ON THE HEIGHT OF TREES

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1. Introduction

In this note we shall deal with the enumeration of labelled trees of given order and given height over a selected point.

An undirected graph is called a tree if it is connected and contains no cycle. If we select any two vertices P and Q of a tree T, there is evidently a uniquely determined path in T leading from P to Q. We shall call the length of this path (i.e. the number of edges in the path) the distance of P and Q in T and denote it by $d_T(P, Q)$. If a vertex P is distinguished as the root of T, we define the height of T over P as the length of the longest path in T starting from P; thus if $h_P(T)$ denotes the height of T over the root P, we have

$$h_P(T) = \max_{Q \in T} d_T(P, Q). \tag{1.1}$$

Let us consider the set $\mathscr{T}_n$ of all possible trees with n given labelled vertices $P_1, P_2, \cdots, P_n$. According to a classical result of Cayley [1] if t_n denotes the number of elements of $\mathscr{T}_n$, we have

$$t_n = n^{n-2} \tag{1.2}$$

Let $t_n(k)$ denote the number of those trees $T \in \mathscr{T}_n$ for which $h_{P_1}(T) \leqq k$. Clearly

$$t_1(0) = 1, \quad t_n(0) = 0 \quad \text{for } n > 1 \tag{1.3}$$

and

$$t_n(k) = t_n \quad \text{for} \quad k \geqq n-1. \tag{1.4}$$

J. Riordan [2] has shown that the enumerator

$$G_k(x) = \sum_{n=1}^{\infty} \frac{t_n(k)}{(n-1)!} x^n \qquad (k = 0, 1, \cdots) \tag{1.5}$$

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satisfies the recursion formula

$$G_{k+1}(x) = x \exp G_k(x) \qquad (k = 0, 1, \cdots) \tag{1.6}$$

with

$$G_0(x) = x; \tag{1.7}$$

the latter follows from (1.3) and (1.5).

From the recursion formula (1.6) one can determine $t_n(k)$ for any k and n $(0 \leqq k \leqq n-1)$. For instance

$$G_1(x) = xe^x, \quad G_2(x) = xe^{xe^x}, \quad \text{etc.,} \tag{1.8}$$

and thus

$$t_n(1) = 1 \qquad (n = 1, 2, \cdots) \tag{1.9}$$

$$t_n(2) = \sum_{m=0}^{n-1} \binom{n-1}{m} m^{n-m-1} \qquad (n = 1, 2, \cdots) \tag{1.10}$$

and generally for $k \geqq 1$

$$t_n(k) = \sum_{\substack{m_1+\cdots+m_k=n-1 \\ m_i \geqq 0}} \frac{(n-1)!}{m_1!m_2!\cdots m_k!} m_1^{m_2} m_2^{m_3} \cdots m_{k-1}^{m_k} \qquad (i = 1, 2, \cdots, k). \tag{1.11}$$

In these formulae 0^0 always means 1.

In view of (1.2) and (1.4) one has

$$\lim_{k\to\infty} G_k(x) = \sum_{n=1}^{\infty} \frac{n^{n-2}}{(n-1)!} x^n \tag{1.12}$$

provided that the series on the right of (1.12) is convergent. But the series

$$y = \sum_{n=1}^{\infty} \frac{n^{n-2}}{(n-1)!} x^n \tag{1.13}$$

converges for $|x| \leqq 1/e$ and represents the inverse function of

$$x = ye^{-y}. \tag{1.14}$$

This equation also follows from (1.6) and (1.12).

Riordan [2] obtained the formula (1.6) as a special case of a more general result on enumerators of trees. In § 2 we shall give direct proofs of (1.6) and (1.11).

In § 3 we shall investigate the asymptotic distribution of

$$d_n(k) = t_n(k) - t_n(k-1), \tag{1.15}$$

i.e. the number of trees $T \in \mathscr{T}_n$ having exact height k over P_1. Let us mention that if $D(T)$ denotes the diameter of T (i.e. the length of the longest path in T) one has evidently

$$\frac{1}{2}D(T) \leqq \min_i h_{P_i}(T) \leqq h_{P_i}(T) \leqq \max_i h_{P_i}(T) \leqq D(T). \tag{1.16}$$

Thus the study of the distribution of $h_{P_i}(T)$ for $T \in \mathscr{T}_n$ gives us also some information on the distribution of $D(T)$.

Our thanks are due to F. Harary and J. W. Moon for calling our attention to the paper [2] of Riordan.

2. Proof of the recursion formula

To prove (1.6) we start from the formula

$$t_n(k) = \sum_{p=1}^{n-1} \binom{n-1}{p} \sum_{m_1+\cdots+m_p=n-1} \frac{(n-1-p)!}{(m_1-1)!\cdots(m_p-1)!} t_{m_1}(k-1)\cdots t_{m_p}(k-1). \tag{2.1}$$

(2.1) can be proved as follows: Let E denote the set of those points of $T \in \mathscr{T}_n$ which are directly connected (i.e. connected by an edge) with P_1. If p is the number of elements of E then $1 \leqq p \leqq n-1$ and denoting these points by $Q_1, \cdots, Q_p$, the points Q_i can be selected in $\binom{n-1}{p}$ ways. All the remaining $n-1-p$ points $P_m (P_m \neq Q_j,\ 1 \leqq j \leqq p;\ P_m \neq P_1)$ can be classified into p classes which are defined as follows: P_m lies in the j-th class ($j = 1, 2, \cdots, p$) if the unique path from P_1 to P_m goes through Q_j. Clearly if the j-th class contains m_j-1 points, these points together with Q_j form a tree of order m_j and height $\leqq k-1$ over the basic point Q_j. Thus (2.1) follows.

Multiplication of (2.1) by $x^n/(n-1)!$ and summation for $n = 1, 2, \cdots$ leads immediately to (1.6). (1.11) can be deduced from (1.6) by using several times the power series of the exponential function. It can also be proved directly as follows:

Let $T \in \mathscr{T}_n$ be a tree the height of which over the basic point P_1 is $\leqq k$. Then all points of T different from P_1 can be classified into k classes, the j-th class $\mathscr{C}_j$ consisting of those points whose distance from P_1 is equal to j $(1 \leqq j \leqq k)$. Let m_j denote the number of points in the class $\mathscr{C}_j$ $(1 \leqq j \leqq k)$; then

$$\sum_{j=1}^{k} m_j = n-1.$$

If the numbers m_j are fixed, the distribution of the $n-1$ points in the classes $\mathscr{C}_j$ can be carried out in $(n-1)!/m_1! \cdots m_k!$ ways. Now evidently each point in the class $\mathscr{C}_1$ is directly connected with P_1, each point in $\mathscr{C}_2$ is directly connected with some point in $\mathscr{C}_1$ etc., each point in $\mathscr{C}_k$ is directly connected with some point of $\mathscr{C}_{k-1}$. As the connections can be established in $m_1^{m_2} m_2^{m_3} \cdots m_{k-1}^{m_k}$ different ways and by choosing these connections the tree T is completely determined, (1.11) follows.

For $d_n(k) = t_n(k) - t_n(k-1)$ the proof of (1.11) gives

$$d_n(k) = \sum_{\substack{m_1+\cdots+m_k=n-1 \\ m_i \geq 1}} \frac{(n-1)!}{m_1! \cdots m_k!} m_1^{m_2} m_2^{m_3} \cdots m_{k-1}^{m_k}, \quad (i = 1, \cdots, k). \tag{2.2}$$

If $d_n(k, m)$ denotes the number of trees $T \in \mathscr{T}_n$ for which $h_{P_1}(T) = k$ and in which there are exactly m points connected with P_1 by an edge then (2.2) gives

$$d_n(k, m) = \frac{1}{m!} \sum_{\substack{m_2+\cdots+m_k=n-1-m \\ m_i \geq 1}} \frac{(n-1)!}{m_2! \cdots m_k!} m^{m_2} \cdots m_{k-1}^{m_k}. \tag{2.3}$$

From here the following recursion formula can be deduced:

$$d_n(k, m) = \binom{n-1}{m} \sum_{p=1}^{n-1-m} m^p d_{n-m}(k-1, p). \tag{2.4}$$

Similarly if $t_n(k, m)$ is the number of those trees $T \in \mathscr{T}_n$ which have height $\leq k$ over P_1 and in which the number of points having distance k from P_1 equals m, then

$$t_n(k, m) = \frac{1}{m!} \sum_{m_1+\cdots+m_{k-1}=n-1-m} \frac{(n-1)!}{m_1! \cdots m_{k-1}!} m_1^{m_2} \cdots m_{k-2}^{m_{k-1}} m_{k-1}^{m}, \tag{2.5}$$

and

$$t_n(k, m) = \binom{n-1}{m} \sum_{p=1}^{n-1-m} p^m t_{n-m}(k-1, p). \tag{2.6}$$

Thus putting

$$F_k(x, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{t_n(k, m)}{(n-1)!} x^{n-m} z^m \tag{2.7}$$

we have the recursion formula

$$F_k(x, z) = F_{k-1}(x, xe^z) \tag{2.8}$$

with $F_0(x, z) = z$. We obtain

$$F_1(x, z) = xe^z, \quad F_2(x, z) = xe^{xe^z}, \text{ etc.},$$

hence

$$F_{k+1}(x, z) = x \exp F_k(x, z) \tag{2.9}$$

further

$$F_k(x, x) = F_{k+1}(x, 0) = G_k(x). \tag{2.10}$$

3. The asymptotic distribution of $d_n(k)$

We consider now the asymptotic distribution of $d_n(k)$ when n and k are large. We shall make use of the generating function

$$G_k(x)-G_{k-1}(x) = \sum_{n=1}^{\infty} \frac{d_n(k)}{(n-1)!} x^n \tag{3.1}$$

where

$$G_0(x) = x, \quad G_k(x) = x \exp G_{k-1}(x) \qquad (k = 1, 2, \cdots) \tag{3.2}$$

by (1.5) and (1.15). From (3.2) it is seen that $G_k(z)-G_{k-1}(z)$ is an entire function and hence

$$\frac{d_n(k)}{(n-1)!} = \frac{1}{2\pi i} \int_{C+} \frac{G_k(z)-G_{k-1}(z)}{z^{n+1}} dz \tag{3.3}$$

where C is any circular path with centre 0. For the radius of C we shall take $r = e^{-1}$; this is the largest positive value of r for which the sequence $G_k(r)$, $k = 1, 2, \cdots$ tends to a finite limit, namely $\lim_{k\to\infty} G_k(e^{-1}) = 1$. Moreover if k is of order $\sqrt{n}$ which is the case of principal interest then the point e^{-1} lies very close to a saddle point of the integrand.

As in (2.9), write for a fixed complex number ζ

$$F_0(\zeta, z) = z, \tag{3.4}$$

$$F_k(\zeta, z) = \zeta \exp F_{k-1}(\zeta, z) \qquad (k = 1, 2, \cdots). \tag{3.5}$$

Thus $F_k(\zeta, z)$ is the k-th iterate of

$$F(\zeta, z) = \zeta e^z \quad (\zeta \text{ fixed}) \tag{3.6}$$

and

$$G_k(\zeta) = F_k(\zeta, \zeta) = F_{k+1}(\zeta, 0), \tag{3.7}$$

as in (2.10).

In the particular case of $\zeta = e^{-1}$, $z = 1$ is a fixed point with multiplier 1 of the function [2] $F(e^{-1}, z) = e^{z-1}$; in fact $F(e^{-1}, 1) = 1$, $F'(e^{-1}, 1) = 1$.

The sequence

$$\gamma_k = F_k(e^{-1}, e^{-1}) = G_k(e^{-1}), \qquad k = 0, 1, 2, \cdots \tag{3.8}$$

satisfies

$$\gamma_k = \exp(\gamma_{k-1}-1), \qquad k = 1, 2, \cdots \tag{3.9}$$

and has an asymptotic expansion

[2] $z = a$ is a fixed-point with multiplier μ of the function $F(z)$ if $F(a) = a$ and $F'(a) = \mu$. If $|\mu| < 1$, the fixed-point is called attractive. (Fatou [4], p. 186).

$$\gamma_k \cong 1 - \frac{2}{k} + \frac{2}{3}\frac{\log k}{k^2} + \frac{c}{k^2} + \cdots \qquad (k \to \infty) \tag{3.10}$$

where c is a certain constant (see e.g. [3], Lemma 3, p. 247). For all other values of ζ on the circle

$$\zeta = e^{-1+it}, \quad -\pi \leqq t \leqq \pi,$$

$F(\zeta, z)$ has an attractive fixed-point [2] $\omega = u+iv$ with multiplier ω ($|\omega| < 1$), given by the equation

$$\omega = \zeta\, e^{\omega} = e^{-1+it+\omega}. \tag{3.11}$$

These fixed-points lie on the curve

$$u^2+v^2 = e^{2(u-1)}, \quad u \leqq 1 \tag{3.12}$$

with

$$\tan\,(v+t) = v/u = u^{-1}\,(e^{2(u-1)}-u^2)^{\frac{1}{2}}. \tag{3.13}$$

Thus to each $\zeta = e^{-1+it}$ there corresponds a unique $\omega = \omega(\zeta) = u+iv$ on the curve (3.12). In the neighbourhood of $t = 0$, i.e. of $u = 1$, $v = 0$, the curve of fixed-points has a double point and satisfies an expansion

$$\begin{aligned} u &= 1-\sqrt{t}+0\cdot t+at\sqrt{t}+\cdots, \\ v &= \sqrt{t}-\tfrac{2}{3}t+bt\sqrt{t}+\cdots \qquad \text{if } t>0 \\ &= -\sqrt{-t}-\tfrac{2}{3}t-bt\sqrt{-t}+\cdots \text{ if } t<0. \end{aligned} \tag{3.14}$$

The fixed-points ω on (3.12) are attractive for all z on the circle $|z| = e^{-1}$; in fact

$$|F(\zeta, z)| = |\zeta e^{z}| < 1 \tag{3.15}$$

for all $|\zeta| = e^{-1}$, Re $z < 1$ and the functions

$$F_k(\zeta, z),\ |\zeta| = e^{-1}, \qquad k = 0, 1, 2, \cdots$$

form a normal family on the half plane Re $z < 1$.

By a more refined argument one can show that if we set

$$D_k(\zeta) = G_k(\zeta)-\omega(\zeta) \tag{3.16}$$

then $D_k(\zeta)/\omega(\zeta)^k$ is uniformly bounded for all $\zeta = e^{-1+it}$, $-\pi \leqq t \leqq \pi$ and $k = 0, 1, 2, \cdots$, i.e.

$$|D_k(\zeta)| < A|\omega(\zeta)|^k \tag{3.17}$$

for a suitable positive constant A. For we have

$$D_k(\zeta) = \omega(\zeta)[\exp D_{k-1}(\zeta)-1], \qquad k = 1, 2, \cdots \tag{3.18}$$

by (3.2), (3.11) and (3.16), and the sequence behaves very nearly like the sequence D_k^* given by the recursion

$$(3.19)\qquad D_k^* = \omega D_{k-1}^*/(1-\tfrac{1}{2}D_{k-1}^*).$$

For this sequence the statement can be verified by direct calculation since

$$(3.20)\qquad D_k^* = -\frac{2\omega^k}{1+\omega+\cdots+\omega^{k-1}-a}, \quad a = 1/D_0^*.$$

We omit details.

For $|t| < ((\log^2 k)/k)^2$ the sequence $G_k(\zeta)$, $k = 0, 1, 2, \cdots$ has a uniform asymptotic expansion

$$(3.21)\qquad \begin{aligned} G_k(e^{-1+it}) \cong 1 &- \frac{2}{k}\,\tau\cot\tau + \tfrac{2}{3}\frac{\tau^2}{\sin^2\tau}\frac{\log k}{k^2} \\ &+ \frac{1}{k^2}\left[c\,\frac{\tau^2}{\sin^2\tau} + \tfrac{2}{3}\tau^2\left(2+\log\frac{\sin\tau}{\tau}\right)\right] + \cdots \end{aligned}$$

where

$$(3.22)\qquad it = 2\tau^2/k^2$$

and c is the constant in (3.10). The expansion becomes (3.10) for $\tau = 0$ and can be verified formally by setting

$$(3.23)\qquad G_k(e^{-1+it}) \cong 1 - \frac{\theta_1(\tau)}{k} + \theta_2(\tau)\frac{\log k}{k^2} + \frac{\theta_3(\tau)}{k^2} + \cdots$$

and using (3.2).

We obtain (for fixed t) by (3.22)

$$\begin{aligned} G_{k+1}(e^{-1+it}) &\cong 1 - \frac{\theta_1(\tau+\tau/k)}{k+1} + \theta_2(\tau+\tau/k)\frac{\log(k+1)}{(k+1)^2} \\ &\quad + \theta_3(\tau+\tau/k)\frac{1}{(k+1)^2} + \cdots \\ &\cong 1 - \theta_1/k + \theta_1/k^2 - \theta_1/k^3 - \tau\theta_1'/k^2 \\ &\quad + \tau\theta_1'/k^3 - \tau^2\theta_1''/2k^3 + \theta_2\log k/k^2 \\ &\quad - 2\theta_2\log k/k^3 + \theta_2/k^3 + \tau\theta_2'\log k/k^3 \\ &\quad + \theta_3/k^2 - 2\theta_3/k^3 + \tau\theta_3'/k^3 + \cdots, \end{aligned}$$

$$\begin{aligned} \exp(G_k - 1 + 2\tau^2/k^2) &\cong \exp\left[-\frac{\theta_1}{k} + \theta_2\frac{\log k}{k^2} + \frac{\theta_3}{k^2} + \frac{2\tau^2}{k^2} + \cdots\right] \\ &\cong 1 - \theta_1/k + \theta_2\log k/k^2 + (\theta_3 + 2\tau^2)/k^2 \\ &\quad + \tfrac{1}{2}\theta_1^2/k^2 - \theta_1\theta_2\log k/k^3 - \theta_1\theta_3/k^3 \\ &\quad - 2\theta_1\tau^2/k^3 - \tfrac{1}{6}\theta_1^3/k^3 + \cdots \end{aligned}$$

where values of the function θ_i and their derivatives are taken at τ. These two expressions are equal by (3.2), hence comparing coefficients

$$\theta_1 - \tau\theta_1' = \tfrac{1}{2}\theta_1^2 + 2\tau^2, \tag{3.24}$$

$$\tau\theta_2' - 2\theta_2 = -\theta_1\theta_2, \tag{3.25}$$

$$-\theta_1 + \tau\theta_1' - \tfrac{1}{2}\tau^2\theta_1'' + \theta_2 - 2\theta_3 + \tau\theta_3' = -\theta_1\theta_3 - 2\tau^2\theta_1 - \tfrac{1}{6}\theta_1^3. \tag{3.26}$$

With the initial conditions $\theta_1(0) = 2$, $\theta_2(0) = \frac{2}{3}$, $\theta_3(0) = c$, obtained from (3.10), the equations (3.24)—(3.26) give

$$\theta_1(\tau) = 2\tau \cot \tau, \quad \theta_2(\tau) = \tfrac{2}{3}\frac{\tau^2}{\sin^2 \tau},$$

$$\theta_3(\tau) = c\frac{\tau^2}{\sin^2 \tau} + \tfrac{2}{3}\tau^2\left(2 + \log \frac{\sin \tau}{\tau}\right), \quad \text{i.e. } (3.21).$$

Thus we only have to prove the existence of an expansion of the form (3.23). This can be achieved by a step by step method such as the one used in [3] for the proof of general expansions of the type (3.11); we omit details. Actually we only need the expansion in the weaker form

$$G_k(e^{-1+it}) = 1 - \frac{2}{k}\tau \cot \tau + O\,(k^{-1-2\delta}) \tag{3.27}$$

for some $\delta > 0$ when $|t| \leqq ((\log^2 k)/k)^2$.

We then get from (3.2), since

$$2\tau \cot \tau = O\,(k\sqrt{|t|}) = O\,(\log^2 k),$$

$$\begin{aligned} G_k(e^{-1+it}) - G_{k-1}(e^{-1+it}) &= \exp\,(-1 + it + G_{k-1}) - G_{k-1} \\ &= it + \tfrac{1}{2}(G_{k-1} - 1)^2 + O(k^{-3+\delta}) \\ &= 2\tau^2(1 + \cot^2 \tau)/k^2 + O(k^{-2-\delta}), \end{aligned}$$

$$G_k(e^{-1+it}) - G_{k-1}(e^{-1+it}) = \frac{2}{k^2}\frac{\tau^2}{\sin^2 \tau} + O(k^{-2-\delta}) \tag{3.28}$$

for $|t| \leqq ((\log^2 k)/k)^2$.

Now from (3.17)

$$\begin{aligned} G_k(e^{-1+it}) - G_{k-1}(e^{-1+it}) &= D_k(e^{-1+it}) - D_{k-1}(e^{-1+it}) \\ &= O\left(\left(1 - \frac{\log^2 k}{k}\right)^k\right) = O(e^{-\log^2 k}) \end{aligned}$$

for $|t| \geqq ((\log^2 k)/k)^2$, by (3.14), hence by (3.3) and (3.28)

$$(3.29)\qquad \begin{aligned} \frac{d_n(k)}{(n-1)!} &= \frac{1}{2\pi i}\,\frac{2e^n}{k^2}\left(\int_{|t|\leq((\log^2 k)/k)^2} \frac{\tau^2}{\sin^2\tau}\, e^{-nit} i dt + O(k^{-2-\delta})\right) \\ &\cong \frac{1}{2\pi i}\,\frac{8e^n}{k^4}\int_\Gamma \frac{\tau^3}{\sin^2\tau}\, e^{-\beta\tau^2} d\tau \end{aligned}$$

by (3.22), where

$$(3.30)\qquad \beta = 2n/k^2$$

and Γ is the path

$$\tau = (i-1)u, \quad -\tfrac{1}{2}\log^2 k \leqq u \leqq 0$$
$$\tau = (i+1)u, \quad 0 \leqq u \leqq \tfrac{1}{2}\log^2 k.$$

Hence

$$\frac{d_n(k)}{(n-1)!} \cong \frac{8e^n}{k^4}\sum_{p=1}^{\infty} \operatorname*{res}_{\tau=p\pi}\left(\frac{\tau^3}{\sin^2\tau}\, e^{-\beta\tau^2}\right).$$

By using Stirling's formula we obtain from here

$$(3.31)\qquad p_n(k) = \frac{d_n(k)}{n^{n-2}} \cong 2\left(\frac{2\pi}{n}\right)^{\frac{1}{2}} \beta^2 \sum_{p=1}^{\infty} (2p^4\pi^4\beta - 3p^2\pi^2) e^{-\beta\pi^2p^2}$$

for large n and k where β is given by (3.30).

This is the required asymptotic probability distribution. Note that

$$\begin{aligned} \sum_{k=1}^{n-1} p_n(k) &\sim 2\left(\frac{2\pi}{n}\right)^{\frac{1}{2}} \int_0^{n-1} \beta^2 \sum_{p=1}^{\infty} (2p^4\pi^4\beta - 3p^2\pi^2)\, e^{-\beta\pi^2p^2} dk \\ &\cong 2\pi^{\frac{1}{2}} \int_0^{\infty} \sum_{p=1}^{\infty} \beta^{\frac{1}{2}}(-3p^2\pi^2 + 2p^4\pi^4\beta) e^{-\beta\pi^2p^2} d\beta \\ &= 2\pi^{\frac{1}{2}} \lim_{\epsilon=0}\left[-\sum_{p=1}^{\infty} 2p^2\pi^2\beta^{\frac{3}{2}}\, e^{-\beta\pi^2p^2}\right]_\epsilon^\infty \\ &= \lim_{\beta\to 0} 4\pi^{\frac{5}{2}}\beta^{\frac{3}{2}} \sum_{p=1}^{\infty} p^2 e^{-\beta\pi^2p^2} \\ &= 4\pi^{-\frac{1}{2}}\int_0^\infty u^2 e^{-u^2} du = 1, \end{aligned}$$

as required.

The maximum of the distribution curve is reached when in (3.31), $(d/d\beta)p_n(k) = 0$, i.e.

$$\sum_{p=1}^{\infty} (9p^4\pi^4\beta^2 - 6p^2\pi^2\beta - 2p^6\pi^6\beta^3) e^{-\beta\pi^2p^2} = 0.$$

Numerically β (max) $= 0.373138525$ and

$$(3.32)\qquad k\ (\text{max}) = 2.31515436\sqrt{n}.$$

4. Conclusion

The result of the previous section can be stated as follows:

Let $\mathscr{H}_n$ be the height over P_1 of a labelled random tree of order n i.e. of a tree selected at random from the set of n^{n-2} elements of $\mathscr{T}_n$ with uniform probability distribution. Then

$$\lim_{n\to\infty} P\left(\frac{\mathscr{H}_n}{\sqrt{2n}} < x\right) = H(x) \tag{4.1}$$

where

$$H(x) = 4x^{-3}\pi^{\frac{5}{2}} \sum_{p=1}^{\infty} p^2 e^{-\pi^2p^2/x^2}. \tag{4.2}$$

This can be transformed (e.g. by means of Poisson's formula) to the form

$$H(x) = \sum_{v=-\infty}^{\infty} e^{-v^2x^2}(1-2v^2x^2) \tag{4.3}$$

whence

$$h(x) = H'(x) = 4x \sum_{v=1}^{\infty} v^2(2v^2x^2-3)e^{-v^2x^2}. \tag{4.4}$$

From (4.4) we can calculate all moments of the distribution function $H(x)$:

$$\begin{aligned} M_s &= \int_0^\infty x^s h(x)dx \\ &= 2\Gamma(\tfrac{1}{2}s+1)(s-1)\zeta(s) \qquad (s > 1) \end{aligned} \tag{4.5}$$

where $\zeta(s) = \sum_{m=1}^{\infty} 1/m^s$. In particular we obtain for M_1, since

$$\lim_{s\to 1} (s-1)\zeta(s) = 1,$$

$$M_1 = \sqrt{\pi}. \tag{4.6}$$

Hence the expectation value of $\mathscr{H}_n$ is

$$E(\mathscr{H}_n) \sim \sqrt{2n\pi} = 2.50663\sqrt{n} \tag{4.7}$$

and the variance is

$$D^2 = M_2 - M_1^2 = \frac{\pi(\pi-3)}{3}. \tag{4.8}$$

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