A GENERALIZATION OF A THEOREM OF E. VINCZE

by

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The functional equation $\varphi(x) = \varphi(ax)\,\varphi(bx)$ $(a, b > 0,\ a^2 + b^2 = 1)$ was solved by E. VINCZE [1] under the assumption that $\varphi(x)$ is a complex valued function of the real variable x which can be differentiated twice at the origin. This equation occurs in certain problems in probability theory and was therefore studied by a number of authors under the restriction that $\varphi(x)$ is a positive definite function.

In the present note we prove the following generalization of VINCZE's result:

Theorem. *Let $\varphi(x)$ be a complex valued function of the real variable x and let $\{a_j\}$ be a sequence of nonnegative real numbers such that $\sum_{j=1}^{\infty} a_j^2 = 1$ and $0 < a_1 < 1$. Suppose that there exist complex constants A and B such that*

$$\lim_{x \to 0} \frac{\varphi(x) - \varphi(0) - Ax}{x^2} = B. \tag{1}$$

Assume further that $\varphi(x)$ satisfies for all real x the functional equation

$$\varphi(x) = \prod_{j=1}^{\infty} \varphi(a_j x) \tag{2}$$

where the infinite product converges.[2] *Then $A = 0$ and $\varphi(x) = e^{Bx^2}$.*

Proof. It follows from (1) that $\varphi(x)$ is continuous at $x = 0$. It follows further from the convergence of the infinite product (2) that $\lim_{j \to \infty} \varphi(a_j x) = 1$, and as clearly $\lim_{j \to \infty} a_j = 0$ we obtain $\varphi(0) = 1$. Thus it follows from (1) that

$$\varphi(x) = 1 + Ax + O(x^2) \qquad \text{for} \quad x \to 0. \tag{3}$$

Thus we have

$$\prod_{j=1}^{N} \varphi(a_j x) = e^{A(\sum_{j=1}^{N} a_j)x + O(x^2)}.$$

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[2] As usually the convergence of an infinite product $\prod_{n=1}^{\infty} z_n$ is understood in the sense that only a finite number of factors may be equal to 0 and if $z_n \neq 0$ for $n \geqq n_0$ then $\lim_{N \to \infty} \prod_{n=n_0}^{N} z_n$ exists and is different from 0.

As the product (2) is convergent, it follows that the series $\sum_{j=1}^{\infty} a_j$ is convergent too. Let us put

$$\sum_{j=1}^{\infty} a_j = C \,. \tag{4}$$

We have evidently $C > 1$. It follows that $\varphi(x) = e^{ACx + O(x^2)}$ and thus

$$\frac{\varphi(x) - 1 - Ax}{x^2} = \frac{A(C-1)}{x} + O(1) \qquad \text{for} \quad x \to 0. \tag{5}$$

Clearly (5) is compatible with (1) only if $A = 0$. Condition (1) now reduces to

$$\lim_{x\to 0} \frac{\varphi(x) - 1}{x^2} = B \tag{6}$$

which yields

$$\lim_{x\to 0} \frac{\log \varphi(x)}{x^2} = B \,. \tag{7}$$

Clearly (6) implies that there exist positive numbers d and D such that

$$|\,\varphi(x) - 1\,| < Dx^2 \qquad \text{for} \quad |\,x\,| \leqq d. \tag{8}$$

As for $|\,z - 1\,| \leqq \frac{1}{2}$ we have $|\arg z\,| \leqq \frac{\pi}{3}\,|\,z - 1\,|$, it follows that for $|\,x\,| \leqq \triangle = \min\left(d, \frac{1}{\sqrt{2D}}\right)$ we have

$$\sum_{j=1}^{\infty} |\arg \varphi(a_j x)\,| < \pi \,.$$

Thus we obtain from (2)

$$\log \varphi(x) = \sum_{j=1}^{\infty} \log \varphi(a_j x) \qquad \text{for} \quad |\,x\,| \leqq D. \tag{9}$$

We obtain from (9) by iteration for any natural number k

$$\log \varphi(x) = \sum_{j_1=1}^{\infty} \dots \sum_{j_k=1}^{\infty} \log \varphi(a_{j_1} \dots a_{j_k} x) \,. \tag{10}$$

Since $\max a_j = \alpha < 1$ we see that

$$\max a_{j_1} \dots a_{j_k} \leqq \alpha^k \,. \tag{11}$$

In view of (11) and (7) for any $\varepsilon > 0$ we can choose the value of k so large that

$$|\log \varphi(a_{j_1} \dots a_{j_k} x) - Bx^2 a_{j_1}^2 \dots a_{j_k}^2| \leqq \varepsilon\, a_{j_1}^2 \dots a_{j_k}^2 \tag{12}$$

for all $|\,x\,| \leqq \triangle$. It follows from (10) and (12) that

$$|\log \varphi(x) - Bx^2\,| \leqq \varepsilon \,. \tag{13}$$

As ε can be chosen arbitrarily small we obtain that $\varphi(x) = e^{Bx^2}$ for $|\,x\,| \leqq \triangle$.

Let us put $\psi(x) = \varphi(x)\, e^{-Bx^2}$. Then clearly $\psi(x)$ satisfies for all x the equation

$$\psi(x) = \prod_{j=1}^{\infty} \psi(a_j x) \tag{14}$$

further $\psi(x) = 1$ for $|x| \leqq \triangle$. Let x_0 denote the upper bound of those positive numbers β for which $\psi(x) = 1$ for $|x| \leqq \beta$. We have already shown that $x_0 \geqq \triangle > 0$. Suppose that x_0 is finite; we shall prove that this leads to a contradiction. Clearly $\psi(\pm x_0) = 1$ because $\max a_j = \alpha < 1$. Let now η be an arbitrary real number such that $1 > \eta > \alpha$; then $\frac{a_j}{\eta} < \frac{a_j}{\alpha} \leqq 1$ so that $\psi\left(\pm \frac{a_j}{\eta} x_0\right) = 1$. It follows that $\psi\left(\pm \frac{x_0}{\eta}\right) = 1$. This however contradicts the definition of x_0. Thus $x_0 = +\infty$ and $\varphi(x) = e^{Bx^2}$ for all x.

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ОБОБЩЕНИЕ ОДНОЙ ТЕОРЕМЫ E. VINCZE

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Резюме

Доказывается следующее обобщение теоремы E. Vincze [1]:

Теорема. *Пусть $\varphi(x)$ — функция вещественной переменной x с комплексными значениями и $\{a_j\}$ — последовательность неотрицательных чисел, для которых $\sum_{j=1}^{\infty} a_j^2 = 1$ и $0 < a_1 < 1$. Предполагаем, что существуют комплексные константы A и B такие, что*

$$\lim_{x\to 0} \frac{\varphi(x) - \varphi(0) - Ax}{x^2} = B\,,$$

и что $\varphi(x)$ удовлетворяет функциональному уравнению

$$\varphi(x) = \prod_{j=1}^{\infty} \varphi(a_j x)\,,$$

где бесконечное произведение сходится. В этом случае $A = 0$ и $\varphi(x) = e^{Bx^2}$.