Remarks on the Poisson process.

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The (inhomogeneous) Poisson process on the real line is usually characterised as a stochastic additive set function $\xi(E)$ defined for each bounded Borel subset E of the real line such that

a) the random variable $\xi(E)$ has for each bounded Borel set E a Poisson distribution, i.e.

$$(1)\quad P(\xi(E) = n) = \frac{[\lambda(E)]^n \cdot e^{-\lambda(E)}}{n!} \qquad (n = 0,1,\dots)$$

where $\lambda(E)$ is a nonatomic measure on the real line such that $\lambda(E)$ is finite for each finite interval E, and

b) if $E_1, E_2, \dots, E_n$ are mutually disjoint bounded Borel sets the random variables $\xi(E_1), \dots, \xi(E_n)$ are independent.

If we put $\xi_t = \xi([0,t))$ for $t > 0$, this means that ξ_t is a process with independent increments such that $\xi_t - \xi_s$ has a Poisson distribution with mean value $\Lambda(t) - \Lambda(s)$ where $\Lambda(t)$ is the λ-measure of the interval $[0,t)$ if $t > 0$ and $-\Lambda(t)$ is the λ-measure of the interval $[t,0)$ if $t < 0$. D.Szász (oral communication) asked the question whether there exists a point process for which a) holds but b) does not hold.

We shall show in this note that such a process does not exist, i.e. the usual supposition about independence in the above characterisation of the Poisson process is unnecessary, as it follows from the Poissonity of the distribution of $\xi(E)$; in other words we prove that the supposition b) is a consequence of the supposition a).
More exactly we prove the following

Theorem 1.
Let $\mathcal{J}$ denote the family of all subsets of the real line which can be obtained as the union of a finite number of disjoint finite intervals $[a,b)$ closed to the right and open to the left. Let $\xi(E)$ be an additive stochastic set function defined for each $E \in \mathcal{J}$, i.e. such that if E_1 and E_2 are disjoint one has $\xi(E_1 + E_2) = \xi(E_1) + \xi(E_2)$

Suppose that for each $E \in \mathcal{J}$ $\xi(E)$ has a Poisson distribution with mean value $\lambda(E)$ where $\lambda(E)$ is a nonatomic measure on the Borel subsets of the real line, which is finite for each $E \in \mathcal{J}$. Then it follows that if $E_1, \ldots, E_n$ are disjoint sets $(E_k \in \mathcal{J})$ the random variables $\xi(E_1), \ldots, \xi(E_n)$ are independent, i.e. $\xi(E)$ is a Poisson process.

Proof of theorem 1. Let $A(E)$ denote the event $\xi(E) = 0$. If E is the union of the disjoint sets $E_j \in \mathcal{J}$ $(j = 1,2,\ldots,n)$ then (*) clearly $A(E) = A(E_1) \ldots A(E_n)$ because $\xi(E) = \sum_{j=1}^{n} \xi(E_j)$ and thus $\xi(E) = 0$ iff $\xi(E_j) = 0$ for $j = 1,2,\ldots,n$
But by supposition

$$(2) \quad P(A(E)) = P(\xi(E) = 0) = e^{-\lambda(E)} = \prod_{j=1}^{n} e^{-\lambda(E_j)} = \prod_{j=1}^{n} P(A(E_j))$$

Thus it follows that if the sets $E_1, \ldots, E_n$ are disjoint, the events $A(E_1), \ldots, A(E_n)$ are independent.

Now let $^1A(E)$ be the indicator of the event $A(E)$. Let $E \in \mathcal{J}$ and $F \in \mathcal{J}$ be two disjoint sets. For any $\varepsilon > 0$ we can clearly decompose E into disjoint intervals E_i $(1 \leq i \leq n)$ and F into disjoint intervals such that

$$\max_i \lambda(E_i) < \varepsilon \qquad \text{and} \max_j \lambda(F_j) < \varepsilon$$

Now evidently $\xi(E) \neq \sum_{i=1}^{n} {}^1A(E_i)$ implies $\max_i \xi(E_i) \geq 2$

and $\xi(F) \neq \sum_{j=1}^{m} {}^1(A(F_j)$ implies

$$\max_j \xi(F_j) \geq 2.$$

On the other hand for any $B \in \mathcal{J}$

$$(3) \quad P(\xi(B) \geq 2) = \sum_{k=2}^{\infty} \frac{\lambda(B)^k e^{-\lambda(B)}}{k!} \leq \lambda^2(B)$$

Thus

$$(4a) \quad P(\xi(E) \neq \sum_{i=1}^{n} {}^1A(E_i)) \leq \sum_{i=1}^{n} \lambda^2(E_i) < \varepsilon \lambda(E)$$

(*) Here and in what follows the product of events denotes the joint occurrence of these events.

and

$$(4b) \qquad P(\xi(F) \neq \sum_{j=1}^{m} {}^{1}A(F_j)) \leq \sum_{j=1}^{m} \lambda^2(F_j) < \varepsilon \lambda(F)$$

This implies, as the sums

$$\sum_{i=1}^{n} {}^{1}A(E_i) \quad \text{and} \quad \sum_{j=1}^{m} {}^{1}A(F_j)$$

are independent, that $\xi(E)$ and $\xi(F)$ are independent too.

As a matter of fact it follows from (4a) and (4b) that for any n and m $(n,m = 0,1,2,\dots)$

$$(5) \qquad \left| P(\xi(E) = n, \xi(F) = m) - P(\xi(E) = n)\, P(\xi(F) = m) \right| \leq 2\varepsilon\lambda(E + F).$$

As $\varepsilon > 0$ can be chosen arbitrarly small, our statement follows. The independence of the variables $\xi(E_i)$ $(i = 1,2, \dots, r)$ with disjoint E_i and $r > 2$ is proved in exactly the same way. Thus our theorem is proved.

<u>Remark1.</u> Note that to prove the independence of $\xi(E_i)$ $(i=1,2, \dots, r)$ for $E_iE_j=\emptyset$ if $i \neq j$ we have not used the full supposition that for each $E \in \mathcal{F}$ $\xi(E)$ has a Poisson distribution, only that

$$(6a) \qquad P(\xi(E) = 0) = e^{-\lambda(E)}$$

and

$$(6b) \qquad P(\xi(E) \geq 2) = O(\lambda(E)) \qquad \text{if} \quad \lambda(E) \to 0$$

uniformly in E.

Thus even these suppositions imply that the process $\xi(E)$ is a process of independent increments. It is easy to show however that this together with (6a) and (6b) implies that $\xi(E)$ has a Poisson distribution.

Thus the following theorem is true.

<u>Theorem 2.</u>

Let $\mathcal{F}$ denote the family of all subsets of the real line which can be obtained as the union of a finite number of disjoint finite intervals $[a,b)$. Let $\xi(E)$ be an additive stochastic set function defined for $E \in \mathcal{F}$, i.e. such that if $E_1 \in \mathcal{F}$ and $E_2 \in \mathcal{F}$ are disjoint one has $\xi(E_1 + E_2) = \xi(E_1) + \xi(E_2)$. Suppose that $\xi(E)$ is for each

$E \in \mathcal{J}$ a non-negative integer valued random variable such that

$$(7a) \qquad P(\xi(E) = 0) = e^{-\lambda(E)} \qquad \text{and}$$

$$(7b) \qquad P(\xi(E) \geq 2) \leq \lambda(E)\,\delta(\lambda(E))$$

where $\delta(x)$ is an increasing positive function defined for $x > 0$ such that $\lim_{x\to 0} \delta(x)=0$ and $\lambda(E)$ a nonatomic measure on $\mathcal{J}$. Then it follows that $\xi(E)$ is a Poisson process, i.e. if E_i $(i=1,2, \dots , r)$ are disjoint sets, $E_i \in \mathcal{J}$ the random variables $\xi(E_i)$ $(i = 1,2, \dots ,r)$ are independent, and (1) holds.

Proof of theorem 2. Put for $E \in \mathcal{J}$

$$\varphi_E(u) = M(e^{iu\,\xi(E)}) \qquad (-\infty < u < +\infty) \qquad \text{then clearly}$$

$$(8) \qquad |\varphi_E(u)| \geq e^{-\lambda(E)} - |u|(1 - e^{-\lambda(E)}) > 0 \qquad \text{if}$$

$$|u| < \frac{1}{e^{\lambda(E)}-1}$$

Thus if

$$E = \sum_{i=1}^{r} E_i, \qquad \text{where } E_i \in \mathcal{J} \text{ and } E_iE_j = \emptyset \text{ if } i \neq j, \text{ then for}$$

$$|u| < \frac{1}{e^{\lambda(E)}-1} \qquad \text{we have}$$

$$(9) \qquad \varphi_E(u) = \prod_{i=1}^{r} \varphi_{E_i}(u) \neq 0 \qquad \text{and therefore}$$

$$(10) \qquad \log \varphi_E(u) = \sum_{i=1}^{r} \log \varphi_{E_i}(u). \qquad \text{As however}$$

$$(11) \qquad \varphi_{E_i}(u) = e^{-\lambda(E_i)} + e^{iu}(1 - e^{-\lambda(E_i)}) + O(\lambda(E_i)\,\delta(\lambda(E_i))) \qquad \text{we get}$$

$$(12) \qquad \log \varphi_{E_i}(u) = \lambda(E_i)(e^{iu} - 1) + O(\lambda(E_i)(\lambda(E_i) + \delta(\lambda(E_i))))$$

It follows that if $\lambda(E_i) < \varepsilon$ for $i = 1,2, \dots , r$

$$(13) \qquad \log \varphi_E(u) = \lambda(E)(e^{iu} - 1) + O(\varepsilon + \delta(\varepsilon))$$

that is, as $\varepsilon > 0$ can be chosen arbitrarily small,

$$(14) \qquad \varphi_E(u) = e^{\lambda(E)\,(e^{iu}-1)}$$

which implies that $\xi(E)$ has a Poisson distribution with mean $\lambda(E)$. Thus theorem 2 follows from theorem 1.

Remark 2. The proof can be carried over without any change to the discussion of a Poisson process in more than one dimension or even in an abstract space. Thus we obtain the following

Theorem 3.
Let X be any space, $\mathcal{J}$ a family of subsets of X and $\lambda(E)$ a non-negative finite valued set function defined on $\mathcal{J}$, such that

1) if $E_1 \in \mathcal{J}$, $E_2 \in \mathcal{J}$ and $E_1E_2 = \emptyset$, then $E_1 + E_2 \in \mathcal{J}$
2) $E_1 \in \mathcal{J}$, $E_2 \in \mathcal{J}$, $E_1E_2 = \emptyset$ then $\lambda(E_1 + E_2) = \lambda(E_1) + \lambda(E_2)$
3) There is a constant α with $0 < \alpha < 1$ such that for every $E \in \mathcal{J}$ with $\lambda(E) > 0$ there exists a subset F of E such that $F \in \mathcal{J}$, $E - F \in \mathcal{J}$ and

$$\alpha < \frac{\lambda(F)}{\lambda(E)} < 1 - \alpha$$

Let us suppose a stochastic set function is defined on $\mathcal{J}$, i.e. to every $E \in \mathcal{J}$ there corresponds a random variable $\xi(E)$ such that if $E_1 \in \mathcal{J}$, $E_2 \in \mathcal{J}$ and $E_1E_2 = \emptyset$ we have $\xi(E_1 + E_2) = \xi(E_1) + \xi(E_2)$ and $\xi(E)$ has a Poisson distribution with mean value $\lambda(E)$. Then the random variables $\xi(E_i)$ $(i=1,2,\ldots,r)$ are independent if the sets $E_i \in \mathcal{J}$ $(i = 1,2, \ldots, r)$ are disjoint, i.e. $\xi(E)$ is Poisson-process.

Note that condition 3) is not quite the same as that λ is nonatomic, because we did not suppose that $\mathcal{J}$ is a σ-algebra of sets.

Remark 3. The question arises whether the condition that the process should be one with independent increments can be deduced from other suppositions for other processes of independent increments too.

The most interesting case is that of the Wiener process; for this process one has the following (almost trivial) analogue of theorem 1.

Theorem 4.

Let ξ_t $(-\infty < t < +\infty)$ be a stochastic process such that $\xi_t - \xi_s$ is normally distributed with mean 0 and variance $c(t-s)$ $(c > 0)$ for $s < t$. Suppose further that if the intervals $[s_j, t_j)$ $(j = 1, 2, \ldots, r)$ are disjoint, any linear combination

$\sum_{j=1}^{r} b_j(\xi_{t_j} - \xi_{s_j})$ of the increments $\xi_{t_j} - \xi_{s_j}$ with real coefficients b_j is normally distributed. Then $\{\xi_t\}$ is the Wiener process, i.e. the random variables $\xi_{t_j} - \xi_{s_j}$ are independent if the intervals $[s_j, t_j)$ are disjoint.

Proof of theorem 4. Clearly putting for $I_k = [s_k, t_k)$ $\xi(I_k) = \xi_{t_k} - \xi_{s_k}$ $(k=1, 2)$ if I_1 and I_2 are adjacent intervals $(s_1 < t_1 = s_2 < t_2)$ $\xi(I_1 + I_2) = \xi(I_1) + \xi(I_2)$ and thus

$$M((\xi(I_1 + I_2))^2) = t_2 - s_1 = t_2 - s_2 + t_1 - s_1 =$$

$$= M(\xi^2(I_1)) + M(\xi^2(I_2))$$

and thus $M(\xi(I_1)\xi(I_2)) = 0$, i.e. $\xi(I_1)$ and $\xi(I_2)$ are uncorrelated. Now let I_1 and I_2 be arbitrary disjoint intervals

$$I_1 = [s_1, t_1), \quad I_2 = [s_2, t_2) \qquad \text{where} \quad s_1 < t_1 < s_2 < t_2$$

and put $I_3 = [t_1, s_2)$. Then, taking into account that

$$M(\xi(I_1)\xi(I_3)) = 0 \quad \text{and} \quad M(\xi(I_3)\xi(I_2)) = 0, \quad \text{we get}$$

$$M(\xi^2(I_1 + I_2 + I_3)) = t_2 - s_1 = M(\xi^2(I_1)) + M(\xi^2(I_2)) + M(\xi^2(I_3)) + $$
$$+ 2M(\xi(I_1)\xi(I_2))$$

Thus $$M(\xi(I_1)\xi(I_2)) = 0$$

(We have used here the following elementary geometrical fact: if a,b,c are vectors in the 3-dimensional Euclidean space for which c is orthogonal both to b and to a + b, then c is orthogonal to a too.)

Thus $\xi(I_1)$ and $\xi(I_2)$ are uncorrelated if I_1 and I_2 are arbitrary disjoint intervals.

It follows that if $I_1, I_2, \ldots, I_r$ are disjoint intervals and I_j has length $|I_j|$, and $b_1, \ldots, b_r$ are arbitrary real constants then

$$M\Big(\Big(\sum_{j=1}^{r} b_j\,\xi(I_j)\Big)^2\Big) = \sum_{j=1}^{r} b_j^2\; c\,|I_j|$$

Thus
$$M\Big(e^{iu\sum_{j=1}^{r} b_j\,\xi(I_j)}\Big) = e^{-\frac{1}{2}u^2\sum_{j=1}^{r} b_j^2\, c\,|I_j|}$$

and thus for any real numbers $u_1, u_2, \dots, u_r$

$$M\Big(e^{i\sum_{j=1}^{r} u_j\,\xi(I_j)}\Big) = \prod_{j=1}^{r} M\big(e^{iu_j\,\xi(I_j)}\big)$$

i. e. the $\xi(I_j)$ $(j = 1,2, \dots, r)$ are independent, i.e. ξ_t is the Wiener-process. (Note that it would have been sufficient to suppose that

$\sum_{j=1}^{r} b_j\,\xi(I_j)$ is normally distributed for $\sum_{j=1}^{r} b_j^2 = 1$.)

Remark 4. Returning to the Poisson-process, the question arises, whether if in theorem 1 instead of the condition that $\xi(E)$ has a Poisson distribution if E is any finite union of intervals, one supposes only that $\xi(I)$ has a Poisson distribution if I is any interval, does this still ensure that the process is a Poisson process? It is easy to show that in this case $\xi(I_1)$ and $\xi(I_2)$ are uncorrelated if I_1 and I_2 are disjoint intervals. The proof of this is essentially the same as the first step of the proof of theorem 4.

Remark added on August 22, 1966:
I have been informed by Jay Goldman, that the answer to the question in Remark 4 is: no. This has been shown by a counterexample by L.Shepp; this example will be published in a forthcoming paper of J.Goldman.