let m denote the least integer for which $A(m)$ is not yet defined. Let us define $A(m)$ as the least integer different from m and from all values n for which $A(n)$ is already defined, for which $P_{A(m)}$ is in the same relation with $P_{n'_j}$ as P_m with P_{n_j}, and in the same relation with P_{n_j} as P_m with $P_{n'_j}$ $(j=1,2,\ldots,s)$, and put $A(A(m))=m$.

In this way a non-trivial automorphism of Γ_∞ is constructed step-by-step, provided that the construction can always be continued. But it is easy to see that with probability 1 the construction can always be continued. This follows from the following

LEMMA 3. *Let $i_1,i_2,\ldots,i_k,j_1,j_2,\ldots,j_l$ be arbitrary different natural numbers. Then with probability* 1 *the number of vertices P_n which are connected in Γ_∞ with each of $P_{i_1},P_{i_2},\ldots,P_{i_k}$ and not connected with $P_{j_1},P_{j_2},\ldots,P_{j_l}$ is infinite for every choice of the indices $i_1,i_2,\ldots,i_k,j_1,j_2,\ldots,j_l$ $(k,l=1,2,\ldots)$.*

PROOF of Lemma 3. The probability of the event E_n that P_n is in the required relation with all vertices $P_{i_1}, \ldots, P_{i_k}, P_{j_1}, \ldots, P_{j_l}$ is clearly equal to $\frac{1}{2^{k+l}}$, further these events E_n are independent. Thus by the Borel-Cantelli lemma E_n takes place for an infinity of values of n with probability 1. As the union of a denumerable set of sets of probability 0 has probability 0 too, with probability 1 in Γ there are infinitely many vertices connected with the vertices $P_{i_1}, \ldots, P_{i_k}$ and not connected with the vertices $P_{j_1}, \ldots, P_{j_l}$ simultaneously for all choices of the indices $i_1, i_2, \ldots, i_k, j_1, \ldots, j_l$. This proves Lemma 3.

Thus we have proved that with probability 1 Γ_∞ admits a non-trivial automorphism, which moreover is involutory (i. e. $A(A(n))=n$ for every n).

This is what we wanted to prove. It can be seen from the proof that Γ_∞ admits with probability 1 an infinity of nontrivial automorphisms. As a matter of fact, instead of putting $A(1)=2$, we could have prescribed $A(1)=k$ with an arbitrary k.

It is easy to see, that our result remains also valid if instead of supposing that the edge $\widehat{P_jP_k}$ is contained in Γ_∞ with probability $\frac{1}{2}$, we suppose only that this probability $p_{j,k}$ is contained between the limits δ and $1-\delta$ where $0<\delta<1$, admitting that this probability should depend on j and k. The result holds also if $p_{j,k}$ is not bounded away from 0 and 1 but is such that the series

$$\sum_{n=1}^{\infty} p_{i_1,n} p_{i_2,n} \cdots p_{i_k,n} (1-p_{j_1,n}) \cdots (1-p_{j_l,n})$$

is divergent for every choice of the integers $i_1, \ldots, i_k, j_1, \ldots, j_l$.

§ 4. Asymmetry of graphs of order n with a fixed number N of edges

In this § we consider only such graphs of order n which contain exactly N edges. If the valencies of the vertices $P_1, \ldots, P_n$ are denoted by $v_1, \ldots, v_n$ then we have by supposition

$$\sum_{l=1}^{n} v_l = 2N. \tag{4. 1}$$

For such a graph we have by Cauchy's inequality

$$\sum_{l=1}^{n} v_l^2 \geq \frac{1}{n}\left(\sum_{l=1}^{n} v_l\right)^2 = \frac{4N^2}{n}. \tag{4. 2}$$

Thus it follows from (1. 3) and (1. 4) that for such a graph

$$A[G] \leq \frac{4N}{n} - \frac{8N^2}{n^2(n-1)}. \tag{4. 3}$$

Thus we have proved the following

THEOREM 3. *If a graph G of order n has* $N=\lambda n$ *edges* $\left(0<\lambda<\frac{n-1}{2}\right)$ *then*

$$A[G] \leqq 4\lambda\left(1-\frac{2\lambda}{n-1}\right). \tag{4.4}$$

(The maximum of the right hand side of (4.4) is clearly attained if $\lambda=\frac{n-1}{4}$, in which case it is equal to $\frac{n-1}{2}$).

We can prove with the same probabilistic method as applied in § 2 combined with methods of our paper [7] that the estimate (4.4) is asymptotically best possible if together with $n\to+\infty$ we have $\lambda\to+\infty$ in such a way that $\lim\limits_{n\to+\infty}\frac{\lambda}{\log n}=+\infty$; moreover $A[G]$ is near to $4\lambda\left(1-\frac{2\lambda}{n-1}\right)$ for most graphs of order n having $N=\lambda n$ edges.

The meaning of the condition $\lim\limits_{n\to+\infty}\frac{\lambda}{\log n}=+\infty$ is that as we have shown in [7] in a random graph of order n and having $N=\lambda n$ edges the valencies of all vertices are asymptotically equal with probability tending to 1 for $n\to\infty$ if $\frac{\lambda}{\log n}\to+\infty$.

§ 5. Further remarks and unsolved problems

The following problems, closely connected with that considered in § 4 can be raised: for a fixed positive integer k, and $n>2k+1$ determine the least value $F(n,k)$ such that there exists a graph G of order n, having $N=F(n,k)$ edges and asymmetry $A[G]=k$; further the least value $C(n,k)$ such that there exists a *connected* graph G of order n, having $N=C(n,k)$ edges, and asymmetry $A[G]=k$. We can not give a full answer to these questions, only some partial results. We prove first

THEOREM 4. *We have* $C(6,1)=6$ *and* $C(n,1)=n-1$ *for* $n\geqq 7$.

REMARK. As shown in the introduction each graph of order $\leqq 5$ is symmetric, thus $C(n,1)$ is defined only for $n\geqq 6$.

PROOF of Theorem 4. For $n=6$ there are, as we have seen, in the introduction, four types of asymmetric graphs, each having the asymmetry 1; as shown by Fig. 3 among these there is one having 6 edges, the others have 7 edges or more. Thus $C(6,1)=6$. As any connected graph G of order n has at least $n-1$ edges, clearly $C(n,1)\geqq n-1$ for $n\geqq 7$ with equality only if there exists an asymmetric tree of order n. Now it is easy to see that for any $n\geqq 7$ there exists an asymmetric tree of order n; such a tree for $n=7$ is shown by Fig. 8; for any $n\geqq 7$, such a tree T_n can be obtained as fol-

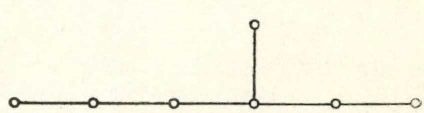

Fig. 8

lows: Let T_n consist of the vertices $P_1, \dots, P_n$ and the edges $\widehat{P_i P_{i+1}}$ $(i=1, 2, \dots, n-2)$ and of the edge $\widehat{P_{n-3} P_n}$.

Thus Theorem 4 is proved. Let us add that the asymmetry of a tree can not exceed 1. As a matter of fact, let T be an arbitrary tree; we may suppose that T has at least 3 vertices, as a tree of order 2 is evidently symmetric. Let us consider a longest path with any fixed starting point P_1 in T and let P_2 be the endpoint of this path. Let P_3 be the (unique) vertex which is connected with P_2 in T. Then two cases are possible. Either $P_3 = P_1$, in this case T is a star with center P_1, and thus is evidently symmetric; or $P_3 \neq P_1$, then again two cases are possible. Either P_3 has valency 2; in this case let P_4 be the unique vertex connected with P_3 besides P_2; by omitting from T the edge $P_3 P_4$ we obtain a graph which has the symmetry interchanging P_2 and P_3. If P_3 has valency larger than 2, then any vertex P_l connected with P_3 which is not on the path $P_1 P_2$ has valency 1 because otherwise the path $P_1 P_2$ would not be the longest. In this case the tree itself is symmetric as it is invariant under the permutation interchanging P_2 and P_l. As P_1 has been chosen arbitrarily, we have incidentally proved the following

THEOREM 5. *Let T be a tree of order $n \geqq 3$; let us select one of the vertices of T, say P_1. Then either there exists a nontrivial permutation Π of the vertices of T which does not move P_1 and under which T is invariant, or one can transform T into a graph having such an automorphism by omitting one of its edges.*

We do not know the exact value of $F(n, 1)$. It is an interesting question also what is the total number of non-isomorphic asymmetric trees of order n? We can not answer this question; we can prove however that in a certain sense „almost all" trees of order n are symmetric, if n is large. This is a consequence of Theorem 6 below. Before formulating this theorem we introduce the following definition. If a graph G contains two vertices P_1, P_2 of valency 1 which are connected with the same vertex P_3, we shall say that G contains the *cherry* $\widehat{P_1 P_3 P_2}$.

A graph containing a cherry is evidently symmetric, as it is invariant under the permutation which interchanges the two vertices of order 1. Thus our assertion that almost all trees of order n are symmetric if n is large, is contained in the following

THEOREM 6. *Let us choose at random a tree from the set of all possible trees which can be formed from a given set of n labelled vertices, so that each of these trees should have the same probability to be chosen. Let γ_n denote the probability that the random tree contains at least one cherry. Then we have*

$$\lim_{n \to +\infty} \gamma_n = 1.$$

PROOF of Theorem 6. Let $P_1, \dots, P_n$ denote the vertices of our random tree T_n. Let us put $\varepsilon(i_1, i_2, j) = 1$ (i_1, i_2, j are different natural numbers not exceeding n) if $\widehat{P_{i_1} P_j P_{i_2}}$ is a cherry in the random tree, i. e. if P_{i_1} and P_{i_2} have the valency 1 in T_n and if both are connected in T_n with P_j; let us put $\varepsilon(i_1, i_2, j) = 0$ otherwise. Taking into account that according to a well-known theorem of A. CAYLEY [8] the total number of trees which can be formed from n given labelled vertices is equal to n^{n-2}, we obtain that

$$M(\varepsilon_{i_1,i_2,j}) = \frac{(n-2)^{n-4}}{n^{n-2}} \tag{5. 1}$$

and

$$M\big(\varepsilon(i_1,i_2,j_1)\varepsilon(i_3,i_4,j_2)\big) = \begin{cases} \dfrac{(n-6)^{n-6}}{n^{n-2}} & \text{if } i_1,i_2,i_3,i_4,j_1,j_2 \text{ are all different,} \\ \dfrac{(n-5)^{n-6}}{n^{n-2}} & \text{if } j_1=j_2=j \text{ and } i_1,i_2,i_3,i_4 \text{ are different,} \end{cases} \tag{5. 2}$$

$$M\big(\varepsilon(i_1,i_2,j_1)\varepsilon(i_3,i_4,j_2)\big) = \begin{cases} \dfrac{(n-3)^{n-4}}{n^{n-2}} & \text{if } i_1=i_3,\ i_2=i_4 \text{ and } j_1=j_2 \text{ or } i_1=i_4,\ i_2=i_3 \text{ and } j_1=j_2 \\ 0 & \text{otherwise.} \end{cases} \tag{5. 3}$$

Let Γ_n denote the number of cherries in T_n. Then by (5. 1), (5. 2) and (5. 3) we obtain

$$M(\Gamma_n) = \frac{n}{2e^3} + O(1)$$

and

$$M(\Gamma_n^2) = \frac{n^2}{4e^6} + O(n)$$

and thus

$$D^2(\Gamma_n) = O(n).$$

It follows by the inequality of Chebyshev that for $n \geqq n_0$

$$1-\gamma_n = O\left(\frac{1}{n}\right).$$

Thus Theorem 6 is proved.

Now we prove the following

THEOREM 7. *Any connected graph of order n having n edges is either symmetric, or its asymmetry is equal to* 1.

REMARK. By other words we have $C(n, 2) > n$ for $n \geqq 7$ (As we have seen any graph of order $\leqq 6$ is either symmetric or has the asymmetry 1.)

PROOF of Theorem 7. Any connected graph of order n having n edges has as well known the following structure: it contains exactly one cycle, and any vertex of this cycle may be the root of one or more trees. Now suppose that contrary to the assertion of Theorem 7 there exists a graph G of order n having n edges, for which $A[G] \geqq 2$. In such a graph any tree attached to a vertex of the single cycle of the graph consists of a single edge only, because otherwise by Theorem 5 we would have $A[G] \leqq 1$. Let us call such an edge a „thorn". We can exclude the case when to a vertex two or more thorns are attached, because two thorns make a cherry which admits a symmetry. Now if to a vertex P of the cycle a thorn PQ is attached, then necessarily a thorn has to be attached to both neighbouring vertices of the cycle too, because if P' would be a neighbour of P which is not the starting point of a thorn, then if P'' is the other neighbour of P' in the cycle by omitting the edge $P'P''$ we would obtain a graph containing a cherry $\widehat{QPP'}$. Thus either a thorn is attached to all vertices of the cycle or to none of them. As in both cases the graph has a cyclic symmetry, we obtained a contradiction, which proves Theorem 7.

It can be shown by a similar argument that $C(n, 2) > n+1$.

Our last result is a lower estimate for $F(n, 3)$. We prove

THEOREM 8. *We have* $F(n, 3) \geqq \frac{4n}{3} - \frac{3}{2}$.

PROOF. Let G be a graph of order n having N edges for which $A[G]=3$. Clearly G can contain only a single vertex having the valency 1. Let n_2 be the number of vertices of G of valency 2 and n_3 the number of vertices of G of valency $\geqq 3$. Clearly two vertices P_1, P_2 of valency 2 can not be connected by an edge, because if P_1 and P_2 were connected by an edge, and P_1 would be connected besides P_2 with P_1' and P_2 besides P_1 with P_2', then omitting the edges $\widehat{P_1P_1'}$ and $\widehat{P_2P_2'}$ the resulting graph would admit the symmetry consisting in interchanging P_1 and P_2.

Further no vertex with valency $\geqq 3$ can be connected with more than one vertex with valency 2; as a matter of fact if P_1 and P_2 were vertices with valency 2 connected with a vertex P_3 with valency $\geqq 3$, then omitting the two edges connecting the vertices P_1 and P_2 with vertices different from P_3 the resulting graph would contain the cherry $\widehat{P_1P_3P_2}$.

It follows that $n_3 \geqq 2n_2 - 1$. As on the other hand $n_2 + n_3 \geqq n-1$, we obtain $3n_3 \geqq 2n-3$. Now we have

$$2N \geqq 2n_2 + 3n_3 \geqq 2n - 2 + n_3 \geqq \frac{8n-9}{3}$$

and therefore $N \geqq \frac{8n-9}{6} = \frac{4n}{3} - \frac{3}{2}$.

Finally we mention a further unsolved problem: is it true that $C(n, k) = F(n, k)$ for $k \geqq 2$?

Remarks, added on November 8. 1963. Prof. R. C. BOSE kindly informed us that in a forthcoming paper he introduced a class of graphs, called by him *strongly regular* graphs, which contains the class of Δ-graphs discussed in the present paper, as a subclass. A graph of order n is called strongly regular with parameters n_1, p_1, p_2 if each vertex of G_n is joined with n_1 other vertices, further any two joined vertices are both joined to exactly p_1 vertices and any two unjoined vertices are both joined to exactly p_2 other vertices. Clearly a Δ-graph of order $n \equiv 1 \bmod 4$ is a strongly regular graph with parameters $n_1 = p_1 = p_2 = \frac{n-1}{2}$.

The notion of strongly regular graphs is closely connected with the concept of an association scheme with two associate classes introduced by R. C. BOSE and T. SHIMAMOTO in their paper: Classification and analysis of partially balanced incomplete block designs with two associate classes (*Journal of the American Statistical Association*, **47** (1952), pp. 151-184).

We should like to add further that the Δ-graph of order p constructed on p. 301 is identical with the graph constructed by H. SACHS on p. 282 of his paper: Über selbstkomplementäre Graphen (*Publicationes Mathematicae*, **9** (1962), pp. 270-288). This paper was not known to us at the time when our paper was written. As shovn by H. SACHS, this graph is isomorphic with its complementary graph.

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References

[1] O. PERRON, Bemerkung über die Verteilung der quadratischen Reste, *Mathematische Zeitschrift*, **56** (1952), pp. 122–130.

[2] J. B. KELLY, A characteristic property of quadratic residues, *Proceedings of the American Mathematical Society*, **5** (1954), pp. 38–46.

[3] A. RÉNYI, On the measure of equidistribution of point sets, *Acta Sci. Math. Szeged*, **13** (1949), pp. 77–92.

[4] A. W. GOODMAN, On sets of acquaintances and strangers at any party, *American Math. Monthly*, **66** (1959), pp. 778–783.

[5] I. L. SAUVÉ, On chromatic graphs, *American Math. Monthly*, **68** (1961), pp. 107–111.

[6] P. ERDŐS, On the number of triangles contained in certain graphs, *Bulletin of the Research Council of Israel*, Section F. (in print).

[7 P. ERDŐS and A. RÉNYI, On the evolution of random graphs, *Publications of the Math. Inst. Hung. Acad. Sci.*, **5** (1960), pp. 17–61.

[8] A. CAYLEY, A theorem on trees, *Quarterly Journal of Pure and Applied Math.*, **23** (1889), pp. 376–378.

[9] A. RÉNYI, Some remarks on the theory of trees, *Publ. Math. Inst. Hung. Acad. Sci.*, **4** (1959), pp. 73–85.

10] A. RÉNYI, On connected graphs, *Publ. Math. Inst. Hung. Acad. Sci.*, **4** (1959), pp. 385–388.