ON THE MEAN VALUE OF NONNEGATIVE MULTIPLICATIVE NUMBER-THEORETICAL FUNCTIONS

Paul Erdös and A. Rényi

INTRODUCTION

A complex-valued function $g(n)$ $(n = 1, 2, 3, \cdots)$ defined on the set of natural numbers is called *multiplicative* if, for all pairs n, m of relatively prime natural numbers,

$$g(n \cdot m) = g(n) \cdot g(m). \tag{0.1}$$

A multiplicative function $g(n)$ is called *strongly multiplicative* if for all primes p and all positive integers k it satisfies the additional condition

$$g(p^k) = g(p),$$

and *completely multiplicative* if (0.1) holds for all pairs n, m of natural numbers. We say that the number-theoretical function $g(n)$ has a *mean value* if the limit

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} g(n) = M(g)$$

exists. The question of the existence of the mean value $M(g)$ has been much studied, but it has been solved only for certain subclasses. One of the definite results is the following theorem, due to H. Delange [2] (throughout the paper, p denotes a prime, and $\sum_p$ and $\prod_p$ denote a sum and a product, respectively, taken over all primes):

If $g(n)$ *is a strongly multiplicative number-theoretical function such that* $|g(n)| \leq 1$ *for* $n = 1, 2, \cdots$, *and such that the series*

$$\sum_p \frac{g(p) - 1}{p} \tag{0.2}$$

converges, then $M(g)$ *exists and*

$$M(g) = \prod_p \left(1 + \frac{g(p) - 1}{p}\right). \tag{0.3}$$

Conversely, if $g(n)$ *is a strongly multiplicative function such that* $|g(n)| \leq 1$ $(n = 1, 2, \cdots)$, $M(g)$ *exists, and* $M(g) \neq 0$, *then the series* (0.2) *converges,* $g(2) \neq -1$, *and* (0.3) *holds.*

In the present paper we shall consider only *real, nonnegative* multiplicative functions.

The following theorem has been proved by P. Erdös [3]:

Received September 10, 1964.

THEOREM 1. *If* g(n) *is a nonnegative, strongly multiplicative function such that not only the series* (0.2) *but also the series*

$$\sum_p \frac{(g(p) - 1)^2}{p} \tag{0.4}$$

converges, then M(g) *exists and* (0.3) *holds.*

Erdös asked whether the convergence of the series (0.2) is in itself sufficient for the existence of M(g), and if not, whether the requirement that the series (0.4) converges can be relaxed. The aim of this paper is to investigate these questions. While the answer to the first question is negative [that is, convergence of the series (0.2) does not by itself ensure the existence of M(g)], the following theorem, to be proved in Section 2, shows that the answer to the second question is affirmative.

THEOREM 2. *Let* g(n) *be a nonnegative and strongly multiplicative function such that the series* (0.2) *together with the series*

$$\sum \frac{g^2(p)}{p^2} \tag{0.5}$$

converges, and such that for each $\varepsilon > 0$ *there exist positive constants* $\delta(\varepsilon)$ *and* $N(\varepsilon)$ *with the property*

$$\sum_{N \leq p \leq N(1+\varepsilon)} \frac{g(p) \log p}{p} \geq \delta(\varepsilon) \quad \textit{for } N \geq N(\varepsilon); \tag{0.6}$$

then M(g) *exists and* (0.3) *holds.*

Theorem 1 is not directly contained in Theorem 2, but Theorem 2 is nevertheless stronger than Theorem 1, in the sense that Theorem 1 can be deduced from Theorem 2. The relation between the two theorems is as follows: Suppose a function g(n) satisfies the conditions in Theorem 1, and let $\mathscr{P}$ denote the set of primes for which $g(p) \leq 1/2$; then

$$\sum_{p \in \mathscr{P}} 1/p < \infty .$$

If for each prime $p \in \mathscr{P}$ we change the value of g(p) to 1, then the function $g_1(n)$ thus obtained already satisfies the conditions of Theorem 2, and thus $M(g_1)$ exists; the existence of M(g) and the validity of (0.3) follow easily from the existence of $M(g_1)$ and the validity of the corresponding formula for $M(g_1)$. On the other hand, Theorem 2 can be applied in many cases in which Theorem 1 gives no information.

In proving Theorem 2 we shall make use of an analytic method that H. Delange devised to prove his theorem mentioned above. The second-named author [6] has recently found a much simpler proof of Delange's theorem, but for the case studied in the present paper, the method of Delange seems more appropriate. Besides this method, we shall need an argument that resembles a step in the elementary proof of the prime number theorem (see [1], [4]).

In order to simplify the application of the method of Delange, we shall deal first with certain functions that we call exponentially multiplicative functions. A multiplicative function is called *exponentially multiplicative* if for all primes p and all natural numbers $k \geq 2$ it satisfies the condition

$$g(p^k) = \frac{(g(p))^k}{k!}.$$

In Section 1, we establish Theorem 3, which concerns a special class of exponentially multiplicative functions. In Section 2, we deduce Theorem 2 from Theorem 3 and show how Theorem 1 can be deduced from Theorem 2, while in Section 3 we deduce from Theorem 2 a corresponding result (Theorem 4) for general nonnegative multiplicative functions. In Section 4 we deal with cases where the mean value of a multiplicative function is 0 or ∞, and we give some counterexamples.

1. EXPONENTIALLY MULTIPLICATIVE FUNCTIONS

THEOREM 3. *Let* $g(n)$ *be a nonnegative and exponentially multiplicative function such that the series* (0.2) *converges and condition* (0.6) *is satisfied. Then the limit*

$$M(g) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} g(n)$$

exists, and

$$M(g) = \prod_p \left(1 - \frac{1}{p}\right) \exp \frac{g(p)}{p}.$$

Proof. The Dirichlet generating series of a multiplicative function $g(n)$ evidently has the product-representation

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_p \left(\sum_{k=0}^{\infty} \frac{g(p^k)}{p^{ks}} \right).$$

If $g(n)$ is exponentially multiplicative, this can be written in the form

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \exp \sum_p \frac{g(p)}{p^s}.$$

Now clearly the convergence of the series (0.2) implies the convergence of the infinite product

$$L(g) = \prod_p \left(1 - \frac{1}{p}\right) \exp \frac{g(p)}{p}.$$

Thus

$$\lim_{s \to +0} s \cdot \sum_{n=1}^{\infty} \frac{g(n)}{n^{1+s}} = \lim_{s \to +0} \frac{\sum_{n=1}^{\infty} \frac{g(n)}{n^{1+s}}}{\zeta(1+s)} = L(g).$$

From a well-known Tauberian theorem of G. H. Hardy and J. E. Littlewood [5] it now follows that

$$\lim_{N \to +\infty} \frac{\sum_{n=1}^{N} \frac{g(n)}{n}}{\log N} = L(g). \tag{1.1}$$

Thus the logarithmic mean value of the sequence $\{g(n)\}_1^\infty$ exists.

We shall need the identity

$$g(n) \log n = \sum_{p \mid n} g(p) g\left(\frac{n}{p}\right) \log p, \tag{1.2}$$

which we shall easily deduce from the definition of an exponentially multiplicative function.

As a matter of fact, if the canonical product representation of n is

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r},$$

then

$$g(n) \log n = \sum_{j=1}^{r} g(n) k_j \log p_j, \tag{1.3}$$

and for each j we have the formula

$$k_j g(n) = k_j g(p_j^{k_j}) g\left(\frac{n}{p_j^{k_j}}\right) = g(p_j) g\left(\frac{n}{p_j}\right). \tag{1.4}$$

From (1.3) and (1.4) we obtain (1.2). Let us now put

$$G(N) = \frac{1}{N} \sum_{n=1}^{N} g(n), \qquad G^*(N) = \frac{1}{N \log N} \sum_{n=1}^{N} g(n) \log n.$$

Then

$$G(N) - G^*(N) = \Delta(N) = \frac{\sum_{n=1}^{N} G(n) n \log\left(1 + \frac{1}{n}\right)}{N \log N}. \tag{1.5}$$

We shall show now that

$$\lim_{N \to \infty} \Delta(N) = 0. \tag{1.6}$$

If we write $H(N) = \sum_{n=1}^{N} \frac{g(n)}{n}$, then

$$G(N) = H(N) - \frac{1}{N} \sum_{n=1}^{N-1} H(n). \tag{1.7}$$

Now (1.1) can be written in the form

$$\lim_{N \to \infty} \frac{H(N)}{\log N} = L(g), \tag{1.8}$$

and it follows from (1.8) that

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N-1} H(n)}{N \log N} = L(g). \tag{1.9}$$

Combining (1.7), (1.8), and (1.9), we conclude that

$$\lim_{N \to \infty} \frac{G(N)}{\log N} = 0. \tag{1.10}$$

From (1.5) and (1.10) it follows immediately that (1.6) holds.

From the identity (1.2) we deduce that

$$G(N) = \frac{1}{\log N} \sum_{p \leq N} \frac{g(p) \log p}{p} G\left(\left[\frac{N}{p}\right]\right) + \Delta(N), \tag{1.11}$$

where $\Delta(N)$ tends to 0 as $N \to +\infty$ ($[x]$ denotes the integral part of x). We shall now show that (1.11) implies that

$$\lim_{N \to \infty} G(N) = L(g), \tag{1.12}$$

which is the assertion of Theorem 3. To prove (1.12), let us put

$$A = \liminf_{N \to \infty} G(N), \qquad B = \limsup_{N \to \infty} G(N).$$

It is easy to see that

$$\frac{H(N)}{\log N} = \frac{G(N)}{\log N} + \frac{\sum_{m=1}^{N-1} \frac{G(m)}{m+1}}{\log N}. \tag{1.13}$$

Taking into account (1.8) and (1.10), we see that $0 \leq A \leq L(g) \leq B \leq +\infty$. Thus if (1.12) were false, at least one of the inequalities $B > L(g)$ and $A < L(g)$ would hold. We shall show that either of these inequalities leads to a contradiction, and thus we shall establish (1.12).

In what follows, $C_1, C_2, \cdots$ will denote positive constants. Let us suppose first that $B = +\infty$. Let $\{N_k\}$ be a sequence of natural numbers such that

$$\lim_{k \to +\infty} G(N_k) = +\infty \quad \text{and} \quad G(N_k) > G(n) \quad \text{for all } n < N_k.$$

Let us choose a number ε $(0 < \varepsilon < 1/2)$, and let $\Gamma_k(\varepsilon)$ denote the set of those natural numbers n in the interval $1 \leq n \leq N_k$ for which $G(n) \leq G(N_k)(1-\varepsilon)$. Putting $\operatorname{Max}_N |\Delta(N)| = C_1$, we obtain the inequality

$$G(N_k) \leq \frac{G(N_k)(1-\varepsilon)}{\log N_k} \sum_{\left[\frac{N_k}{p}\right] \in \Gamma_k(\varepsilon)} \frac{g(p)\log p}{p} + \frac{G(N_k)}{\log N_k} \sum_{\left[\frac{N_k}{p}\right] \notin \Gamma_k(\varepsilon)} \frac{g(p)\log p}{p} + C_1 .$$

It follows that

$$\sum_{\left[\frac{N_k}{p}\right] \in \Gamma_k(\varepsilon)} \frac{g(p)\log p}{p} \leq \frac{1}{\varepsilon}\left(\sum_{p \leq N_k} \frac{g(p)\log p}{p} - \log N_k \right) + \frac{C_1 \log N_k}{\varepsilon\, G(N_k)} . \tag{1.14}$$

As is well known, it follows from the prime number theorem that

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + \alpha + o(1),$$

where α is a constant; together with the convergence of the series (0.2), this implies that

$$\sum_{p \leq x} \frac{(g(p)-1)\log p}{p} = o(\log x). \tag{1.15}$$

Thus we obtain the estimate

$$\sum_{p \leq x} \frac{g(p)\log p}{p} \sim \log x . \tag{1.16}$$

Therefore we can find a natural number $k_1(\varepsilon)$ such that

$$\left| \sum_{p \leq N_k} \frac{g(p)\log p}{p} - \log N_k \right| \leq \varepsilon \cdot \delta\left(\frac{\varepsilon}{1-2\varepsilon}\right) \log N_k \quad \text{for } k \geq k_1(\varepsilon). \tag{1.17}$$

As $G(N_k) \to \infty$, we can by (1.14) and (1.16) choose $k_2(\varepsilon)$ so that

$$\sum_{\left[\frac{N_k}{p}\right] \in \Gamma_k(\varepsilon)} \frac{g(p)\log p}{p} \leq C_2 \cdot \delta\left(\frac{\varepsilon}{1-2\varepsilon}\right) \log N_k \qquad \text{for } k \geq k_2(\varepsilon), \tag{1.18}$$

where (for example) we can choose $C_2 = 2$.

We shall now show that (1.18) holds also (with some appropriate value of the constant C_2) if $B > L(g)$ is a finite number.

Let $\{N_k\}$ denote a sequence such that $G(N_k) \to B$. Let us choose $n_0(\varepsilon)$ so that

$$G(n) \leq B\left(1 + \varepsilon \cdot \delta\left(\frac{\varepsilon}{1-2\varepsilon}\right)\right) \qquad \text{for } n \geq n_0(\varepsilon),$$

and let us put $\sup_n G(n) = C_3$. Let us further choose $k_3(\varepsilon)$ so that

$$|\Delta(N_k)| \leq \varepsilon \cdot \delta\left(\frac{\varepsilon}{1-2\varepsilon}\right) \qquad \text{for } k \geq k_3(\varepsilon).$$

We obtain the inequality

$$G(N_k) \leq \frac{C_3}{\log N_k} \sum_{\frac{N_k}{p} < n_0(\varepsilon)} \frac{g(p)\log p}{p} + \frac{B\left(1+\varepsilon\cdot\delta\left(\frac{\varepsilon}{1-2\varepsilon}\right)\right)}{\log N_k} \sum_{\left[\frac{N_k}{p}\right] \notin \Gamma_k(\varepsilon)} \frac{g(p)\log p}{p}$$
$$+ \frac{G(N_k)(1-\varepsilon)}{\log N_k}\left(\sum_{\left[\frac{N_k}{p}\right] \in \Gamma_k(\varepsilon)} \frac{g(p)\log p}{p}\right) + \varepsilon\cdot\delta\left(\frac{\varepsilon}{1-2\varepsilon}\right) \qquad \text{for } k \geq k_3(\varepsilon).$$

Now we choose $k_4(\varepsilon)$ so that

$$G(N_k) \geq B\left(1 - \varepsilon\cdot\delta\left(\frac{\varepsilon}{1-2\varepsilon}\right)\right) \quad \text{and} \quad N_k \geq n_0(\varepsilon) \qquad \text{for } k \geq k_4(\varepsilon).$$

Then, for $k \geq k_5(\varepsilon) = \max(k_3(\varepsilon), k_4(\varepsilon))$, we have the inequality

$$\sum_{\left[\frac{N_k}{p}\right] \in \Gamma_k(\varepsilon)} \frac{g(p)\log p}{p} \leq \left(2+\frac{1}{B}\right)\delta\left(\frac{\varepsilon}{1-2\varepsilon}\right)\log N_k + \frac{1}{\varepsilon}\left(\sum_{p<N_k} \frac{g(p)\log p}{p} - \log N_k\right)$$
$$+ \frac{C_3}{B\varepsilon} \sum_{\frac{N_k}{n_0(\varepsilon)} < p < N_k} \frac{g(p)\log p}{p}.$$

Since the quantity

$$\sum_{\frac{N_k}{n_0(\varepsilon)} < p < N_k} \frac{\log p}{p}$$

is bounded (by a constant depending on ε), it follows from (1.15) that

$$\sum_{\frac{N_k}{n_0(\varepsilon)} < p < N_k} \frac{g(p)\log p}{p} = o(\log N_k).$$

Thus we can find a $k_6(\varepsilon)$ such that for $k \geq k_6(\varepsilon)$

$$\sum_{\frac{N_k}{n_0(\varepsilon)} < p < N_k} \frac{g(p)\log p}{p} \leq \frac{B\varepsilon}{C_3}\,\delta\left(\frac{\varepsilon}{1-2\varepsilon}\right).$$

Thus, by (1.17), if $k \geq k_7(\varepsilon) = \max(k_1(\varepsilon), k_5(\varepsilon), k_6(\varepsilon))$, then (1.18) holds also for the case where B is finite, if we choose the constant C_2 sufficiently large (in fact, $C_2 = 4 + 1/B$ is sufficient).

Let us now call those values of $n \le N_k$ for which $n \in \Gamma_k(\varepsilon)$, that is, for which $G(n) \le (1-\varepsilon)G(N_k)$, ε-*bad* values. Clearly if n is 2ε-bad and $\frac{1-2\varepsilon}{1-\varepsilon} n \le n' < n$, then

$$G(n') \le \frac{n}{n'} G(n) \le \left(\frac{1-\varepsilon}{1-2\varepsilon}\right)(1-2\varepsilon)G(N_k) = (1-\varepsilon)G(N_k),$$

and thus n' is ε-bad. Let $n_1, n_2, \cdots, n_Z$ denote all 2ε-bad values of n. Let us consider the sum

$$S_k(\varepsilon) = \sum_{n_j \in \Gamma_k(2\varepsilon)} \frac{1}{n_j} \sum_{\frac{1-2\varepsilon}{1-\varepsilon} n_j \le \left[\frac{N_k}{p}\right] < n_j} \frac{g(p)\log p}{p}.$$

Since all values $\left[\frac{N_k}{p}\right]$ lying in an interval $\left[\frac{1-2\varepsilon}{1-\varepsilon} n_j, n_j\right)$ are ε-bad, it is evident that

$$S_k(\varepsilon) \le \sum_{\left[\frac{N_k}{p}\right] \in \Gamma_k(\varepsilon)} \left(\frac{g(p)\log p}{p} \sum_{\frac{N_k}{p} < n < \frac{N_k}{p}\left(\frac{1-\varepsilon}{1-2\varepsilon}\right)} \frac{1}{n}\right).$$

Clearly, the sum over n in the right-hand member is less than $C_4\varepsilon$, and therefore

$$S_k(\varepsilon) \le C_4\varepsilon \sum_{\left[\frac{N_k}{p}\right] \in \Gamma_k(\varepsilon)} \frac{g(p)\log p}{p}. \tag{1.19}$$

On the other hand,

$$S_k(\varepsilon) = \sum_{n_j \in \Gamma_k(2\varepsilon)} \left(\frac{1}{n_j} \sum_{\frac{N_k}{n_j} < p < \frac{N_k}{n_j}\left(1+\frac{\varepsilon}{1-2\varepsilon}\right)} \frac{g(p)\log p}{p}\right),$$

and thus, by our supposition (0.6), if k is sufficiently large, then

$$S_k(\varepsilon) \ge \delta\left(\frac{\varepsilon}{1-2\varepsilon}\right) \cdot \sum_{n_j \in \Gamma_k(2\varepsilon), n_j \le \frac{N_k}{N(\varepsilon)}} \frac{1}{n_j}. \tag{1.20}$$

Comparing (1.18), (1.19), and (1.20), we obtain the estimate

$$\sum_{n_j \in \Gamma_k(2\varepsilon), n_j < \frac{N_k}{N(\varepsilon)}} \frac{1}{n_j} \le \frac{C_4\varepsilon}{\delta\left(\frac{\varepsilon}{1-2\varepsilon}\right)} C_2\,\delta\left(\frac{\varepsilon}{1-2\varepsilon}\right)\log N_k = C_5\varepsilon\log N_k.$$

It follows by the definition of the set $\Gamma_k(2\varepsilon)$ that for sufficiently large k

$$(1.21)\quad \sum_{n=1}^{N_k} \frac{G(n)}{n} \geq \left(\sum_{1}^{\left[\frac{N_k}{N(\varepsilon)}\right]} \frac{1}{n} - C_5 \varepsilon \log N_k \right)(1-\varepsilon) G(N_k) \geq (1 - C_6 \varepsilon) G(N_k) \log N_k .$$

In view of (1.8), (1.10), and (1.13), we have however the relation

$$(1.22)\qquad \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{G(n)}{n} = L(g).$$

Comparing (1.21) and (1.22), we conclude that

$$L(g) \geq (1 - C_6 \varepsilon) B,$$

which is impossible of $B = +\infty$, and also if B is finite and $B > L(g)$, provided that ε is chosen sufficiently small. This contradiction proves that $B = L(g)$.

Now we prove that the assumption $A < L(g)$ also leads to a contradiction. The proof is similar to that given above for the impossibility of $B > L(g)$, but is somewhat simpler because $0 \leq A \leq L(g)$ and thus A is always finite, wherefore we need not distinguish between two cases (as before between $B = +\infty$ and $B < +\infty$).

Let $\{N_k\}$ be a sequence of natural numbers such that $\lim_{k\to\infty} G(N_k) = A$. Let $\gamma_k(\varepsilon)$ denote the set of those integers n $(1 \leq n \leq N_k)$ for which $G(n) > (1+\varepsilon)A$, and let us call the values n belonging to $\gamma_k(\varepsilon)$ the ε-*bad* values. We choose $k_9(\varepsilon)$ so large that

$$G(N_k) \leq A\left(1 + \varepsilon \cdot \delta\left(\frac{\varepsilon}{1+\varepsilon}\right)\right) \quad \text{and} \quad |\Delta(N_k)| \leq \varepsilon \cdot \delta\left(\frac{\varepsilon}{1+\varepsilon}\right)$$

for $k \geq k_9(\varepsilon)$, and $D(\varepsilon)$ so that

$$G(n) \geq A\left(1 - \varepsilon \cdot \delta\left(\frac{\varepsilon}{1+\varepsilon}\right)\right) \qquad \text{for } n \geq D(\varepsilon).$$

It follows from (1.11) that if $k \geq k_{10}(\varepsilon)$, then

$$\sum_{\left[\frac{N_k}{p}\right] \in \gamma_k(\varepsilon)} \frac{g(p)\log p}{p} \leq C_7\, \delta\left(\frac{\varepsilon}{1+\varepsilon}\right) \log N_k .$$

Now if n is 2ε-bad and $n \leq n' \leq \left(\frac{1+2\varepsilon}{1+\varepsilon}\right) n$, then

$$G(n') \geq \frac{n}{n'} G(n) \geq \left(\frac{1+\varepsilon}{1+2\varepsilon}\right)(1+2\varepsilon)A = (1+\varepsilon)A ,$$

and thus n' is ε-bad. For the sum

$$\overline{S}_k(\varepsilon) = \sum_{n_j \in \gamma_k(2\varepsilon)} \left(\frac{1}{n_j} \sum_{n_j < \frac{N_k}{p} < n_j\left(\frac{1+2\varepsilon}{1+\varepsilon}\right)} \frac{g(p)\log p}{p} \right)$$

we obtain the inequalities

$$\bar{S}_k(\varepsilon) \leq C_8 \varepsilon \sum_{\left[\frac{N_k}{p}\right] \epsilon \gamma_k(\varepsilon)} \frac{g(p) \log p}{p}$$

and

$$\bar{S}_k(\varepsilon) \geq \delta \left(\frac{\varepsilon}{1+\varepsilon}\right) \sum_{\substack{n_j \epsilon \gamma_k(2\varepsilon) \\ n_j < \frac{N_k(1+\varepsilon)}{(1+2\varepsilon)N(\varepsilon)}}} \frac{1}{n_j},$$

and thus it follows that

$$\sum_{\substack{n_j \epsilon \gamma_k(2\varepsilon) \\ n_j < \frac{N_k(1+\varepsilon)}{(1+2\varepsilon)N(\varepsilon)}}} \frac{1}{n_j} \leq C_9 \varepsilon \log N_k .$$

This implies (since from the first part of the proof we already know that $G(n)$ is bounded) that for $k \geq k_{11}(\varepsilon)$

$$\frac{\sum_{n=1}^{N_k} \frac{G(n)}{n}}{\log N_k} \leq (1+\varepsilon)A + C_{10}\varepsilon,$$

and thus, in view of (1.22),

$$L(g) \leq (1+\varepsilon)A + C_{10}\varepsilon .$$

This contradicts the inequality $L(g) > A$, if ε is sufficiently small. Thus $L(g) > A$ is impossible, and Theorem 3 is proved.

Let us mention that the condition (0.6) is certainly satisfied if $g(p)$ has a positive lower bound.

2. STRONGLY MULTIPLICATIVE FUNCTIONS

Corresponding to two number-theoretical functions $g_1 = g_1(n)$ and $g_2 = g_2(n)$ we define the function $g_3 = g_1 * g_2$ (called the *convolution* of g_1 and g_2) by putting

$$g_3(n) = \sum_{d|n} g_1(d) g_2\left(\frac{n}{d}\right).$$

Clearly, if any two of g_1, g_2, g_3 are multiplicative, so is the third, and

$$\sum_{n=1}^{\infty} \frac{g_3(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{g_1(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g_2(n)}{n^s}\right). \tag{2.1}$$

We shall use the following simple lemma, which is due to A. Wintner [8]:

LEMMA. *If* $g_3 = g_1 * g_2$, *and if further the mean value* $M(g_1)$ *of* g_1 *exists and the series*

$$\sum_{n=1}^{\infty} \frac{|g_2(n)|}{n} \tag{2.2}$$

converges, then $M(g_3)$ *exists and*

$$M(g_3) = M(g_1) \sum_{n=1}^{\infty} \frac{g_2(n)}{n}.$$

Proof of Theorem 2. Let $g_1(n)$ denote the exponentially multiplicative function whose values for primes are the same as those of $g(n)$. Then by virtue of (2.1),

$$g(n) = g_1(n) * g_2(n),$$

where

$$\sum_{n=1}^{\infty} \frac{g_2(n)}{n^s} = \prod_p \left(1 + \sum_{\ell=2}^{\infty} \frac{\sum_{k=0}^{\ell-1} \frac{(-1)^k g(p)^{k+1}}{k!} + \frac{(-1)^\ell g(p)^\ell}{\ell\,!}}{p^{\ell s}}\right).$$

It follows from the convergence of the series (0.5) that the series (2.2) converges; thus by our lemma $M(g)$ exists and (0.3) holds. This proves Theorem 2.

Clearly the conditions (0.5) and (0.6) of Theorem 2 are satisfied if $g(p)$ has a positive lower bound and is also bounded from above.

Deduction of Theorem 1 *from Theorem* 2. Suppose that $g(n)$ satisfies the conditions of Theorem 1, and define the strongly multiplicative function $g_1(n)$ by putting

$$g_1(p) = \begin{cases} g(p) & \text{if } g(p) \geq 1/2, \\ 1 & \text{if } g(p) < 1/2. \end{cases}$$

Since the series (0.4) is supposed to converge, it follows that if S denotes the set of those primes p for which $g(p) < 1/2$, then the series

$$\sum_{p \in S} \frac{1}{p}$$

converges. Putting $g(n) = g_1(n) * g_2(n)$, we see that

$$\sum \frac{g_2(n)}{n^s} = \prod_{p \in S} \left(1 + \frac{g(p) - 1}{p^s}\right).$$

Thus the series

$$\sum \frac{|g_2(n)|}{n}$$

is convergent.

Clearly $g_1(n)$ satisfies the conditions of Theorem 2, because

$$\sum \frac{g_1^2(p)}{p^2} = \sum_{p \in S} \frac{1}{p^2} + \sum \frac{g^2(p)}{p^2} \leq \sum \frac{2(g(p) - 1)^2 + 3}{p^2}.$$

It follows that $M(g_1)$ exists; thus Lemma 1 can be applied, and we conclude that $M(g)$ exists and (0.3) holds. This proves Theorem 1.

3. GENERAL MULTIPLICATIVE FUNCTIONS

THEOREM 4. *Let* $g(n)$ *be a nonnegative multiplicative function for which the series* (0.2) *and the series*

$$\sum_p \frac{g^2(p)}{p^2} + \sum_p \sum_{k=2}^{\infty} \frac{g(p^k)}{p^k} \tag{3.1}$$

converge, and suppose that to every positive ε *there correspond positive constants* $\delta(\varepsilon)$ *and* $N(\varepsilon)$ *such that condition* (0.6) *is satisfied. Then* $M(g)$ *exists and*

$$M(g) = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{g(p^k) - g(p^{k-1})}{p^k}\right). \tag{3.2}$$

Proof. We deduce Theorem 4 from Theorem 2 by using again the lemma in Section 2. Let $g_1(n)$ be the exponentially multiplicative function that takes the same value as $g(n)$, when n is prime, and put $g = g_1 * g_2$. Then clearly g_1 satisfies the conditions of Theorem 2, and thus $M(g_1)$ exists. Further,

$$\sum_{n=1}^{\infty} \frac{g_2(n)}{n^s} = \prod_p \left(1 + \sum_{k=2}^{\infty} \frac{\sum_{\ell=0}^{k} g(p^\ell)(-1)^{k-\ell} \frac{g^{k-\ell}(p)}{(k-\ell)!}}{p^{ks}}\right).$$

Thus it follows from the convergence of the series (3.1) that

$$\sum \frac{|g_2(n)|}{n} < \infty;$$

therefore the lemma can be applied, and the existence of $M(g)$ and the validity of (3.2) follow.

COROLLARY. *Let* $g(n)$ *be a nonnegative multiplicative function, and suppose there exist positive constants* a *and* b *such that*

$$g(p) \geq a \quad \textit{and} \quad g(p^k) \leq b$$

for all primes p *and for* $k = 1, 2, \cdots$. *Suppose further that the series* (0.2) *converges. Then* $M(g)$ *exists and* (3.2) *holds.*

4. MULTIPLICATIVE FUNCTIONS WHOSE MEAN VALUE IS 0 OR $+\infty$

The condition in Theorem 2 that the series (0.5) should converge is necessary for the convergence (to a positive limit) of the infinite product on the right-hand side of (0.3). As a matter of fact, it is easy to see that if the series (0.2) converges and the series (0.5) diverges, then

$$\lim_{x \to +\infty} \prod_{p \leq x} \left(1 + \frac{g(p) - 1}{p}\right) = 0.$$

Thus one is led to the conjecture that if all the other conditions of Theorem 2 are satisfied but the series (0.5) diverges, then $M(g) = 0$. Similarly, if for instance $g(n)$ is completely multiplicative and the series (0.5) diverges, then the product on the right of (3.2) diverges to $+\infty$, and thus one is again led to the conjecture that if all the other conditions of Theorem 4 are satisfied for a completely multiplicative function but the series (3.1) diverges, then $M(g) = +\infty$. Both these conjectures are true, as is shown by the following theorems.

THEOREM 5. *Let* $g(n)$ *be a nonnegative, completely multiplicative function such that the series* (0.2) *converges and the series* (0.5) *diverges. Suppose further that condition* (0.6) *holds. Then* $M(g) = +\infty$, *that is,*

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} g(n) = +\infty. \tag{4.1}$$

THEOREM 6. *Let* $g(n)$ *be a nonnegative, strongly multiplicative function such that the series* (0.2) *converges and the series* (0.5) *diverges. Suppose that condition* (0.6) *holds. Then* $M(g) = 0$.

It should be mentioned that if under the conditions of Theorem 5 we put

$$E(p) = 1 + \sum_{k=1}^{\infty} \frac{g(p^k) - g(p^{k-1})}{p^k}, \tag{4.2}$$

then

$$E(p) = \frac{1 - \frac{1}{p}}{1 - \frac{g(p)}{p}}$$

and thus

$$\lim_{x \to \infty} \prod_{p < x} E(p) = \infty,$$

while under the conditions of Theorem 6

$$E(p) = 1 + \frac{g(p) - 1}{p}$$

and thus

$$\lim_{x \to +\infty} \prod_{p < x} E(p) = 0.$$

This shows that Theorems 2 and 6 can be combined and expressed in the form of the following single statement:

THEOREM 7. *If* g(n) *is a nonnegative and strongly multiplicative function such that the series* (0.2) *converges and condition* (0.6) *is fulfilled, then* M(g) *exists and*

$$M(g) = \lim_{x\to\infty} \prod_{p\leq x} \left(1 + \frac{g(p)-1}{p}\right), \tag{4.3}$$

this limit being positive or zero according to whether the series (0.5) *converges or diverges.*

Similarly, combining Theorem 5 with what follows from Theorem 4 for completely multiplicative functions, we obtain the following result.

THEOREM 8. *Let* g(n) *be a nonnegative and completely multiplicative function for which the series* (0.2) *converges and condition* (0.6) *is fulfilled; suppose further that* $g(p) < p$ *for all primes* p. *Then*

$$M(g) = \lim_{x\to\infty} \prod_{p<x} \frac{\left(1-\frac{1}{p}\right)}{\left(1-\frac{g(p)}{p}\right)},$$

this limit being either finite and positive or infinite according to whether the series (0.5) *converges or diverges.*

These theorems suggest the following conjecture: The mean value of a nonnegative multiplicative function g(n) exists if and only if the limit

$$\lim_{x\to\infty} \prod_{p<x} E(p)$$

exists (where E(p) is defined by (4.2)), and the two are equal whenever they exist, including the case where the limit is $+\infty$.

Proof of Theorem 5. Let g(n) be a completely multiplicative function satisfying the conditions of Theorem 5. Let $g_1(n)$ be the exponentially multiplicative function for which $g_1(p) = g(p)$ for all primes. Then, by Theorem 3, $M(g_1)$ exists and is positive. Now clearly if

$$n = p_1^{\alpha_1}\cdots p_r^{\alpha_r} p_{r+1}\cdots p_s,$$

where $\alpha_i \geq 2$ $(i = 1, 2, \cdots, r)$, then

$$g(n) = \frac{1}{r}\sum_{i=1}^{r} g(p_i^2)\, g\left(\frac{n}{p_i^2}\right).$$

Since however $1/r \geq 1/2^{r-1} \geq \alpha_i!/\prod_{j=1}^{r} \alpha_j!$, it follows that

$$g(n) \geq \sum_{i=1}^{r} g^2(p_i)\, g_1\left(\frac{n}{p_i^2}\right),$$

and thus, for any fixed $D > 0$,

$$\frac{1}{N}\sum_{n=1}^{N} g(n) \geq \frac{1}{N}\sum_{p\leq D}\left(g(p^2) \sum_{m\leq \frac{N}{p^2}} g_1(m)\right).$$

However, for each fixed $p \leq D$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m\leq\frac{N}{p^2}} g_1(m) = \frac{M(g_1)}{p^2},$$

and therefore

$$\liminf_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} g(n) \geq M(g_1) \sum_{p\leq D} \frac{g^2(p)}{p^2} .$$

Since this holds for arbitrarily large values of D and the series (0.5) diverges, by hypothesis, it follows that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g(n) = +\infty ,$$

and thus Theorem 5 is proved.

Proof of Theorem 6. Let $g(n)$ be a strongly multiplicative function satisfying the conditions of Theorem 6. Let $g_1(n)$ be the exponentially multiplicative function for which $g_1(p) = g(p)$, and $g_0(n)$ the multiplicative function defined by

$$g_0(p) = g(p) \quad \text{and} \quad g_0(p^k) = 0 \quad \text{for } k \geq 2 .$$

First we prove that $M(g_0)$ exists and is equal to 0. To show this, let $g^{(D)}(n)$ be the multiplicative function defined as follows:

$$g^{(D)}(p) = g(p) \quad \text{for all primes } p ,$$

$$g^{(D)}(p^2) = 0 \quad \text{for } p \leq D,$$

$$g^{(D)}(p^k) = g_1(p^k) \text{ if } p \leq D \text{ and } k \geq 3 \text{ or } p > D \text{ and } k \geq 2 .$$

Clearly, for all n and D,

$$g_0(n) \leq g^{(D)}(n) \leq g_1(n) .$$

For each square-free integer d, let us further define $g_1^{(d)}(n)$ as the exponentially multiplicative function for which $g_1^{(d)}(p) = 0$ if $p \mid d$ and $g_1^{(d)}(n) = g_1(n)$ if n is relatively prime to d. Then

$$\frac{1}{N}\sum_{n=1}^{N} g^{(D)}(n) = \frac{1}{N}\sum_{n=1}^{N} g_1(n) - \sum_{p\le D}\left(\frac{g(p^2)}{2}\sum_{m\le\frac{N}{p^2}} g_1^{(p)}(m)\right)$$

$$+ \sum_{p\le D,\, q\le D,\, p\ne q}\left(\frac{g(p^2)g(q^2)}{4}\sum_{m\le\frac{N}{p^2q^2}} g_1^{(pq)}(m)\right) - \cdots .$$

For each square-free d, $M(g_1^{(d)})$ clearly exists by Theorem 3, and

$$M(g_1^{(d)}) = M(g_1)\exp\left(-\sum_{p|d}\frac{g(p)}{p}\right);$$

it follows that $M(g^{(D)})$ exists and

$$M(g^{(D)}) = \prod_{p\le D}\left(1 - \frac{g(p^2)}{2p^2}\exp\left(-\frac{g(p)}{p}\right)\right) M(g_1).$$

Since

$$\lim_{D\to\infty}\prod_{p\le D}\left(1 - \frac{g(p^2)}{2p^2}\exp\left(-\frac{g(p)}{p}\right)\right) = 0,$$

it follows that $M(g_0) = 0$. Using the lemma in Section 2, we easily see that $M(g) = 0$. Thus Theorem 6 is proved.

In all our theorems we have supposed that the series (0.2) converges. If this condition is dropped, then $M(g)$ does not exist in general. As a matter of fact, it is easy to construct multiplicative (and strongly multiplicative) functions, bounded by positive constants both from below and above, for which the series (0.2) diverges and the logarithmic means

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{g(n)}{n}$$

oscillate between different upper and lower limits. For this purpose it is sufficient to put

$$g(p) = \begin{cases} 1/2 & (n_{2k}\le p < n_{2k+1}), \\ 2 & (n_{2k+1}\le p < n_{2k+2}), \end{cases}$$

for some rapidly increasing sequence $\{n_k\}$. It is well known that

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g(n) \le \liminf_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{g(n)}{n}$$

$$\le \limsup_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{g(n)}{n} \le \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g(n),$$

and it follows that $M(g)$ does not exist.

We can also give a counterexample showing that the convergence of the series (0.2) alone does not ensure the existence of $M(g)$, not even if the values 0 and $+\infty$ are admitted, and not even if $g(p)$ has a positive lower bound. Let p_k denote the least prime p for which

$$p \geq e^{e^{\sqrt{k}}},$$

and let

$$g(p_k) = \frac{p_k}{4\sqrt{k}}, \qquad g(p) = \frac{1}{2} \quad (p \notin \{p_k\}).$$

Then

$$\sum_{e^{e^{\sqrt{k}}} < p < e^{e^{\sqrt{k+1}}}} \frac{g(p) - 1}{p} = O(k^{-3/2}),$$

and thus the series (0.2) converges. Let us now put

$$g(p^2) = 0 \qquad (p \notin \{p_k\}),$$

$$g(p_k^2) = \begin{cases} \frac{3}{4}\, g^2(p_k) & (n_{2j} \leq k < n_{2j+1}), \\ \frac{1}{4}\, g^2(p_k) & (n_{2j+1} \leq k < n_{2j+2}), \end{cases}$$

$$g(p^\ell) = g(p^2) \qquad (p \text{ prime}, \ \ell \geq 3).$$

Then the series $\sum g(p_k^2)/p_k^2$ diverges, and Theorem 4 is not applicable.

It is easy to see that the product

$$\prod_{p<x} E(p) = \prod_{p<x} \left(1 + \frac{g(p) - 1}{p} + \frac{g(p^2) - g(p)}{p^2}\right)$$

oscillates between 0 and ∞, if the sequence $\{n_j\}$ increases fast enough. It follows that the logarithmic mean values

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{g(n)}{n}$$

and thus the arithmetic mean values

$$\frac{1}{N} \sum_{n=1}^{N} g(n)$$

do the same. In our example $g(p)$ is bounded from below, and thus condition (0.6) is of course fulfilled.

Remark. After finishing this paper, we were informed by E. Wirsing that he has constructed a nonnegative multiplicative function g(n) for which the series (0.2) and (3.1) converge but M(g) does not exist. This example shows that the condition (0.6) in Theorem 4 is necessary. By modifying slightly the example one can show that the condition is necessary also in Theorems 2 and 3. Wirsing's example concerning Theorem 4 is as follows: Let p_k be defined as the least prime greater than e^k, and let

$$g(p_k) = \frac{e^k}{k} \qquad (k = 1, 2, \cdots),$$

$$g(p) = 0 \qquad (p \notin \{p_k\}),$$

$$g(p^\ell) = 0 \qquad (p \text{ prime}, \ \ell \geq 2).$$

Wirsing further proved that if g(n) is a nonnegative multiplicative function for which $g(p^k)$ is bounded from above, then the convergence of the series (0.2) alone is sufficient for the existence of the mean value M(g). His results will be published in a forthcoming paper.

REFERENCES

1. J. G. van der Corput, *Demonstration élémentaire du théorème sur la distribution des nombres premiers*, Math. Centrum Amsterdam, Scriptum no. 1 (1948), 32 pp.

2. H. Delange, *Sur les fonctions arithmetiques multiplicatives*, Ann. Sci. École Norm. Sup. (3) 78 (1961), 273-304.

3. P. Erdős, *Some asymptotic formulas for multiplicative functions*, Bull. Amer. Math. Soc. 53 (1947), 536-544.

4. ———, *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem*, Proc. Nat. Acad. Sci. U.S.A. 35 (1949), 374-384.

5. G. H. Hardy and J. E. Littlewood, *Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive*, Proc. London Math. Soc. (2) 13 (1914), 174-191.

6. A Renyi, *A new proof of a theorem of Delange*, Publ. Math. Debrecen (to appear).

7. A. Wintner, *Eratosthenian averages*, Waverly Press, Baltimore, 1943.

8. ———, *The theory of measure in arithmetical semi-groups*, Waverly Press, Baltimore, 1944.

The Mathematical Institute of the
Hungarian Academy of Sciences