ON „SMALL" COEFFICIENTS OF THE POWER SERIES OF ENTIRE FUNCTIONS

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§ 1. Introduction

The second named author has proved (see [1]) some years ago the following

THEOREM A. *Let $f(z)$ be a transcendental entire function and let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

$$f(z) = \sum_{n=0}^{\infty} b_n (z-1)^n$$

be the power series of $f(z)$ around the points[1] *$z=0$ and $z=1$ respectively. Let $Z_0(n)$ resp. $Z_1(n)$ denote the number of zeros in the sequence $a_0, a_1, \ldots, a_{n-1}$ resp. $b_0, b_1, \ldots, b_{n-1}$. Then*

$$\liminf_{n \to \infty} \frac{Z_0(n)+Z_1(n)}{n} \leq 1.$$

Theorem A can be formulated somewhat vaguely in the following way: if there are „many" zeros among the coefficients of the power series of a transcendental entire function about the point $z=0$, then there can not be „too many" zeros among the coefficients of its power series about the point $z=1$.

In the present paper we shall prove a generalization of Theorem A, whose statement can be expressed somewhat vaguely as follows: if „many" of the coefficients of the power series of an entire transcendental function about the point $z=0$ are „small", then there can not be „too many" „small" coefficients in the power series of the same function about the point $z=1$. Of course one has to give a precise meaning to the word „small" in this context.

Roughly speaking we shall call a coefficient of the power series of an entire function $f(z)$ „small" if it is smaller in absolute value than the corresponding coefficient of the power series of an other entire function $g(z)$ which is of smaller order of magnitude than $f(z)$. To make this definition quite definite, we have to decide how to compare the orders of magnitude of the entire functions $f(z)$ and $g(z)$. This could be done for example by comparing the rate of increase for $r \to \infty$ of the maximum modulus $M_f(r) = \max_{|z|=r} |f(z)|$ of $f(z)$ with that of $g(z)$, i. e. with $M_g(r) = \max_{|z|=r} |g(z)|$.

[1] The theorem remains valid if we consider the power series of $f(z)$ around two arbitrary different points α and β.

For our purposes it is however more convenient to compare the orders of magnitude of $f(z)$ and $g(z)$ by comparing the maximal terms of their power series. We shall prove the following

THEOREM 1. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n (z-1)^n$ *be a transcendental entire function and* $g(z)=\sum_{n=0}^{\infty} c_n z^n$ *be an arbitrary entire function. Put*

$$\mu_f(r) = \operatorname{Max}_n |a_n| r^n$$

and

$$\mu_g(r) = \operatorname{Max}_n |c_n| r^n.$$

Let us suppose that there exists a number δ *such that* $0<\delta<1$ *and*

$$\liminf_{r\to\infty} \frac{\mu_g(r^{1/\delta})}{\mu_f(r)} < \infty. \tag{1.1}$$

Let $S_0(n)$ *denote the number of indices* $k<n$ *for which* $|a_k| \leq |c_k|$ *and* $S_1(n)$ *the number of indices* $k<n$ *for which* $|b_k| \leq |c_k|$. *Then we have*[2]

$$\liminf_{n\to\infty} \frac{S_0(n)+S_1(n)}{n} \leq 1+2\delta. \tag{1.2}$$

Let us make some remarks.

REMARK 1. If we choose $g(z)\equiv 0$, then (1.1) holds for any $\delta>0$, further in this case $S_0(n)=Z_0(n)$ and $S_1(n)=Z_1(n)$, where $Z_0(n)$ resp. $Z_1(n)$ denote the number of zeros in the sequence $a_0, a_1, \ldots, a_{n-1}$ resp. $b_0, b_1, \ldots, b_{n-1}$. Thus in this special case Theorem 1 reduces to Theorem A, i. e. Theorem 1 is a generalization of Theorem A.

REMARK 2. If $f(z)$ is an entire function of order ϱ_f and $g(z)$ an entire function of order ϱ_g, then according to a well known theorem (see e. g. [2], Chapter IV. Problem 51)

$$\limsup_{r\to\infty} \frac{\log\log \mu_f(r)}{\log r} = \varrho_f$$

and similarly

$$\limsup_{r\to\infty} \frac{\log\log \mu_g(r)}{\log r} = \varrho_g. \tag{1.3}$$

Thus if $\varrho_g<\varrho_f$ and we choose a sequence r_k $(k=1, 2, \ldots)$ such that $\lim_{k\to\infty} r_k=\infty$ and $\lim_{k\to\infty} \frac{\log\log \mu_f(r_k)}{\log r_k} = \varrho_f$ then we have for any δ with $\delta > \frac{\varrho_g}{\varrho_f}$

$$\lim_{k\to\infty} \frac{\mu_g(r_k^{1/\delta})}{\mu_f(r_k)} = 0.$$

[2] Clearly only the case $\delta<\frac{1}{2}$ is interesting as otherwise (1.2) is trivial.

It follows that the conditions of Theorem 1 hold with any δ for which $\delta > \frac{\varrho_g}{\varrho_f}$ and thus we obtain

$$\liminf_{n\to\infty} \frac{S_0(n)+S_1(n)}{n} \leq 1+2\frac{\varrho_g}{\varrho_f}. \tag{1.4}$$

In case $\varrho_g = 0$ and $\varrho_f > 0$ further also in case $\varrho_f = \infty$ and $\varrho_g < \infty$ it follows

$$\liminf_{n\to\infty} \frac{S_0(n)+S_1(n)}{n} \leq 1.$$

We shall show later (Theorem 4) that (1.4) can be improved, namely the factor 2 on the right hand side of (1.4) is unnecessary.

Of course, Theorem 1 can also be applied to certain pairs of entire functions which are either both of order 0 or both of order ∞.

REMARK 3. Clearly instead of considering the power series of $f(z)$ about $z=0$ and $z=1$ we could consider its power series about $z=\alpha$ and $z=\beta$ where α and β are arbitrary complex numbers and $\alpha\neq\beta$. As a matter of fact in this case we have to apply Theorem 1 to the function $f[(\beta-\alpha)z+\alpha]$ instead of $f(z)$ and $g[(\beta-\alpha)z]$ instead of $g(z)$.

§ 2 contains estimations of certain interpolatory polynomials. In **§ 3** we give the proof of Theorem 1, further of the related Theorems 2,3 and 4. The method of proof of Theorem 1 is the same as that of Theorem A given in [1] but besides the tools used there, the results of **§ 2** are also needed. **§ 4** contains the discussion of an example, which shows that our results are not far from being best possible.

§ 2. Estimates of certain interpolatory polynomials

Let $Q_n(z)$ $(n=1, 2, \ldots)$ be the unique polynomial of degree $2n-1$ which satisfies the following conditions:

$$\frac{Q_n^{(j)}(0)}{j!} = p_j \quad \text{for} \quad j = 0, 1, \ldots, n-1 \tag{2.1}$$

and

$$\frac{Q_n^{(j)}(1)}{j!} = q_j \quad \text{for} \quad j = 0, 1, \ldots, n-1 \tag{2.2}$$

where p_j and q_j $(j=0, 1, \ldots, n-1)$ are arbitrary complex numbers. An explicit formula for $Q_n(z)$ has been given by P. JOHANSEN [3]; this can be written as follows:

$$\begin{aligned} Q_n(z) = (1-z)^n \sum_{k=0}^{n-1} z^k \left[\sum_{s=0}^{k} p_{k-s}\binom{n+s-1}{s}\right] + \\ + z^n \sum_{k=0}^{n-1} (z-1)^k \left[\sum_{s=0}^{k} (-1)^s q_{k-s}\binom{n+s-1}{s}\right]. \end{aligned} \tag{2.3}$$

Formula (2.3) can be transfromed as follows:

$$Q_n(z) = \sum_{m=0}^{2n-1} z^m \left\{ \sum_{k=0}^{n-1} \binom{n}{m-k} (-1)^{m-k} \left[\sum_{s=0}^{k} p_{k-s} \binom{n+s-1}{s} \right] \right\} + \tag{2.4}$$

$$+ \sum_{t=0}^{n-1} z^{n+t} \left\{ \sum_{k=t}^{n-1} \binom{k}{t} (-1)^{k-t} \left[\sum_{s=0}^{k} (-1)^s q_{k-s} \binom{n+s-1}{s} \right] \right\}.$$

Thus we obtain

$$\frac{Q_n^{(n)}(z)}{n!} = \sum_{t=0}^{n-1} z^t \binom{n+t}{n} Q_{n,t}$$

where

$$Q_{n,t} = (-1)^n \sum_{k=t}^{n-1} \binom{n}{n+t-k} (-1)^{k-t} \left[\sum_{s=0}^{k} p_{k-s} \binom{n+s-1}{s} \right] +$$

$$+ \sum_{k=t}^{n-1} \binom{k}{t} (-1)^{k-t} \left[\sum_{s=0}^{k} (-1)^s q_{k-s} \binom{n+s-1}{s} \right].$$

Now evidently

$$|Q_{n,t}| \leq \left(\operatorname{Max}_{0 \leq u \leq n-1} |p_u| \right) \sum_{k=t}^{n-1} \binom{n}{n+t-k} \left[\sum_{s=0}^{k} \binom{n+s-1}{s} \right] +$$

$$+ \left(\operatorname{Max}_{0 \leq v \leq n-1} |q_v| \right) \sum_{k=t}^{n-1} \binom{k}{t} \left[\sum_{s=0}^{k} \binom{n+s-1}{s} \right].$$

As however

$$\sum_{s=0}^{k} \binom{n+s-1}{s} = \binom{n+k}{k} < 2^{n+k}$$

we obtain

$$\sum_{k=t}^{n-1} \binom{n}{n+t-k} \left[\sum_{s=0}^{k} \binom{n+s-1}{s} \right] < 2^{n+t} \cdot 3^n < 12^n$$

further

$$\sum_{k=t}^{n-1} \binom{k}{t} \left[\sum_{s=0}^{k} \binom{n+s-1}{s} \right] < 8^n.$$

Thus we obtain

$$|Q_{n,t}| < 12^n \left(\operatorname{Max}_{0 \leq u \leq n-1} |p_u| + \operatorname{Max}_{0 \leq v \leq n-1} |q_v| \right) \qquad \text{for} \qquad t = 0, 1, \ldots, n-1.$$

Thus we have proved the following

LEMMA 1. *Let $Q_n(z)$ $(n=1, 2, \ldots)$ be the unique polynomial of degree $2n-1$ determined by the conditions* (2.1) *and* (2.2). *Then we have*

$$\operatorname{Max}_{|z|=\varrho} \frac{|Q_n^{(n)}(z)|}{n!} < 12^n \left(\operatorname{Max}_{0 \leq u \leq n-1} |p_u| + \operatorname{Max}_{0 \leq v \leq n-1} |q_v| \right) \sum_{t=0}^{n-1} \binom{n+t}{n} \varrho^t.$$

§ 3. Proof of Theorems 1–4

We can suppose without the restriction of generality, that $f(z)$ is real for real z, i. e. that all the coefficients a_n (and thus all the b_n) are real. As a matter of fact suppose that Theorem 1 is already proved for such functions. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an arbitrary entire function, put $f_1(z) = \sum_{n=0}^{\infty} a_n' z^n$ resp. $f_2(z) = \sum_{n=0}^{\infty} a_n z^n$ where a_n' and a_n'' denotes the real and imaginary part of a_n respectively ($n=0, 1, \ldots$). Clearly

$$\mu_f(r) \leq \mu_{f_1}(r) + \mu_{f_2}(r)$$

thus if $f(z)$ satisfies (1.1), at least one of the functions $f_1(z)$ and $f_2(z)$ satisfies it too. Suppose e. g. that $f_1(z)$ satisfies (1.1) i. e.

$$\liminf_{r\to\infty} \frac{\mu_g(r^{1/\delta})}{\mu_{f_1}(r)} < \infty.$$

Put $f(z) = \sum_{n=0}^{\infty} b_n(z-1)^n$, further $b_n = b_n' + ib_n''$; now if $|a_n| \leq |c_n|$ then $|a_n'| \leq |c_n|$, and similarly if $|b_n| \leq |c_n|$ then $|b_n'| \leq |c_n|$. Thus if $S_0'(n)$ resp. $S_1'(n)$ denotes the number of indices $k < n$ for which $|a_n'| \leq |c_n|$ resp. $|b_n'| \leq |c_n|$ then $S_0(n) \leq S_0'(n)$ and $S_1(n) \leq S_1'(n)$. As by supposition the statement of Theorem 1 is valid for $f_1(z)$, it follows that it is valid for $f(z)$ too. Therefore in what follows we shall suppose that $f(z)$ is real on the real axis, i. e. that the coefficients a_n and b_n are all real numbers.

To prove Theorem 1 we shall need the following Lemma A, which has been used in proving Theorem A in [1].

LEMMA A. *If $h(x)$ is a real function which has n continuous derivatives in the closed interval* $[0, 1]$ *and if $N(n)$ denotes the number of zeros of $h^{(n)}(x)$ in the interval* $[0, 1]$ *further $Z_0(n, h)$ and $Z_1(n, h)$ denotes the number of zeros in the sequence $h(0), h'(0), \ldots, h^{(n-1)}(0)$ and in the sequence $h(1), h'(1), \ldots, h^{(n-1)}(1)$ respectively, then we have*

$$Z_0(n, h) + Z_1(n, h) - n \leq N(n).$$

We shall apply Lemma A to the function

$$h_n(z) = f(z) - Q_n(z) \qquad (n = 1, 2, \ldots) \tag{3.1}$$

where $Q_n(z)$ is the unique polynomial of order $2n-1$ defined by the following conditions:

$$\frac{Q_n^{(j)}(0)}{j!} = \begin{cases} 0 & \text{for } 0 \leq j < \delta' n \\ a_j & \text{if } \delta' n \leq j \leq n-1 \text{ and } |a_j| \leq |c_j| \\ 0 & \text{if } \delta' n \leq j \leq n-1 \text{ and } |a_j| > |c_j| \end{cases} \tag{3.2a}$$

$$\frac{Q_n^{(j)}(1)}{j!} = \begin{cases} 0 & \text{for } 0 \leq j < \delta' n \\ b_j & \text{if } \delta' n \leq j \leq n-1 \text{ and } |b_j| \leq |c_j| \\ 0 & \text{if } \delta' n \leq j \leq n-1 \text{ and } |b_j| > |c_j|. \end{cases} \tag{3.2b}$$

Here δ' is an arbitrary number such that $\delta < \delta' < 1$.

Clearly if $h_n(z)$ is defined by (3.1) and $Q_n(z)$ by (3.2a–b) we have

$$S_0(n) \leq Z_0(n, h_n) + \delta' n \tag{3.3}$$

and

$$S_1(n) \leq Z_1(n, h_n) + \delta' n. \tag{3.4}$$

Thus it follows by Lemma A that denoting by $N_n(n)$ the number of zeros of $h_n^{(n)}(z)$ in the interval [0, 1], we have

$$S_0(n) + S_1(n) \leq N_n(n) + n(1 + 2\delta'). \tag{3.5}$$

Now we shall need *Jensen*'s theorem which we shall use in the following form (see [4], p. 127, formula (13)):

LEMMA B. *Let $H(z)$ be an analytic function which is regular in a circle $|z| \leq R$ $(R > 1)$ and $H(o) \neq 0$. Let $\mathcal{N}_0$ denote the number of zeros of $H(z)$ in the (closed) unit circle, and choose a number ϱ with $1 < \varrho < R$. Then*

$$\mathcal{N}_0 \leq \frac{\operatorname*{Max}_{|z|=\varrho} \log \left| \dfrac{H(z)}{H(0)} \right|}{\log \varrho}. \tag{3.6}$$

It follows from (3.5) and (3.6) that if $h_n^{(n)}(o) \neq 0$, then for any $\varrho > 1$ we have

$$\frac{S_0(n) + S_1(n)}{n} \leq \frac{\operatorname*{Max}_{|z|=\varrho} \log \left| \dfrac{h_n^{(n)}(z)}{h_n^{(n)}(0)} \right|}{n \log \varrho} + (1 + 2\delta'). \tag{3.7}$$

Now let us choose a sequence r_s of positive numbers, such that $\lim\limits_{s\to\infty} r_s = \infty$ and

$$\lim_{s\to\infty} \frac{\mu_g\left(r_s^{\frac{1}{\delta}}\right)}{\mu_f(r_s)} = G < \infty.$$

Such a sequence r_s exists by supposition. Let n_s denote the index of the maximal term of the series $\sum\limits_{n=0}^{\infty} a_n z^n$ for $|z| = r_s$, i. e. suppose

$$|a_{n_s}| r_s^{n_s} \geq |a_m| r_s^m \qquad \text{for} \qquad m = 0, 1, \ldots \tag{3.8}$$

It follows by Lemma 1 that

$$\left| \frac{Q_{n_s}^{(n_s)}(0)}{f^{(n_s)}(0)} \right| \leq \frac{2 \operatorname*{Max}_{t \geq \delta' n_s} |c_t|}{|a_{n_s}|} 12^{n_s} \leq C_1 (12 \lambda_{n_s})^{n_s} \tag{3.9}$$

where $C_1 > 0$ is a constant and

$$\lambda_{n_s} = \frac{1}{r_s^{\frac{\delta' - \delta}{\delta}}}.$$

As clearly

(3.10) $$\lim_{s\to\infty} \lambda_{n_s} = 0$$

if follows that

(3.11) $$\lim_{s\to\infty} \left| \frac{Q_{n_s}^{(n_s)}(0)}{f^{(n_s)}(0)} \right| = 0$$

and thus also $h_{n_s}^{(n_s)}(0) \neq 0$ for sufficiently large s.

By Lemma 1 we have also for $|z| = \varrho > 1$

$$\left| \frac{Q_{n_s}^{(n_s)}(z)}{f^{(n_s)}(0)} \right| \leq \frac{2\left(\operatorname{Max}_{t \geq \delta' n} |c_t|\right) 12^{n_s} \sum_{j=0}^{n_s-1} \binom{n_s+j}{j} \varrho^j}{|a_{n_s}|}.$$

Similarly as in (3.9) we obtain

$$\operatorname{Max}_{|z|=\varrho} \left| \frac{Q_{n_s}^{(n_s)}(z)}{f^{(n_s)}(0)} \right| \leq C_2 (12\lambda_{n_s})^{n_s} \sum_{j=0}^{n_s-1} \binom{n_s+j}{j} \varrho^j.$$

where $C_2 > 0$ is a constant.

Now if s is sufficiently large, we have $12\lambda_{n_s}\varrho < 1$ and therefore

$$\operatorname{Max}_{|z|=\varrho} \left| \frac{Q_{n_s}^{(n_s)}(z)}{f^{(n_s)}(0)} \right| \leq \frac{C_2}{(1-12\lambda_{n_s}\varrho)^{n_s+1}}.$$

It follows by (3.10) that

(3.12) $$\lim_{s\to\infty} \frac{\operatorname{Max}_{|z|=\varrho} \log \left| \frac{Q_{n_s}^{(n_s)}(z)}{f^{(n_s)}(0)} \right|}{n_s} = 0.$$

Finally we have for $|z| = \varrho$ by (3.8)

$$\left| \frac{f^{(n_s)}(z)}{f^{(n_s)}(0)} \right| \leq \sum_{l=0}^{\infty} \left| \frac{a_{n_s+l}}{a_{n_s}} \right| \binom{n_s+l}{l} \varrho^l \leq \frac{1}{\left(1 - \frac{\varrho}{r_s}\right)^{n_s+1}}$$

and thus

(3.13) $$\lim_{s\to\infty} \frac{\operatorname{Max}_{|z|=\varrho} \log \left| \frac{f^{(n_s)}(z)}{f^{(n_s)}(0)} \right|}{n_s} = 0.$$

As we have evidently

(3.14) $$\operatorname{Max}_{|z|=\varrho} \left| \frac{h_{n_s}^{(n_s)}(z)}{h_{n_s}^{(n_s)}(0)} \right| \leq \frac{\operatorname{Max}_{|z|=\varrho} \left| \frac{f^{(n_s)}(z)}{f^{(n_s)}(0)} \right| + \operatorname{Max}_{|z|=\varrho} \left| \frac{Q_{n_s}^{(n_s)}(z)}{f^{(n_s)}(0)} \right|}{1 - \left| \frac{Q_{n_s}^{(n_s)}(0)}{f^{(n_s)}(0)} \right|}$$

comparing (3.11), (3.12), (3.13) and (3.14) it follows that

$$(3.15) \qquad \lim_{s\to\infty} \frac{\operatorname{Max}_{|z|=\varrho} \log \left| \dfrac{h_{n_s}^{(n_s)}(z)}{h_{n_s}^{(n_s)}(0)} \right|}{n_s} = 0.$$

Combining (3.7) with (3.15) we obtain

$$(3.16) \qquad \limsup_{s\to\infty} \frac{S_0(n_s)+S_1(n_s)}{n_s} \leq 1+2\delta'$$

and as (3.16) holds for any $\delta'>\delta$, we obtain finally

$$\liminf_{s\to\infty} \frac{S_0(n)+S_1(n)}{n} \leq 1+2\delta.$$

Thus Theorem 1 is proved.

New we prove the following

THEOREM 2. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n (z-1)^n$ *be a transcendental entire function and* $g(z)=\sum_{n=0}^{\infty} c_n z^n$ *an arbitrary entire function for which the suppositions of Theorem 1 hold. Let* $Z_0(n)$ *denote the number of zeros in the sequence* $a_0, a_1, \ldots, a_{n-1}$ *and* $S_1(n)$ *the number of indices* $k<n$ *for which* $|b_k|\leq|c_k|$. *Then*

$$\liminf_{n\to\infty} \frac{Z_0(n)+S_1(n)}{n} \leq 1+\delta.$$

The proof of Theorem 2 is almost the same as that of Theorem 1; as however we do not use now (3.3), only (3.4), we get δ instead of 2δ.

The following result contains each of Theorem 1 and Theorem 2 as special cases.

THEOREM 3. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n (z-1)^n$ *be a transcendental entire function, further* $g_1(z)=\sum_{n=0}^{\infty} c_n z^n$ *and* $g_2(z)=\sum_{n=0}^{\infty} d_n z^n$ *be entire functions, such that putting*

$$\mu_{g_1}(r) = \operatorname{Max}_n |c_n| r^n$$

and

$$\mu_{g_2}(r) = \operatorname{Max}_n |d_n| r^n$$

we have for some δ_1 *and* δ_2 $(\delta_1>0, \delta_2>0, \delta_1+\delta_2<1)$

$$\liminf_{r\to\infty} \frac{\mu_{g_1}(r^{1/\delta_1})+\mu_{g_2}(r^{1/\delta_2})}{\mu_f(r)} < \infty.$$

Let $S_0(n)$ *denote the number of indices* $k < n$ *for which* $|a_k| \le |c_k|$ *and* $S_1(n)$ *the number of indices* $k < n$ *for which* $|b_k| \le |c_k|$. *Then*

$$\liminf_{n\to\infty} \frac{S_0(n)+S_1(n)}{n} \le 1+\delta_1+\delta_2 .$$

The proof of Theorem 3 is essentially the same as that of Theorem 1; the only differences consist in that now we define the polynomial $Q_n(z)$ of order $2n-1$ $(n=1, 2, \ldots)$ as follows:

$$\frac{Q_n^{(j)}(0)}{j!} = \begin{cases} 0 & \text{for } 0 \le j < \delta_1' n \\ a_j & \text{if } \delta_1' n \le j \le n-1 \text{ and } |a_j| \le |c_j| \\ 0 & \text{if } \delta_1' n \le j \le n-1 \text{ and } |a_j| > |c_j| \end{cases}$$

$$\frac{Q_n^{(j)}(1)}{j!} = \begin{cases} 0 & \text{for } 0 \le j < \delta_2' n \\ b_j & \text{if } \delta_2' \le j \le n-1 \text{ and } |b_j| \le |d_j| \\ 0 & \text{if } \delta_2' n \le j \le n-1 \text{ and } |b_j| > |d_j|. \end{cases}$$

Here δ_1 is an arbitrary number for which $\delta_1 < \delta_1' < 1$ and δ_2' an arbitrary number for which $\delta_2 < \delta_2' < 1$.

Evidently if we choose $g_1(z) = g_2(z) = g(z)$, then Theorem 3 reduces to Theorem 1, while if $g_1(z) \equiv 0, g_2(z) = g(z)$, then Theorem 3 reduces to Theorem 2.

Now we shall prove the following Theorem 4, which gives a sharper inequality than (1.4).

THEOREM 4. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n (z-1)^n$ *is an entire function of order* $\varrho_f (0 < \varrho_f \le \infty)$ *and* $g(z) = \sum_{n=0}^{\infty} c_n z^n$ *an entire function of order* ϱ_g *where* $0 \le \varrho_g < \varrho_f$, *and if* $S_0(n)$ *denotes the number of indices* $k < n$ *for which* $|a_k| \le |c_k|$ *and* $S_1(n)$ *the number of indices* $k<n$ *for which* $|b_k| \le |c_k|$ *then*

$$\liminf_{n\to\infty} \frac{S_0(n)+S_1(n)}{n} \le 1 + \frac{\varrho_g}{\varrho_f}.$$

PROOF OF THEOREM 4. Let us put

$$\varphi(z) = \sum_{|a_n| \le |c_n|} a_n z^n$$

and

$$f^*(z) = f(z) - \varphi(z) = \sum_{|a_n| > |c_n|} a_n z^n = \sum_{n=0}^{\infty} a_n^* z^n = \sum_{n=0}^{\infty} b_n^* (z-1)^n.$$

Then clearly $f^*(z)$ has the same order ϱ_f as $f(z)$ further $\varphi(z)$ is at most of order ϱ_g. Let $Z_0^*(n)$ denote the number of zeros in the sequence $a_0^*, a_1^*, \ldots, a_{n-1}^*$, then clearly

$$Z_0^*(n) = S_0(n) \tag{3.17}$$

We can without restricting the generality suppose that $c_n \geq 0$ as evidently $\sum_{n=0}^{\infty}|c_n|z^n$ is an entire function of the same order as $g(z)$ [This follows e. g. by (1.3)]. Put

$$g^*(z) = 2\sum_{n=0}^{\infty} z^n \left[\sum_{m=n}^{\infty} \binom{m}{n} c_m\right] = \sum_{n=0}^{\infty} c_n^* z^n.$$

It is easy to see that $g^*(z)$ is also an entire function of order ϱ_g. As a matter of fact, one has

$$g^*(z) = 2g(z+1).$$

Now as

$$f^*(z) = \sum_{n=0}^{\infty}\Bigg[b_n - \sum_{\substack{m=n \\ |a_m| \leq |c_m|}}^{\infty} \binom{m}{n} a_m\Bigg](z-1)^n$$

we obtain

$$|b_k^*| \leq |b_k| + \sum_{m=k}^{\infty} \binom{m}{k} c_k \qquad (k = 0, 1, \ldots).$$

Thus if $|b_k| \leq |c_k|$ we have $|b_k^*| \leq |c_k^*|$. It follows that if $S_1^*(n)$ denotes the number of indices $k<n$ for which $|b_k^*| \leq |c_k^*|$, then

$$S_1(n) \leq S_1^*(n). \tag{3.18}$$

Now we can apply Theorem 2 to the function $f^*(z)$ of order ϱ_f and the function $g^*(z)$ of order ϱ_g with any $\delta > \frac{\varrho_g}{\varrho_f}$ and thus we obtain, taking (3.17) and (3.18) into account

$$\liminf_{n\to\infty} \frac{S_0(n)+S_1(n)}{n} \leq \liminf_{n\to\infty} \frac{Z_0^*(n)+S_1^*(n)}{n} \leq 1 + \frac{\varrho_g}{\varrho_f}$$

Thus Theorem 4 is proved.

§ 4. An example

We shall discuss now an example, which shows that Theorem 4 is in a certain sense best possible.

Let us consider the function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{a^n}}{(a^n)!}$$

where $a > 1$ is an integer. Clearly $f(z)$ is an entire transcendental function of order 1. Let us choose for $g(z)$ the function

$$g(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/\vartheta}}$$

where $\frac{1}{a} < \vartheta < 1$. It is easy to see that $g(z)$ is an entire function of order ϑ. We consider now the power series of $f(z)$ around the point $z = 1$; we have

$$f(z) = \sum_{n=0}^{\infty} \frac{[1+(z-1)]^{a^n}}{(a^n)!} = \sum_{m=0}^{\infty} b_m(z-1)^m$$

where

$$b_m = \frac{1}{m!} \sum_{a^n \geq m} \frac{1}{(a^n-m)!}.$$

Let us compare now b_m with

$$c_m = \frac{1}{(m!)^{1/\vartheta}}$$

for a value of m which lies in the interval

$$a^{n-1} \leq m < \frac{a^n \vartheta}{1+\varepsilon} \qquad (n = 1, 2, \ldots)$$

where ε is an arbitrary positive number.

We have clearly for any m satisfying (4.1)

$$b_m \leq \frac{2}{m!\left[\left(\frac{1+\varepsilon}{\vartheta}-1\right)m\right]!}.$$

It follows by *Stirling*'s formula that if $n \geq n_0$ where n_0 is a suitably chosen large number and if m satisfies (4.1) we have

$$b_m \leq c_m.$$

It follows that

$$\liminf_{n\to\infty} \frac{S_0(n)+S_1(n)}{n} \geq 1 + \frac{\frac{\vartheta}{1+\varepsilon} - \frac{1}{a}}{1-\frac{1}{a}}. \tag{4.2}$$

As for ε we can choose an arbitrary small positive number and a can be chosen arbitrarily large, it follows that the lower bound in (4.2) can be arbitrarily near to $1+\vartheta$. As however by Theorem 4

$$\liminf_{n\to\infty} \frac{S_0(n)+S_1(n)}{n} \leqq 1+\vartheta \tag{4.3}$$

it follows that Tehorem 4 (and thus also Theorem 2) is best possible in the sense that $1+\vartheta$ can not be replaced in general by any smaller number in (4.3).

Literature

[1] C. Rényi, On a conjecture of G. Pólya, *Acta Math. Acad. Sci. Hung.*, **7** (1956), 145 – 150.
[2] G. Pólya und G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, Vol. 2., Berlin, Springer 1925.
[3] P. Johansen, Über osculierende Interpolation, *Skandinavisk Aktuarietidskrift*, 1931.
[4] P. Erdős and A. Rényi, On the number of zeros of successive derivatives of analytic functions, *Acta Math. Acad. Sci. Hung.*, **7** (1956), 125 – 144.