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ON TWO MATHEMATICAL MODELS OF THE TRAFFIC ON A DIVIDED HIGHWAY

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Introduction

In the present paper we deal with two models of the traffic on a divided highway. In both models traffic is flowing in one direction only, in two lanes, of which one (the left hand lane) is used only for overtaking. We suppose that vehicles enter the highway at the same entrance so that the instants at which a vehicle enters the highway form a homogeneous Poisson process, with density λ. Thus λ is the rate at which vehicles enter the highway per unit time. We suppose that there are no junctions, or exits (i.e. the highway extends in one direction to infinity).

In the first model, discussed in Section 1, we suppose that each driver chooses a speed v, and drives constantly at this speed. We suppose in this model that if a vehicle B approaches a slower vehicle A ahead of it, B overtakes A without delay. This means that we neglect the case when a third, still faster car, C, has already begun overtaking B on the left hand lane, so that B has to slow down and wait until C passes, and only after this can itself go over to the left hand lane to pass A. In reality the cars lose some time because of overtaking, even on a divided highway. The second model, discussed in Section 2, differs from the first only in that this effect is taken into account, and the corresponding average decrease of speed is estimated. A more detailed analysis of the second model will be given elsewhere.

1. Discussion of the first model

We suppose that denoting by v_k the speed of the vehicle entering the highway at the moment τ_k, the random variables v_k $(k = 1,2,\dots)$ are independently and identically distributed with the cumulative distribution function $F(v) = \mathbf{P}(v_k < v)$, and are also independent of the process $\{\tau_k\}$. Clearly we have to suppose $F(0) = 0$. Besides this, the only restriction necessary for $F(v)$ is that

$$\frac{1}{w} = \int_0^\infty \frac{dF(v)}{v} < +\infty, \tag{1.1}$$

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i.e. that the mean value of $1/v_k$ should be finite; without this condition a traffic jam would arise and make all traffic flow impossible. Of course, in reality, there exists a positive lower limit to the velocities, i.e. a positive number a such that $F(a) = 0$; this ensures that condition (1.1) holds. (On many highways there is in fact a prescribed lower speed limit.) In Model 1 we suppose that overtaking a slower car takes place without delay; thus each car travels with constant velocity. (In the second model we shall drop this simplifying assumption.) In Model 1, if we consider the distribution of cars along the highway at a given instant of time, it can be easily seen that the spatial distribution of the vehicles along the highway is also a homogeneous Poisson process with density

$$\Lambda = \frac{\lambda}{w} , \tag{1.2}$$

this shows why $1/w$ has to be finite).

The fact that the spatial process at any moment is again a Poisson process follows from a general theorem of Ryll-Nardzewski [1] (see also Prékopa [9]).

Recently Breiman [2], [3] considered[1] the problem of one-way traffic flow, starting from the spatial process. If one supposes that at a certain moment t_0 the spatial process is a Poisson process with density μ and the velocities of the vehicles are random variables with the identical distribution function $F(v)$, and are independent of each other and of the position of the vehicles at the moment t_0, it follows from a more general result of Doob [5] that the process will have the same properties at any other moment t. Breiman has proved that the Poisson process is the only process having this time-invariance property.

Our main aim is to take into account (in Model 2 which will be discussed in Section 2) the delays in overtaking; for this reason we have found it more convenient to start from the process of entrance times of the vehicles rather than from the spatial process, as was done by Breiman.

We shall prove now some results concerning Model 1, which will be needed also in discussing Model 2. Let us imagine that we are sitting in a car travelling along the highway with velocity v_0. Let τ_k^+ $(k=1,2,\dots)$ denote the instants when our car overtakes a car with a lower speed, and τ_k^- $(k=1,2,\dots)$ the instants when a faster car overtakes our car. The following theorem is valid.

Theorem 1. The instants $\{\tau_k^+\}$ and $\{\tau_k^-\}$ form two homogeneous Poisson processes, with densities

$$\lambda^{(+)}(v_0) = \lambda \int_0^{v_0} \frac{v_0 - v}{v}\, dF(v) \text{ and } \lambda^{(-)}(v_0) = \lambda \int_{v_0}^{\infty} \left(\frac{v - v_0}{v}\right) dF(v), \tag{1.3}$$

[1] For further literature, we refer to the paper by Hammersley [4] where a bibliography up to 1961 is given. A complete bibliography of the scientific study of road traffic, containing more than 700 items, has been prepared by Frank A. Haight and will be issued by the International Statistical Institute in 1964.

Moreover these two processes are independent of each other.

The following corollary of Theorem 1 should be noted:

Corollary. We have $\lambda^{(+)}(v_0) = \lambda^{(-)}(v_0)$ if and only if

$$v_0 = \left(\int_0^\infty \frac{dF(v)}{v} \right)^{-1} = w. \tag{1.4}$$

In other words a car travelling with speed v_0 is as frequently overtaken by faster cars as it overtakes slower cars if its speed v_0 is equal to the harmonic mean w of the speeds of all the vehicles on the given highway.

If we make notes of the speeds of the cars which overtake us, and the speeds of the cars which we overtake, we naturally obtain a biased picture of the velocity distribution of the cars on the highway; however from this "apparent" velocity distribution, the real distribution can be easily determined. As a matter of fact, supposing for the sake of simplicity that the density function $f(v) = F'(v)$ of the velocity distribution exists, we obtain for the apparent density $f_a(v)$ the respective formulae

$$f_a(v) = \frac{\lambda f(v) \left(\frac{v_0}{v} - 1 \right)}{\lambda^{(+)}(v_0)} \qquad \text{for } 0 < v \leqq v_0 \tag{1.5}$$

for the slower cars, and

$$f_a(v) = \frac{\lambda f(v) \left(1 - \frac{v_0}{v} \right)}{\lambda^{(-)}(v_0)} \qquad \text{for } v_0 \leqq v < +\infty \tag{1.6}$$

for the faster cars, and from these formulae, knowing $f_a(v)$, we can determine $f(v)$ easily.

It is possible to determine the velocity distribution also if we observe only one of the densities $\lambda^{(+)}(v_0)$ and $\lambda^{(-)}(v_0)$, but this must be done for every value of v_0. As a matter of fact, if $\lambda^{+}(v_0)$ as a function of v_0 is given for all v_0, we obtain from (1.3)

$$\lambda^{(+)}(v_0) = \lambda v_0 \int_0^{v_0} \frac{F(v)\, dv}{v^2} \tag{1.7}$$

and thus

$$F(x) = \frac{1}{\lambda} \left(\frac{x\, d\lambda^{(+)}(x)}{dx} - \lambda^{(+)}(x) \right). \tag{1.8}$$

[(1.7) shows that not only $\lambda^{+}(x)$ but also $\lambda^{+}(x)/x$ is an increasing function of x.]

Thus if we travel for sufficiently long periods with different velocities and count the number of cars we overtake (or which overtake us), we can estimate from these data the velocity distribution and the total flow of vehicles fairly accurately. We shall not go into the details of these estimation problems here.

The proof of Theorem 1 is based on the following two properties of Poisson processes:

(A) If $0 < \tau_1 < \tau_2 < \ldots$ are the instants of time when an event occurs in a homogeneous Poisson process with density λ, and $\zeta_1, \zeta_2, \ldots$ is a sequence of independent positive random variables, having the same distribution function $F(v)$, and independent of the process $\{\tau_k\}$, then the time instants $\tau_k \zeta_k$ $(k=1,2\ldots)$ also form a homogeneous Poisson process with density

$$\lambda^* = \lambda \int_0^\infty \frac{dF(v)}{v} \quad . \tag{1.9}$$

(B) If a subsequence $\{\tau_{v_k}\}$ of the instants τ_k, in which an event occurs in a Poisson process with density λ, is selected at random in such a way that for each j the probability of the event A_j that j should belong to the subsequence $\{v_k\}$ is equal to p $(0 < p < 1)$ and the events A_j $(j = 1,2,\ldots)$ are independent, and if $\{\tau_{\mu_k}\}$ are the instants which are not selected, (i.e., j belongs to the sequence $\{\mu_k\}$ if and only if it does not belong to the sequence $\{v_k\}$), then both $\{\tau_{v_k}\}$ and $\{\tau_{\mu_k}\}$ are Poisson processes, with densities λp and λq, where $q = 1 - p$; moreover the two processes are independent.

Property (A) can be proved easily; it follows also from the mentioned theorem of Ryll-Nardzewski [1] or of Prékopa [9]. Property (B) is well known.

Let us mention that in a certain sense property (B) is characteristic of Poisson processes, as the following theorem shows.

Theorem 2. Let $\mathscr{P}_\lambda$ be a class of time-homogeneous point processes such that for each $\lambda > 0$ the process $\mathscr{P}_\lambda$ has density λ. Let us split at random the process $\mathscr{P}_\lambda$ into two processes $\mathscr{P}_\lambda^{(1)}$ and $\mathscr{P}_\lambda^{(2)}$ in such a way that for each point of $\mathscr{P}_\lambda$ the probability that it will belong to $\mathscr{P}_\lambda^{(1)}$ is equal to p, and the probability that it will belong to $\mathscr{P}_\lambda^{(2)}$ is equal to $q = 1 - p$, independently of what happens to the other points. Let us suppose that for every value of p the process $\mathscr{P}_\lambda^{(1)}$ will be governed by the same laws as the process $\mathscr{P}_{\lambda p}$, and $\mathscr{P}_\lambda^{(2)}$ by the same laws as $\mathscr{P}_{\lambda q}$. Further let the processes $\mathscr{P}_\lambda^{(1)}$ and $\mathscr{P}_\lambda^{(2)}$ be independent. Then for each λ, $\mathscr{P}_\lambda$ is a Poisson process with density λ.

Proof. Let $P_n(t, \lambda)$ denote the probability of the event that an interval of length t contains exactly n events of the process $\mathscr{P}_\lambda$. From the assumption that the probability distribution of the number of events in an interval in the process $\mathscr{P}_\lambda^{(1)}$ and the process $\mathscr{P}_{\lambda p}$ are identical, it follows that

$$P_k(t,\lambda p) = \sum_{n=k}^{\infty} \binom{n}{k} p^k q^{n-k} P_n(t,\lambda) \qquad (k = 0,1,\ldots). \tag{1.10}$$

Putting

$$G(t,\lambda,u) = \sum_{n=0}^{\infty} P_n(t,\lambda)(1+u)^n \;, \tag{1.11}$$

where $G(t,\lambda,u)$ is certainly defined if $|1+u| \leqq 1$, we obtain from (1.10) that the functional equation

$$G(t,\lambda p,u) = G(t,\lambda,pu) \tag{1.12}$$

holds. The identity (1.12) can be used for the analytic continuation of $G(t,\lambda,\mu)$ for all complex values of u such that $\mathrm{Re}(u) < 0$. It follows further from (1.12) that for $\mathrm{Re}(u) < 0$,

$$G(t,\lambda,u) = G(t,1,\lambda u) = H(t,\lambda u). \tag{1.13}$$

Now from the assumption that the processes $\mathscr{P}_\lambda^{(1)}$ and $\mathscr{P}_\lambda^{(2)}$ are independent, it follows that

$$P_n(t,\lambda) = \sum_{k=0}^{n} P_k(t,\lambda p) P_{n-k}(t,\lambda q) \tag{1.14}$$

and thus that

$$H(t,\lambda u) = H(t,p\lambda u) \cdot H(t,q\lambda u). \tag{1.15}$$

As $H(t,v)$ is clearly analytic in v, we obtain

$$H(t,\lambda u) = e^{\lambda u K(t)} \tag{1.16}$$

and thus

$$\sum_{n=0}^{\infty} P_n(t,\lambda) z^n = e^{\lambda(z-1)K(t)}. \tag{1.17}$$

Let us now compute the mean value $M_\lambda(t)$ of the number of events in an interval of length t in the process $\mathscr{P}_\lambda$. We obtain from (1.17) that

$$M_\lambda(t) = \sum_{n=0}^{\infty} P_n(t,\lambda) \cdot n = \lambda K(t). \tag{1.18}$$

But as we suppose that the process $\mathscr{P}_\lambda$ is time-homogeneous and has density λ, we have $M_\lambda(t) = \lambda t$. Thus we obtain from (1.18) that $K(t) \equiv t$, and hence

$$\sum_{n=0}^{\infty} P_n(t,\lambda) z^n = e^{\lambda(z-1)t} \;, \tag{1.19}$$

which implies that

$$P_n(t,\lambda) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \qquad (n = 0, 1, \dots), \tag{1.20}$$

so that the process $\mathscr{P}_\lambda$ is a process with independent increments.

Thus $\mathscr{P}_\lambda$ is a Poisson process with density λ, and Theorem 2 is proved.

Let us mention that if we suppose only that $\mathscr{P}_\lambda^{(1)}$ and $\mathscr{P}_\lambda^{(2)}$are governed by the same laws as $\mathscr{P}_{\lambda p}$ and $\mathscr{P}_{\lambda q}$ but do not assume the independence of the processes $\mathscr{P}_\lambda^{(1)}$ and $\mathscr{P}_\lambda^{(2)}$, there are other solutions too, e.g. the compound Poisson processes, for which

$$P_n(t,\lambda) = \int_0^\infty \frac{(\lambda t y)^n e^{-\lambda t y}}{n!}\, dL(y),$$

where $L(y)$ is a probability distribution function in $(0, \infty)$ with mean value 1.

Theorem 2 is related to investigations concerning the rarification of time-homogeneous point processes (see [6], [7] and [8]).

2. Discussion of the second model

Let us pass now to Model 2. In fact, if a car B in the right hand lane approaches a slower car A, it can overtake it only if the left hand lane behind it is free for a certain distance; that is if a still faster car C has not already begun to overtake the car B. If, however, a faster car C has started overtaking B, then B has to slow down to the velocity of the car A and travel with this velocity until C has passed. To make our model precise, we have to introduce quite strict rules on how overtaking has to be effected. First we suppose that if a car B with velocity v_2 approaches in the right hand lane from behind a slower car A, with velocity $v_1 < v_2$, it has to go over to the left hand lane at the instant when the distance between the two cars becomes equal to $T(v_2 - v_1)$ (provided that at this instant there is no third car behind it which has already started overtaking it) and remains there until it has passed A; then it goes back immediately to the right hand lane. B does the same in the case when at the moment it starts overtaking A, A is in the left hand lane in the course of overtaking a third, still slower car; clearly this does not cause any difficulty, since the car A will spend a time less than T in the left hand lane, and thus A will go back to the right hand lane before B passes it, so that B can continue in the left hand lane without slowing down.

This rule is reasonable, as it corresponds to the practice of a cautious driver. Of course, one does not really return to the right hand lane immediately after passing another car, but only after passing it by a certain distance; in order to simplify the theory we neglect this second phase of overtaking. We did not incorporate into our model the fact that most drivers accelerate during overtaking;

these two neglected factors have opposite effects: the first means a lengthening, the second a shortening of the time spent on the left hand lane and their joint effect is practically negligible.

Our rule is convenient from a mathematical point of view because—as shown above—it ensures that no car has to slow down when it is in the left hand lane in the course of overtaking another car.

In our second model each vehicle (except the vehicles which travel either with the minimum or the maximum possible velocity) is from time to time forced to travel slower than its own chosen velocity v and thus its effective average velocity $\bar{v}$ is smaller than v. We want now to calculate $\bar{v}$ as a function of v. This problem is rather difficult. Therefore we deal here only with the case when all cars on the highway are travelling either with the velocity v_1 or with the velocity $v_3 > v_1$ except for the car in which we are sitting, which travels with an intermediate velocity v_2, i.e. we have $v_1 < v_2 < v_3$.

More exactly we suppose that the process of arrival times of the cars at the entrance of the highway is a time-homogeneous Poisson process with density λ, and further that each car chooses at random the velocity v_1 or v_3 $(v_1 < v_3)$ with the corresponding probabilities p and $q = 1 - p$. Let us call for the sake of brevity the cars with speed v_1 the slow cars, and those with speed v_3 the fast cars. Let us suppose further that we arrive at the entrance of the highway at time $t = 0$ and want to travel with the speed v_2 where $v_1 < v_2 < v_3$.

Clearly neither the slow nor the fast cars lose speed; the slow ones because they cannot overtake any other car, the fast ones because no vehicle overtakes them. Thus we have to compute only the velocity loss of our own car. As was shown in Section 2, so long as we maintain the velocity v_2 the time instants τ_k^+ when we overtake a car with velocity v_1 form a Poisson process $\mathscr{P}^+$ with density

(2.1) $$\lambda^{(+)} = \lambda p \left(\frac{v_2 - v_1}{v_1}\right),$$

while the instants when a car with velocity v_3 overtakes us form a Poisson process $\mathscr{P}^-$ with density

(2.2) $$\lambda^{(-)} = \lambda q \left(\frac{v_3 - v_2}{v_3}\right),$$

these two processes being independent.

Let us now compute the distribution of the random variable χ which is the length of the time interval from the moment we entered the highway until the moment when, for the first time, we should like to start overtaking a slow car in front of us, but cannot do this because a fast car has already started overtaking us. Clearly if τ_k^+ is the least instant $> T$ belonging to the process $\mathscr{P}^+$ such that there exists a τ_l^- belonging to $\mathscr{P}^-$ for which $\tau_l^- < \tau_k^+ < \tau_l^- + T$ then $\chi = \tau_k^+ - T$.

Now the time instants τ_k^+ of the process $\mathscr{P}^+$, for which there is such a τ_l^- in the process $\mathscr{P}^-$, themselves clearly form a Poisson process with density

$$D = \lambda^{(+)} \cdot (1 - e^{-\lambda^{(-)} T}) . \tag{2.3}$$

Thus τ_k^+ has an exponential distribution with mean $1/D$. Now

$$\boldsymbol{P}(\chi \geqq x) = \boldsymbol{P}(\tau_k^+ \geqq x + T \mid \tau_k^+ \geqq T) = \boldsymbol{P}(\tau_k^+ \geqq x) ,$$

and thus χ has also an exponential distribution with mean value $1/D$, i.e.

$$E(\chi) = \frac{1}{D} = \frac{1}{\lambda^{(+)}(1 - e^{-\lambda^{(-)} T})} . \tag{2.4}$$

If we are forced to slow down to the velocity v_1 because we cannot overtake the slow car in front of us until the fast car behind us, which has already started overtaking us, has passed, we have to continue to drive with velocity v_1 for at least a time interval of length

$$[T - (\tau_k^+ - \tau_l^-)] \frac{v_3 - v_2}{v_3 - v_1} = \delta_0. \tag{2.5}$$

However it may happen that we cannot start to overtake the slow car in front of us at the instant $\tau_k^+ - T + \delta_0$ because in the meantime another fast car has started to overtake us; in this case we have to wait until this fourth car has passed, and then look out to see whether there is a fifth car which has in the meantime already started overtaking us, and so on. Thus in computing the distribution of the length of the time interval in which we are forced to travel with the velocity v_1 of the slow car in front of us, we have to take into account these various possibilities. In dealing with this question we cannot work with the quantities $\tau_{l+1}^-, \tau_{l+2}^-, \ldots$ because these do not give us the instants when a fast car overtakes us, because we have slowed down. Instead of this, we must work with the instants τ_j^* at which a fast car overtakes us now that we are travelling with the velocity v_1. These instants τ_j^* form, according to what has been said in Section 1, a Poisson process with density

$$\lambda^* = \lambda q \left(\frac{v_3 - v_1}{v_3} \right). \tag{2.6}$$

As the first fast car passes us at the moment $\tau_k^+ - T + \delta_0$, we clearly have to let n fast cars pass before being able to pass the slow car in front of us if and only if

$$\tau^*_{j+1} - T < \tau^+_k - T + \delta_0 < \tau^*_{j+1} \;,$$

$$\tau^*_{j+2} - T < \tau^*_{j+1} < \tau^*_{j+2} \;,$$

$$\tau^*_{j+n-1} - T < \tau^*_{j+n-2} < \tau^*_{j+n-1} \;,$$

and

$$\tau^*_{j+n} - T \geqq \tau^*_{j+n-1} \,.$$

As the variables $\tau^*_j - (\tau^+_k - T + \delta_0), \tau^*_{j+1} - \tau^*_j, \ldots$ are all independent and have an exponential distribution with the mean value $1/\lambda^*$, we have, if we denote by r the number of fast cars which we are forced to let pass us, that

$$P(r = n) = (1 - e^{-\lambda^* T})^{n-1} e^{-\lambda^* T} \tag{2.7}$$

and thus

$$E(r) = e^{\lambda^* T} \;. \tag{2.8}$$

Let us now compute the mean value of the length of the time interval in which we have to drive slowly and wait for an opportunity for overtaking. Let us denote this time interval by η. Thus we have

$$\eta = \delta_0 + \delta_1 + \ldots + \delta_{r-1} \;, \tag{2.9}$$

where δ_0 is given by (2.5), $\delta_1 = \tau^*_{j+1} - (\tau^4_k - T + \delta_0)$ and $\delta_{i+1} = \tau^*_{j+i+1} - \tau^*_{j+i}$ for $i = 1, 2, \ldots r-2$. We have evidently

$$E(\delta_1 + \ldots + \delta_{r-1}) = \sum_{n=2}^{\infty} P(r = n) E(\delta_1 + \ldots + \delta_{n-1} \mid r = n) \,. \tag{2.10}$$

Now for $k < n$

$$E(\delta_k \mid r = n) = E(\delta_k \mid \delta_k \leqq T) = \frac{1}{\lambda^*} - \frac{T}{e^{T\lambda^*} - 1} \,. \tag{2.11}$$

Thus we obtain

$$E(\delta_1 + \ldots + \delta_{r-1}) = \frac{e^{\lambda^* T} - 1}{\lambda^*} - T. \tag{2.12}$$

Let us now compute $E(\delta_0)$. From (2.5) we obtain

$$E(\delta_0) = \left(\frac{v_3 - v_2}{v_3 - v_1}\right) \left(\frac{1}{\lambda^{(-)}} - \frac{T}{e^{\lambda(-)T} - 1}\right)$$

and hence

$$E(\eta) = \left(\frac{v_3 - v_2}{v_3 - v_1}\right) \left(\frac{1}{\lambda^{(-)}} - \frac{T}{e^{\lambda(-)T} - 1}\right) + \left(\frac{e^{\lambda^* T} - 1}{\lambda^*} - T\right) \,. \tag{2.13}$$

Now we are in a position to compute our average effective velocity. As we travel alternatively with velocities v_2 and v_1, the mean values of the corresponding time intervals being $E(\chi)$ and $E(\eta)$ and the time intervals in question being independent, we obtain by the law of large numbers that in the long run our average velocity will be $\bar{v}_2$ where

$$\bar{v}_2 = \frac{v_2 \cdot E(\chi) + v_1 \cdot E(\eta)}{E(\chi) + E(\eta)}, \tag{2.14}$$

where $E(\chi)$ and $E(\eta)$ are given by (2.4) and (2.13) respectively, and the quantities $\lambda^{(+)}, \lambda^{(-)}, \lambda^*$ figuring in these expressions are defined by (2.1), (2.2) and (2.6) respectively.

It is easy to see that a considerable loss in velocity is to be expected only if the traffic on the highway is heavy, more exactly, if both $\lambda^* T$ and $\lambda^{(-)}T$ are large.

As regards the general case, one can get an approximate solution for the effective velocity $\bar{v}$ of a car which travels with velocity v when this is possible, by replacing the actual velocity distribution by a distribution in which there are only two velocities present, namely

$$v_1 = \frac{1}{F(v)} \int_0^v t dF(t)$$

and

$$v_3 = \frac{1}{1 - F(v)} \int_v^\infty t dF(t),$$

these having corresponding probabilities $p = F(v)$ and $q = 1 - F(v)$; in other words we replace the velocity of every car slower or faster than our car by the average velocity of such cars. However in the case when the traffic density is large, this approximation is rather crude. To make this clear, let us mention that in general, when we have to slow down behind a slow car to let a faster car (which has already started overtaking us) pass, we may then be overtaken by another car whose speed is less than our own original speed. This also happens quite often in heavy traffic, as everyone who has some experience of driving on divided highways knows. However the above approximation excludes this possibility, and so gives too high an estimate of the average speed (i.e. too optimistic a view on the loss of speed). We intend to return to the general problem in another paper.

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