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ON STABLE SEQUENCES OF EVENTS

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SUMMARY. A sequence $\{A_n\}$ of events is called a stable sequence if for every event B the limit $\lim_{n \to +\infty} P(A_n B) = Q(B)$ exists. It is shown that in this case Q is a bounded measure which is absolutely continuous with respect to the underlying probability measure P. The Radon-Nikodym derivative $\frac{dQ}{dP} = \alpha$ is called the local density of the stable sequence $\{A_n\}$. Criteria for a sequence of events being stable are given, further examples of stable sequences are discussed The notion of a stable sequence of events generalizes the notion of a mixing sequence of events, introduced in a previous paper of the author. A stable sequence is mixing if its local density is constant almost everywhere.

1. Introduction

Let $[\Omega, \mathcal{A}, P]$ be a probability space in the sense of Kolmogoroff, i.e. let Ω be an arbitrary set whose elements shall be denoted by ω and called elementary events, $\mathcal{A}$ a σ-algebra of subsets of Ω whose elements will be denoted by capital letters A, B etc., and called random events or simply events and $P = P(A)$ a measure, i.e. a non-negative and σ-additive set function defined on $\mathcal{A}$ and normed by the condition $P(\Omega) = 1$; $P(A)$ will be called the probability of the event $A \epsilon \mathcal{A}$.

We shall denote by ϕ the empty set which represents the impossible event, further by $A+B$ the union and by $A.B$ the intersection of the sets A and B. If $A_n (n = 1, 2, \ldots)$ is a sequence of sets, we shall use also the notation $\sum_n A_n$ for the union of the sets A_n. We denote by $A-B$ the set of elements which belong to A but not to B and put $\Omega - A = \bar{A}$. If A and B are arbitrary events such that $P(B) > 0$, we shall denote by $P(A \mid B)$ the conditional probability of the event A with respect to the condition B, i.e. we put

$$P(A \mid B) = \frac{P(AB)}{P(B)}.$$

We shall denote by $a \epsilon A$ that a is an element of the set A and by $A \subset B$ that the set A is a subset of the set B.

As usual a real function $\xi = \xi(\omega)$ defined on Ω is called a random variable if it is measurable with respect to $\mathcal{A}$, that is, if denoting by $\xi^{-1}(I)$ the set of those $\omega \epsilon \Omega$ for which $\xi(\omega) \epsilon I$, then $\xi^{-1}(I)$ belongs to $\mathcal{A}$ if I is an arbitrary interval of the real line.

We denote by $E(\xi)$ the mean value (expectation) of the random variable ξ, i.e. we put $E(\xi) = \int_\Omega \xi dP$.

The infinite sequence of events $A_1, A_2, \ldots, A_n, \ldots$, i.e. of subsets of Ω belonging to $\mathcal{A}$ will be called a *stable* sequence, if the limit

$$\lim_{n \to +\infty} P(A_n B) = Q(B) \qquad \ldots \quad (1.1)$$

exists for every $B \epsilon \mathcal{A}$. We shall show that in this case $Q(B)$ is a bounded measure on $\mathcal{A}$, which is absolutely continuous with respect to the measure P, and thus

$$Q(B) = \int_B \alpha dP \quad \ldots \quad (1.2)$$

for every $B \epsilon \mathcal{A}$ where $\alpha = \alpha(\omega)$ is a measurable function on Ω such that $0 \leqslant \alpha(\omega) \leqslant 1$. We shall call $\alpha(\omega)$ the *local density* of the stable sequence of events $\{A_n\}$.

As well known $\alpha(\omega)$ is not uniquely determined, but if (1.2) holds both with $\alpha = \alpha_1(\omega)$ and $\alpha = \alpha_2(\omega)$ then $\alpha_1(\omega)$ and $\alpha_2(\omega)$ are almost everywhere equal to another.

In the special case when the local density is constant, i.e. $\alpha(\omega) \equiv \alpha$, then $Q(B) = \alpha P(B)$ for every $B \epsilon \mathcal{A}$, i.e. in this case

$$\lim_{n \to +\infty} P(A_n B) = \alpha P(B). \quad \ldots \quad (1.3)$$

Sequences $\{A_n\}$ for which (1.3) holds have been considered already in a previous paper (Rényi, 1958) and have been called strongly *mixing* sequences of events with density α. Thus the notion of a stable sequence of events is a generalization of the notion of a mixing sequence.

The definition of a stable sequence of events can be formulated also in the following equivalent form: The sequence of events $\{A_n\}$ $(n = 1, 2, \ldots)$ is called stable if for every event $B \epsilon \mathcal{A}$ such that $P(B) > 0$ the conditional probability $P(A_n | B)$ tends to a limit, i.e.

$$\lim_{n \to +\infty} P(A_n | B) = q(B) \quad \ldots \quad (1.4)$$

exists. Clearly, if $P(B) > 0$ then (1.1) and (1.4) with $q(B) = \dfrac{Q(B)}{P(B)}$ are equivalent, while if $P(B) = 0$ then (1.1) holds with $Q(B) = 0$ for any sequence $\{A_n\}$.

We shall show that stable sequences of events can be simply characterized in terms of Hilbert space theory. Let H denote the Hilbert space of all random variables ξ, defined on the probability space $[\Omega, \mathcal{A}, P]$, for which $E(\xi^2)$ is finite, the inner product (ξ, η) being defined by $(\xi, \eta) = E(\xi . \eta)$. Let $\alpha_n = \alpha_n(\omega)$ denote the indicator of the set A_n, i.e. $\alpha_n(\omega) = 1$ for $\omega \epsilon A_n$ and $\alpha_n(\omega) = 0$ for $\omega \epsilon \bar{A}_n$. Then the sequence $\{A_n\}$ of events is stable if and only if the sequence α_n converges weakly; the weak limit of the sequence α_n being equal to the local density of the sequence $\{A_n\}$. It follows that the sequence $\{A_n\}$ is mixing if and only if α_n converges weakly to a constant.

We introduce further the notion of a stable sequence of random variables. The sequence of random variables $\xi_n = \xi_n(\omega)$ $(n = 1, 2, \ldots)$ will be called *stable* if for any event B with $P(B) > 0$ the conditional distribution of ξ_n with respect to B tends to a limiting distribution, i.e.

$$\lim_{n \to +\infty} P(\xi_n < x | B) = F_B(x) \quad \ldots \quad (1.5)$$

for every x which is a continuity point of the distribution function $F_B(x)$.

Expressed in terms of Hilbert space theory this means that for every bounded and continuous function $g(x)$ the sequence $g(\xi_n)$ converges weakly.

An other equivalent definition of a stable sequence of random variables is the following : the sequence of random variables $\xi_n (n = 1, 2, \ldots)$ is called *stable*, if for every $x \epsilon X$ where X is a set of real numbers which is everywhere dense on the real line, the sequence of events $\xi_n < x$ $(n = 1, 2, \ldots)$ is stable.

Clearly such a sequence $\{\xi_n\}$ of random variables is stable in the sense of (1.5), because if (1.5) holds for x belonging to an everywhere dense set X then it holds for every continuity point x of $F_B(x)$. On the other hand (1.5) implies the stability of the sequence of events $\xi_n < x$ for $x \epsilon X$ where the set X is everywhere dense on the real line.

In the special case when the limiting distribution $F_B(x)$ does not depend on the choice of B we arrive at the notion of a strongly mixing sequence of random variables, introduced previously (Rényi and Révész, 1958).

The aim of the present paper is to study general properties of stable sequences of events and to give criteria for the stability of a sequence of events which are discussed in Section 2; some examples and applications of these notions in probability theory are discussed in Section 3.

2. Some general theorems on stable sequences of events

Let $\alpha_n = \alpha_n(\omega)$ $(n = 1, 2, \ldots)$ be the indicator of the set A_n, i.e. $\alpha_n(\omega) = 1$ if $\omega \epsilon A_n$ and $\alpha_n(\omega) = 0$ if $\omega \epsilon \bar{A}_n$.

Let H denote the Hilbert space of all random variables ξ for which $E(\xi^2)$ exists, the inner product (ξ, η) being defined by $(\xi, \eta) = E(\xi \, . \, \eta)$. We put further $||\xi|| = (\xi, \xi)^{1/2}$. All definitions and theorems from Hilbert space theory which will be needed in the sequel can be found, e.g., in Szőkefalvi-Nagy (1942).

We prove first the following theorem.

Theorem 1 : *$\{A_n\}$ is a stable sequence of events, i.e. the limit*

$$\lim_{n \to +\infty} P(A_n B) = Q(B) \qquad \ldots \quad (2.1)$$

exists for every $B \epsilon \mathcal{A}$, if and only if $\{\alpha_n\}$ is a weakly convergent sequence of elements of the Hilbert space H, i.e. if for any $\eta \epsilon H$ the limit

$$\lim_{n \to +\infty} (\alpha_n, \eta) = A(\eta) \qquad \ldots \quad (2.2)$$

exists.

Proof of Theorem 1 : Clearly, if β is the indicator of the set B then $(\alpha_n, \beta) = P(A_n B)$. Thus (2.2) reduces to (2.1) if we substitute β instead of η. Thus to prove Theorem 1 it suffices to show that if the limit (2.2) exists whenever η is the indicator of a set B then it exists for every $\eta \epsilon H$. Clearly, if the limit (2.2) exists for every indicator η, it exists also if η is an arbitrary element of H which takes on only a finite number of values. As to every random variable η for which $E(|\eta|) < +\infty$ and to every $\varepsilon > 0$ one can find, by the definition of the Lebesgue integral (Halmos, 1950) a random variable η_1 which takes on only a finite number of values, such that $E(|\eta - \eta_1|) < \varepsilon$, it follows easily that (2.2) holds not only for every $\eta \epsilon H$ but also for every η for which $E(\eta)$ is finite. Thus Theorem 1 is proved.

As a consequence of Theorem 1 we obtain the following theorem.

Theorem 2 : *If $\{A_n\}$ is a stable sequence of events, i.e. if*

$$\lim_{n\to+\infty} P(A_nB) = Q(B) \qquad \dots \quad (2.3)$$

exists for every $B\in\mathcal{A}$, then $Q(B)$ is a measure on $\mathcal{A}$ which is absolutely continuous with respect to the measure P, and thus can be represented in the form

$$Q(B) = \int_B \alpha dP \qquad \dots \quad (2.4)$$

where $\alpha = \alpha(\omega)$ is a random variable; we have further $0 \leqslant \alpha \leqslant 1$.

Proof of Theorem 2 : Clearly $A(\eta)$ defined by (2.2) is a bounded linear operation on H, and thus by a well-known theorem [Szőkefalvi-Nagy (1942)] there exists an $\alpha\in H$ such that $A(\eta)$ can be represented in the form $A(\eta) = (\alpha, \eta)$. It is easy to see that $0 \leqslant \alpha \leqslant 1$. It follows that, denoting by β the indicator of the event B, we have

$$Q(B) = (\alpha, \beta) = \int_B \alpha dP. \qquad \dots \quad (2.5)$$

Thus $Q(B)$ is a measure, which is absolutely continuous with respect to the measure $P(B)$. We shall call $\alpha = \alpha(\omega)$ the *local density* of the stable sequence $\{A_n\}$.

Now we shall prove a criterion of the stability of a sequence of events which is the generalization of a corresponding criterion for mixing sequences, proved by the author (Rényi, 1958).

Theorem 3 : *Let $\{A_n\}$ $(n = 1, 2, \dots)$ be a sequence of events such that the limit*

$$\lim_{n\to+\infty} P(A_n A_k) = Q_k \qquad \dots \quad (2.6)$$

exists for $k = 1, 2, \dots$. Then the sequence $\{A_n\}$ is stable, i.e. (2.1) *holds for every $B\in\mathcal{A}$.*

Proof of Theorem 3 : Let H_1 denote the subspace of H spanned by the sequence $\{\alpha_n\}$ where α_n is the indicator of the event $A_n (n = 1, 2, \dots)$, i.e. the closure with respect to the distance $\|\xi-\eta\|$ of the set of all finite linear combinations $\sum_{\kappa=1} c_\kappa\alpha_\kappa$ where $c_1, c_2, \dots, c_n$ are arbitrary real numbers. Let H_2 denote the set of those elements ξ_2 of H which are orthogonal to every $\xi_1\in H_1$.

According to a well-known theorem (see Szőkefalvi-Nagy, 1942, p. 8) each element ξ of H can be represented in the form $\xi = \xi_1+\xi_2$ where $\xi_1\in H_1$ and $\xi_2\in H_2$. Now we shall prove that if the limit (2.6) exists for $k = 1, 2, \dots$ then the limit

$$\lim_{n\to+\infty} (\alpha_n, \xi) = A(\xi) \qquad \dots \quad (2.7)$$

exists for every $\xi\in H$. To prove this it suffices to show that (2.7) exists if $\xi = \xi_1\in H_1$, because of the above mentioned decomposition of every $\xi\in H$ into the sum of a $\xi_1\in H_1$ and a $\xi_2\in H_2$; as a matter of fact if $\xi = \xi_2\in H_2$ then (2.7) holds with $A(\xi_2) = 0$ while if the limit (2.7) exists for $\xi = \xi_1$ and $\xi = \xi_2$ it clearly exists for $\xi = \xi_1+\xi_2$ also.

Now if ξ_1 is a linear combination of a finite number of the α_k's then clearly the limit (2.7) exists. Let now ξ_1 be an arbitrary element of H_1. Then to every $\varepsilon > 0$ one can find a finite linear combination $\sum_{\kappa=1}^{N} c_\kappa\alpha_\kappa$ such that

$$\| \xi_1 - \sum_{\kappa=1}^{N} c_\kappa\alpha_\kappa \| < \varepsilon. \qquad \dots \quad (2.8)$$

But (2.8) implies in view of $\|\alpha_n\| \leqslant 1$ and the inequality $|(\xi, \eta)| \leqslant \|\xi\| . \|\eta\|$, that

$$|(\alpha_n, \xi_1) - \sum_{k=1}^{N} c_k(\alpha_n, \alpha_k)| \leqslant \varepsilon. \qquad \dots \quad (2.9)$$

Thus it follows that

$$|\overline{\lim_{n\to+\infty}} (\alpha_n, \xi_1) - \underline{\lim_{n\to+\infty}} (\alpha_n, \xi_1)| \leqslant 2\varepsilon. \qquad \dots \quad (2.10)$$

As $\varepsilon > 0$ is arbitrary it follows from (2.10) that the limit (2.7) exists. Thus Theorem 3 is proved.

Let us put (supposing $P(A_k) > 0$ for $k = 1, 2, \dots$, which is no essential restriction)

$$q_k = \frac{Q_k}{P(A_k)} \; (k = 1, 2, \dots) \qquad \dots \quad (2.11)$$

where Q_k is defined by (2.6).

Note that even if all the numbers q_k ($k = 1, 2, \dots$) are equal, it is not sure that the sequence $\{A_n\}$ is mixing. See for instance examples 1 and 2 of Section 3. This is true, however, in case $A_1 = \Omega$ as it has been shown by the author (Rényi, 1958).

Another way to express this fact is contained in Theorem 4.

Theorem 4 : *Let $\{A_n\}$ be a stable sequence of events such that* $\lim_{n\to+\infty} P(A_n) = q_0$ *and* $\lim_{n\to+\infty} P(A_nA_k) = q_kP(A_k)$ ($k = 1, 2, \dots$). *Then the sequence $\{A_n\}$ is mixing if and only if the numbers q_k ($k = 0, 1, \dots$) are all equal to another.*

Proof of Theorem 4 : The necessity of the condition follows immediately from the definition of mixing sequences.

The sufficiency can be proved as follows : Let α denote the local density of the sequence $\{A_n\}$. If $q_k = q \neq 0 (k = 0, 1, \dots)$ then clearly

$$(\alpha, \alpha_k) = q(1, \alpha_k) \qquad \dots \quad (2.12)$$

which can be also expressed as follows :

$$(\alpha, \alpha_k) = qP(A_k). \qquad \dots \quad (2.13)$$

It follows by passing to the limit from (2.12) that

$$(\alpha, \alpha) = q(1, \alpha) \qquad \dots \quad (2.14)$$

and from (2.13) that $$(\alpha, \alpha) = q^2. \qquad \dots \quad (2.15)$$

It follows that $$q = (1, \alpha) \qquad \dots \quad (2.16)$$

and therefore that $$(\alpha, \alpha) = (1, \alpha)^2 \qquad \dots \quad (2.17)$$

i.e., $$\int_\Omega \alpha^2 dP = (\int_\Omega \alpha dP)^2. \qquad \dots \quad (2.18)$$

This implies that α is constant almost everywhere and thus by (2.15)

$$\alpha \equiv q. \qquad \dots \quad (2.19)$$

The case $q = 0$ is trivial.

We should like to add the following remark:

In view of Theorem 1, Theorem 3 is a special case of the following.

Theorem : *A bounded sequence $\{\alpha_n\}$ ($n = 1, 2, \dots$) of elements of a Hilbert space H is weakly convergent if and only if the limits* $\lim_{n\to+\infty} (\alpha_n, \alpha_k)$ *exist for* $k = 1, 2, \dots$.

The proof of this assertion is exactly the same as that of Theorem 3. This useful theorem, which is due to E. Schmidt (see Schmeidler, 1954) is well known in Hilbert space theory; e.g. the proof in Szőkefalvi-Nagy, (1942, p. 10) of the theorem that every bounded set is weakly compact, is based essentially on this fact.[1]

Let us add that if $\{A_n\}$ is a stable sequence of events and α_n the indicator of A_n, then the sequence α_n converges strongly in H only in the trivial case when the weak limit of α_n is almost everywhere equal either to 1 or to 0, e.g., in the case mentioned in Example 1. As a matter of fact a necessary and sufficient condition of the strong convergence of α_n to α is $\lim_{n\to+\infty} ||\alpha_n|| = ||\alpha||$. But

$$\lim_{n\to+\infty} ||\alpha_n||^2 = \lim_{n\to+\infty} (\alpha_n, 1) = \int_\Omega \alpha dP$$

and $||\alpha||^2 = \int_\Omega \alpha^2 dP$. Thus α_n is strongly convergent to α if and only if $\int_\Omega \alpha(1-\alpha)dP = 0$ i.e. if $\alpha(1-\alpha) = 0$ almost everywhere.

In view of Theorem 4 and of the well-known theorem of Hilbert space theory according to which every bounded set is weakly compact, the following theorem holds :

Theorem 5 : *Any sequence $\{A_n\}$ of events contains a subsequence which is stable.*

An interesting feature of the stability of a sequence of events is that unlike such properties as independence, equivalence etc., it remains invariant when the underlying measure is replaced by another which is absolutely continuous with respect to the original measure. Moreover the local density of a stable sequence of events remains also unchanged.

Theorem 6 : *Let $[\Omega, \mathcal{A}, P] = \mathcal{S}$ be a probability space and $\{A_n\}$ a stable sequence of events in $\mathcal{S}$. Let P^* be another probability measure on $\mathcal{A}$ which is absolutely continuous with respect to P. Then the sequence $\{A_n\}$ is stable on the probability space $\mathcal{S}^* = [\Omega, \mathcal{A}, P^*]$ also, with the same local density, i.e. if* (2.1) *and* (2.4) *hold, one has also*

$$\lim_{n\to+\infty} P^*(A_nB) = \int_B \alpha dP^*. \qquad \dots \quad (2.20)$$

Proof of Theorem 6 : By supposition

$$P^*(A) = \int_A \rho dP \qquad \text{for} \quad A \epsilon \mathcal{A} \qquad \dots \quad (2.21)$$

where $\rho = \rho(\omega)$ is a nonnegative random variable and $\int_\Omega \rho dP = 1$. It follows that

$$P^*(A_n, B) = \int_B \alpha_n \rho dP = (\alpha_n, \rho\beta) \qquad \dots \quad (2.22)$$

where β is the indicator of the event $B \epsilon \mathcal{A}$. Thus the existence of the limit (2.20) follows evidently from the remark made in course of proving Theorem 1 that the limit (2.2) exists not only if $\eta \epsilon H$ but also under the single condition that $E(\eta)$ exists and that we have in this case also $A(\eta) = (\eta, \alpha)$. Thus it follows that

$$\lim_{n\to+\infty} P^* (A_nB) = (\alpha, \rho\beta) = \int_B \alpha dP^*.$$

[1] Prof. B. Sz. Nagy has kindly called my attention to this proof.

This proves Theorem 6. Let us mention that for mixing sequences a more general result supposing only the semi-continuity of P^* with respect to P has been proved by Sucheston (1962).

We want to make some further general remarks. It is impossible, except in trivial cases that the convergence in (2.1) should be uniform in B for all $B \epsilon \mathcal{A}$. As a matter of fact, putting $B = A_n$ one has

$$P(A_nB)-Q(B) = P(A_n)-Q(A_n) = \int_{\Omega} (1-\alpha)\alpha_n dP$$

and this difference tends to $\int_{\Omega} \alpha(1-\alpha)dP$. Thus the convergence in (2.1) can be uniform only if the local density α is almost everywhere equal to 1 or 0 as in the trivial case of Example 1 in Section 3. Nevertheless the convergence in (2.1) may be uniform in B for $B \epsilon \mathcal{B}$ where $\mathcal{B}$ is some proper subset of $\mathcal{A}$ which does not contain the sets A_n themselves or only a finite number of them. For instance, if there exist events B which are independent from all the events A_n, then these may all be contained in $\mathcal{B}$. If B is such an event then clearly the indicator β of B is uncorrelated with the local density α of the sequence $\{A_n\}$.

3. Examples of stable sequences of events

Example 1 : A sequence of identical events $A_n = A (n = 1, 2, \ldots)$ is evidently stable. Note that in this case the local density $\alpha(\omega)$ is equal to 1 for $\omega \epsilon A$ and to 0 for $\omega \epsilon \bar{A}$. Let us mention that the sequence $A, A, \ldots$ is trivially mixing if $P(A) = 0$ or $P(A) = 1$ but not if $0 < P(A) < 1$.

More generally, if A'_n is a mixing sequence of events with density α, and A any event, the sequence $A_n = A'_n A$ is a stable sequence of events, with local density equal to α on A and 0 on $\bar{A}$.

Example 2 : Let $\{A_n\}$ be a sequence of equivalent events (called also symmetrically dependent events) i.e. suppose

$$P(A_{i_1} A_{i_2} \ldots A_{i_k}) = W_k \qquad \ldots \quad (3.1)$$

for any choice of the different indices $i_1 < i_2 < \ldots < i_k$ and for $k = 1, 2, \ldots$ where W_k depends on k only, but not on the choice of the indices $i_1, i_2, \ldots, i_k$. Such a sequence $\{A_n\}$ is evidently stable. As a matter of fact, it is sufficient to suppose the independence of W_k from the indices $i_1, i_2, \ldots, i_k$ for $k = 2$ only. This follows clearly from Theorem 3.

Thus sequences of equivalent events are always stable. Note that in view of Theorem 4 a sequence of equivalent events is mixing if and only if

$$W_2 = W_1^2. \qquad \ldots \quad (3.2)$$

It is easy to see however that (3.2) is satisfied if and only if the sequence $\{A_n\}$ is a sequence of independent events.

As a matter of fact according to a well-known theorem due to Khintchine (1952) if $\{A_n\}$ is a sequence of equivalent events and W_k is defined by (3.1) then there exists a distribution function $G(x)$ in the interval [0, 1] such that

$$W_k = \int_0^1 x^k dG(x). \qquad \ldots \quad (3.3)$$

As a matter of fact this follows from the following theorem of Hausdorff (1923). If a sequence $\{W_k\}$ is monotonic of every order, i.e. putting $W_0 = 1$

$$\sum_{j=1}^{k} (-1)^j \binom{k}{j} W_{n+j} \geqslant 0$$

for all $n \geqslant 0$ and $k \geqslant 0$ then W_k can be represented in the form (3.3). Now clearly $W_2 = W_1^2$ means by (3.3) that

$$\int_0^1 x^2 dG(x) = \left(\int_0^1 x dG(x)\right)^2$$

which implies evidently that $G(x)$ is the distribution function of a constant c, i.e.

$$G(x) = \begin{cases} 1 & \text{for} \quad x > c \\ 0 & \text{for} \quad x \leqslant c \end{cases}$$

where of course $c = W_1$; but then according to (3.3) $W_k = W_1^k$ that is

$$P(A_{i_1} A_{i_2} \ldots A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j})$$

for every sequence $i_1 < i_2 < \ldots < i_k$ and therefore the events are independent. Thus we have proved the following theorem.

Theorem 7 : *A sequence of equivalent events is always stable, but it is mixing if and only if the events are completely independent.*

Example 3 : Let Ω be the interval (0, 1), $\mathcal{A}$ the set of measurable subset of Ω, and P the Lebesgue-measure. Let the set A_n be defined as the union of the intervals $\left(\frac{k}{n}, \frac{k+\lambda(k/n)}{n}\right)$ $(k = 0, 1, \ldots n-1)$ where $\lambda(x)$ is a continuous function in the interval [0, 1] such that $0 \leqslant \lambda(x) \leqslant 1$. Then clearly the sequence $\{A_n\}$ is stable with local density $\lambda(x)$.

This follows evidently as for any subinterval I of [0, 1] we have

$$P(A_n I) = \frac{1}{n} \sum_{(k/n)\varepsilon I} \lambda\left(\frac{k}{n}\right) + 0\left(\frac{1}{n}\right) \qquad \ldots \quad (3.4)$$

and the first term on the right of (3.4) is a Riemann sum of the integral $\int_I \lambda(x)dx$.

Thus
$$\lim_{n \to +\infty} P(A_n I) = \int_I \lambda(x)dx. \qquad \ldots \quad (3.5)$$

It follows easily (e.g. by Theorem 3) that

$$\lim_{n \to +\infty} P(A_n B) = \int_B \lambda(x)dx \qquad \ldots \quad (3.6)$$

for every measurable set B, which proves our assertion.

Example 4 : Let us consider a stationary Markov chain with a finite number of states 1, 2, ..., s. Let ξ_0, the state of the chain at time $t = 0$, have an arbitrary distribution over the set of states. Let ξ_n denote the state of the chain at time $t = n$ $(n = 1, 2, \ldots)$. Let A_n denote the event that the value of ξ_n belongs to a set E where E is a proper subset of the set of states. Let $p_{ij}^{(k)}$ denote the probability of a transition from state i into state j in k steps. Let us suppose that the limits

$$\lim_{k \to +\infty} p_{ij}^{(k)} = \pi_{ij}$$

exist for all i and j; π_{ij} may depend in general on i. It follows that

$$\lim_{n\to+\infty} P(A_nA_k) = \sum_{i=1}^{s} W_i \left[\sum_{j\epsilon E} p_{ij}^{(k)} \left(\sum_{h\epsilon E} \pi_{jh} \right) \right] \quad \ldots \quad (3.7)$$

where W_i is the probability that the chain started at time $t = 0$ in the state i. Thus by Theorem 3 the sequence $\{A_n\}$ is stable. Note that in case π_{ij} does not depend on i, the sequence $\{A_n\}$ is mixing.

Example 5 : Let $S = [\Omega, \mathscr{A}, P]$ be a probability space. Suppose that $\Omega = \sum_{j=1}^{\infty} \Omega_j$; where $\Omega_j \epsilon \mathscr{A}$ and $P(\Omega_j) > 0$ $(j = 1, 2, \ldots)$. Let $S_j = [\Omega_j, \mathscr{A}_j, P_j]$ be the probability space obtained by putting

$$P_j(A) = P(A \mid \Omega_j) \quad \text{for} \quad A \epsilon \mathscr{A}_j, \qquad j = 1, 2, \ldots.$$

where $\mathscr{A}_j$ denotes the set of all $A \epsilon \mathscr{A}$ such that $A \subset \Omega_j$.

Let $\{A_n^{(j)}\}$ be a mixing sequence of sets in the space S_j, with density α_j, and put $A_n = \sum_{j=1}^{\infty} A_n^{(j)} \Omega_j$. Then $\{A_n\}$ is a stable sequence of sets in S, with local density $\alpha(\omega) = \alpha_j$ for $\omega \epsilon \Omega_j$ $(j = 1, 2, \ldots)$.

Clearly, Example 5 covers all cases in which the local density α of a stable sequence of sets has a discrete distribution. As a matter of fact let $[\Omega, \mathscr{A}, P]$ be a probability space and $\{A_n\}$ a stable sequence in this space with local density α where α is a discrete random variable, taking on the different values α_j $(j = 1, 2, \ldots)$ with positive probabilities. Let Ω_j denote the set of those ω for which $\alpha(\omega) = \alpha_j$. Put $P_j(A) = P(A \mid \Omega_j)$ for $j = 1, 2, \ldots$. Then clearly for any $B \epsilon \mathscr{A}$.

$$\lim_{n\to+\infty} P_j(A_n B) = \lim_{n\to+\infty} \frac{P(A_n B \Omega_j)}{P(\Omega_j)} = \frac{Q(B\Omega_j)}{P(\Omega_j)} = \alpha_j P_j(B).$$

Thus the sequence $\{A_n\}$ is mixing in the probability space $[\Omega, \mathscr{A}, P_j]$ with density α_j $(j = 1, 2, \ldots)$.

Remarks : One can generalize Example 5 by splitting the probability space into a non-denumerable instead of a denumerable set of probability spaces, but in this case some care is necessary to avoid measure-theoretical difficulties. However, in this way we arrive at a decomposition of a stable sequence of events into the union of mixing sequences of events, in the most general case. This can be seen as follows. Let $\{A_n\}$ be a stable sequence of sets with local density $\alpha(\omega)$. Then one can define as usual the conditional probability of the event B with respect to a given value of α, which will be denoted by $P_\alpha(B)$. $P_\alpha(B)$ is a random variable such that

$$P(AB) = \int_A P_\alpha(B) dP \quad \text{for every} \quad B \epsilon \mathscr{A}$$

and every $A \epsilon \mathscr{A}_\alpha$ where $\mathscr{A}_\alpha$ is the least σ-algebra on which α is measurable. By other words $P_\alpha(B)$ is the Radon-Nykodim derivative of the set function $P(AB)$ with B fixed with respect to $P(A)$ on the σ-algebra $\mathscr{A}_\alpha$.

As well known, while in case $B_k \epsilon \mathscr{A}$, $B_k B_l = \phi$ for $k \neq l$ the relation

$$P_\alpha \left(\sum_{k=1}^{\infty} B_k \right) = \sum_{k=1}^{\infty} P_\alpha(B_k)$$

holds with probability 1, i.e., except for $\omega \epsilon C$ where C is a set such that $P(C) = 0$ nevertheless one cannot say that $P_\alpha(B)$ is with probability 1 a measure, because the set C of exceptional values of ω may depend on the sequence $\{B_k\}$ and the union of all possible such sets C may have positive measure or even be of measure 1. As however, $P_\alpha(B)$ is not uniquely determined and its value may be changed on a set of measure 0, it is often possible to find a determination of $P_\alpha(B)$ such that it is with probability one a measure. If this is the case it is easy to see that the sequence $\{A_n\}$ is almost surely mixing with respect to the measure $P_\alpha(B)$, with density α.

Such examples can be constructed by means of the theory of measurable decompositions of Lebesgue-spaces, developed by Rochlin (1949). We do not propose to go into details here, but shall return to this question in another paper.

However, we give one example of a stable sequence of random variables constructed by the same principle as applied in the above Example 5 of stable sequence of events.

Example 6 : Let $\{\xi_n\}$ be a mixing sequence of random variables with limiting distribution $F(x)$ and η an arbitrary random variable having a discrete distribution. Let further $g(u, v)$ be a continuous function of two variables. Then the sequence of random variables

$$G_n = g(\xi_n, \eta) \quad (n = 1, 2, \ldots)$$

is strictly stable. As a matter of fact if the values taken on by η with positive probability are denoted by y_k $(k = 1, 2, \ldots)$ and B_k denotes the event $\eta = y_k$ we have

$$\lim_{n\to+\infty} P(G_n < z \mid B) = \sum_{k=1}^{\infty} P(B_k \mid B) \int_{g(x,\, y_k) < z} dF(x). \qquad \ldots \quad (3.8)$$

We may take for instance $g(u, v) = u+v$ in which case we get

$$\lim_{n\to+\infty} P(G_n < z \mid B) = \sum_{k=1}^{\infty} P(B_k \mid B) F(z-y_k) \qquad \ldots \quad (3.9)$$

respectively, we may take $g(u, v) = uv$, in which case, supposing that $y_k > 0$ for $k = 1, 2, \ldots,$ we obtain

$$\lim_{n\to+\infty} P(G_n < z/B) = \sum_{k=1}^{\infty} P(B_k/B) F\left(\frac{z}{y_k}\right) \qquad \ldots \quad (3.10)$$

for all values of z for which the function on the right hand side of (3.9) respectively (3.10) is continuous.

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