Studia Scientiarum Mathematicarum Hungarica 3 (1968) 351—358.

ON QUADRATIC INEQUALITIES IN PROBABILITY THEORY

by

J. GALAMBOS and A. RÉNYI

Summary

In this paper quadratic inequalities in the probabilities of Boolean functions of n variable events are considered. For a special class of such inequalities — called exact inequalities — a necessary and sufficient condition is given; this general theorem is applied to deduce certain special inequalities. Generalization to inequalities of degree higher than 2 is also considered.

§ 0. Notations

Let $S=(\Omega, \mathscr{A}, \mathsf{P})$ denote a probability space, i.e. let Ω be an arbitrary non-empty set, $\mathscr{A}$ a σ-algebra[1] of subsets of Ω and P a measure on $\mathscr{A}$ such that $\mathsf{P}(\Omega)=1$. We call the elements of $\mathscr{A}$ events and denote them by capital letters. We denote by $A+B$ the union and by AB the intersection of the sets A and B, and by $\bar{A}$ the complement of the set A with respect to Ω. As usual, $\bar{A}$ is interpreted as the event consisting in the non-occurrence of the event A, while $A+B$ and AB respectively, are interpreted as the event that at least one of the events A, B occurs, resp. that both the events A, B occur.

Let $p_1, p_2, \ldots, p_r$ be any set of positive numbers such that

$$\sum_{j=1}^{r} p_j = 1$$

We shall denote by $S_r(p_1, \ldots, p_r)$ that (finite) probability space in which the set Ω consists of r elements $\omega_1, \omega_2, \ldots, \omega_r$. $\mathscr{A}$ is the set of all 2^r subsets of Ω, and P is defined by

$$\mathsf{P}(A) = \sum_{\omega_j \in A} p_j \tag{0.1}$$

Especially $S_1(1)$ is the trivial probability space which contains only two events: the "certain event" Ω and the "impossible event" $\emptyset$ (the empty set). Further $S_2(\frac{1}{2}, \frac{1}{2})$ is the probability space (describing e.g. the throw of a fair coin) which contains only four events: $\Omega, \emptyset, \alpha=\{\omega_1\}$ and $\beta=\{\omega_2\}$ and $\mathsf{P}(\alpha)=\mathsf{P}(\beta)=\frac{1}{2}$.

A Boolean function $F=F(A_1, A_2, \ldots, A_n)$ of n variable events $A_1, \ldots, A_n$ is a function of these events which can be expressed by means of the variables

[1] All results of this paper are valid also if $\mathscr{A}$ is only an algebra of subsets of Ω and P a finitely additive nonnegative set function on $\mathscr{A}$ for which $\mathsf{P}(\Omega)=1$.

$A_1, \ldots, A_n$ and a finite number of Boolean operations, i.e. the operations $A+B$, AB, $\bar{A}$. We introduce the notation

$$A^1 = A, \quad A^{-1} = \bar{A}.$$

Let us denote by $\delta_k(m)$ the k-th digit of the binary representation of the non-negative integer m, i.e. we put

$$m = \sum_{k \geqq 0} \delta_k(m) 2^k \tag{0.2}$$

Let us put further

$$\varepsilon_k(m) = 2\delta_{k-1}(m) - 1 \qquad (k = 1, 2, \ldots). \tag{0.3}$$

Clearly $\varepsilon_k(m) = \pm 1$, and if m runs over the integers $0, 1, \ldots, 2^n - 1$, the n-tuple $\{\varepsilon_1(m), \ldots, \varepsilon_n(m)\}$ runs over all 2^n possible n-tuples of the signs $+1$ and -1.

Let us put

$$B_n(m) = A_1^{\varepsilon_1(m)} A_2^{\varepsilon_2(m)} \ldots A_n^{\varepsilon_n(m)} \qquad (0 \leqq m \leqq 2^n - 1) \tag{0.4}$$

We call the $B_n(m)$ the basic Boolean functions of the variables $A_1, \ldots, A_n$. Clearly

$$B_n(m_1) B_n(m_2) = \emptyset \quad \text{if} \quad m_1 \neq m_2 \tag{0.5}$$

and

$$\sum_{m=0}^{2^n - 1} B_n(m) = \Omega \tag{0.6}$$

It is well known that every Boolean function $F(A_1, \ldots, A_n)$ can be uniquely represented in a „canonical form" as the sum of certain basic functions $B_n(m)$; thus there are only 2^{2^n} different Boolean functions of n variable events.

§ 1. Introduction

Some time ago, the second named author has proved ([1], see also [2]) the following

THEOREM 1. *Let $F_j = F_j(A_1, A_2, \ldots, A_n)$ $(j = 1, 2, \ldots, N)$ be arbitrary Boolean functions of the n variable events $A_1, \ldots, A_n$. The linear inequality*

$$\sum_{j=1}^{N} c_j \mathsf{P}(F_j) \geqq 0 \tag{1.1}$$

(where $c_1, \ldots, c_N$ are real constants) is valid in every probability space S if it is valid in the trivial probability space $S_1(1)$.

This simple theorem is useful because it makes it possible to reduce the proof of any linear inequality among probabilities of Boolean functions to a corresponding combinatorial inequality.

To make this paper self-contained we reproduce here the proof of Theorem 1, especially as the proof is very short.

PROOF OF THEOREM 1. Let the expression of the functions $F_1, \ldots, F_N$ in canonical form be

$$F_j = \sum_{m \in E_j} B_n(m) \qquad (j = 1, 2, \ldots, N) \tag{1.2}$$

where E_j is some subset of the set $\{0, 1, \dots, 2^n-1\}$. It follows from (0. 5) that

$$\mathsf{P}(F_j)=\sum_{m\in E_j}\mathsf{P}(B_n(m)) \tag{1.3}$$

and thus

$$\sum_{j=1}^{N} c_j\mathsf{P}(F_j)=\sum_{m=0}^{2^n-1} d_m\mathsf{P}(B_n(m)) \tag{1.4}$$

where

$$d_m=\sum_{m\in E_j} c_j \tag{1.5}$$

Now evidently if $A_k=\Omega$ if $\varepsilon_k(m)=1$ and $A_k=\emptyset$ if $\varepsilon_k(m)=-1$, then $B_n(m)=\Omega$ and $B_n(l)=\emptyset$ for $l\neq m$, $0\leqq l\leqq 2^n-1$, thus for this special choice of the values of the variables $A_1, \dots, A_n$ we have

$$\sum_{j=1}^{N} c_j\mathsf{P}(F_j)=d_m \tag{1.6}$$

Thus if (1.1) holds on $S_1(1)$ we have $d_m\geqq 0$ for $m=0, 1, \dots, 2^n-1$ and thus it follows from (1. 4) that (1.1) holds for every choice of the values of the events $A_1, \dots, A_n$ in every probability space S. Thus Theorem 1 is proved.

It is evident that Theorem 1 can be used also to prove identities. To prove that a relation

$$\sum_{j=1}^{N} c_j\mathsf{P}(F_j)=0 \tag{1.7}$$

is valid, according to Theorem 1 it is sufficient to verify that (1. 7) holds if all A_k are equal either to Ω or to $\emptyset$.

A typical example of an inequality which can be obtained as a special case of Theorem 1 is the following inequality, due to GUMBEL ([3]): Putting

$$\sigma_k^{(n)}=\sum_{1\leqq i_1<i_2<\dots<i_k\leqq n}\mathsf{P}(A_{i_1}A_{i_2}\dots A_{i_k}) \qquad (k=1,2,\dots,n) \tag{1.8}$$

one has for $2\leqq k\leqq n$

$$(n-k+1)\sigma_{k-1}^{(n)}\leqq\binom{n}{k}+(k-1)\sigma_k^{(n)}. \tag{1.9}$$

By means of Theorem 1 the proof of (1. 9) is reduced to a simple inequality between binomial coefficients (see [2], p. 30).

The aim of this paper is to prove a theorem similar to Theorem 1 for *quadratic* (instead of linear) inequalities. This will be done in § 2. In § 3 we give some applications of the general theorem of § 2. In § 4 we discuss the possibility of generalizing the result of § 2 to polynomial inequalities of the third and still higher degrees.

§ 2. A General Theorem on Quadratic Inequalities

In this § we consider quadratic inequalities of the form

$$\sum_{i=1}^{N}\sum_{j=1}^{N} c_{i,j}\mathsf{P}(F_i)\mathsf{P}(F_j)\geqq 0 \tag{2.1}$$

where the $c_{i,j}$ are real constants, and $F_1, F_2, \dots, F_N$ are Boolean functions of the variable events $A_1, \dots, A_n$.

Note that it is no restriction that in (2. 1) no linear terms occur, because one of the F_i may be equal to Ω (which is also a Boolean function, namely a constant function) and thus inequalities which contain both quadratic and linear terms can be also written in the form (2.1).

We shall call an inequality (2.1) *exact*, if in (2.1) the equality sign is valid every time when each A_k is equal either to Ω or to $\emptyset$. By other words (2.1) is exact if equality is valid in (2.1) when the variables $A_1, \dots, A_n$ are restricted to events in the trivial probability space $S_1(1)$.

We shall prove now the following

THEOREM 2. *Let* (2.1) *be an exact inequality. In order that* (2.1) *should be valid on every probability space S it is sufficient (and of course also necessary) that it should be valid on the probability space* $S_2(\frac{1}{2}, \frac{1}{2})$.

PROOF OF THEOREM 2. Let again (1. 2) be the expression of the function $F_j (1 \leqq j \leqq N)$ in canonical form. In view of (1. 3) we get

$$\sum_{i=1}^{N} \sum_{j=1}^{N} c_{i,j} \mathsf{P}(F_i) \mathsf{P}(F_j) = \sum_{r=0}^{2^n-1} \sum_{s=0}^{2^n-1} d_{r,s} \mathsf{P}(B_n(r)) \mathsf{P}(B_n(s)), \tag{2. 2}$$

where

$$d_{r,s} = \sum_{\substack{r \in E_i \\ s \in E_j}} c_{i,j} \tag{2. 3}$$

Now let us choose $A_k = \Omega$ if $\varepsilon_k(r) = 1$ and $A_k = \emptyset$ if $\varepsilon_k(r) = -1$ $(k = 1, 2, \dots, n)$.

It follows that $\mathsf{P}(B_n(r)) = 1$ and $\mathsf{P}(B_n(s)) = 0$ if $s \neq r$; thus for this special choice of the values of the variables $A_1, \dots, A_n$ we have

$$\sum_{i=1}^{N} \sum_{j=1}^{N} c_{i,j} \mathsf{P}(F_i) \mathsf{P}(F_j) = d_{r,r} \tag{2. 4}$$

As we have supposed that the inequality (2.1) is exact, it follows that

$$d_{r,r} = 0 \quad \text{for} \quad 0 \leqq r \leqq 2^n - 1. \tag{2. 5}$$

Putting

$$D_{r,s} = d_{r,s} + d_{s,r} \quad \text{for} \quad r \neq s \tag{2. 6}$$

we obtain

$$\sum_{i=1}^{N} \sum_{j=1}^{N} c_{i,j} \mathsf{P}(F_i) \mathsf{P}(F_j) = \sum_{0 \leqq r < s \leqq 2^n - 1} D_{r,s} \mathsf{P}(B_n(r)) \mathsf{P}(B_n(s)) \tag{2. 7}$$

Now let us choose an arbitrary pair (r, s) of integers, $0 \leqq r < s \leqq 2^n - 1$, and let us choose the values of the events A_k as follows:

$$\begin{aligned} A_k &= \Omega \quad \text{if} \quad \varepsilon_k(r) = \varepsilon_k(s) = 1 \\ A_k &= \alpha \quad \text{if} \quad \varepsilon_k(r) = 1 \quad \text{and} \quad \varepsilon_k(s) = -1 \\ A_k &= \beta \quad \text{if} \quad \varepsilon_k(r) = -1 \quad \text{and} \quad \varepsilon_k(s) = +1 \\ A_k &= \emptyset \quad \text{if} \quad \varepsilon_k(r) = \varepsilon_k(s) = -1 \end{aligned} \tag{2. 8}$$

where α and β are the events $\alpha=\{\omega_1\}$, $\beta=\{\omega_2\}$ of the probability space $S_2(\frac{1}{2}, \frac{1}{2})$. For this special choice of the values of the variables A_k we have clearly

(2. 9) $$B_n(r)=\alpha,\ B_n(s)=\beta \quad \text{and} \quad B_n(t)=\emptyset \quad \text{for} \quad t\neq r,\ t\neq s.$$

Thus we obtain for this choice of the values of the A_k

(2. 10) $$\mathsf{P}(B_n(r))=\mathsf{P}(B_n(s))=\tfrac{1}{2}, \qquad \mathsf{P}(B_n(t))=0 \quad \text{for} \quad t\neq r,\ t\neq s,$$

and therefore

(2. 11) $$\sum_{i=1}^{N}\sum_{j=1}^{N} c_{i,j}\mathsf{P}(F_i)\mathsf{P}(F_j)=\tfrac{1}{4}D_{r,s}$$

Thus if (2.1) is valid on $S_2(\frac{1}{2}, \frac{1}{2})$, then we must have $D_{r,s}\geqq 0$ for all pairs (r, s) and thus in view of (2. 7) it follows that (2.1) is valid on every probability space S and for every choice of the value of the variables A_k.

Thus Theorem 2 is proved.

Similarly as Theorem 1, Theorem 2 can be used also to prove identities. As a matter of fact we obtain from Theorem 2 the following

COROLLARY. *If*

(2. 12) $$\sum_{i=1}^{N}\sum_{j=1}^{N} c_{i,j}P(F_i)P(F_j)=0$$

holds on $S_1(1)$ *and on* $S_2(\frac{1}{2}, \frac{1}{2})$, *then it holds identically on every probability space.*

§ 3. Some Applications of the General Theorem of § 2

In this § we consider some examples of quadratic inequalities which can be easily proved by means of Theorem 2.

EXAMPLE 1. Let us put $\sigma_0^{(n)}=1$ and

(3. 1) $$\sigma_k^{(n)}=\sum_{1\leqq i_1<i_2<\ldots<i_k\leqq n}\mathsf{P}(A_{i_1}A_{i_2}\ldots A_{i_k})$$

We shall prove that the inequality

(3. 2) $$k\sigma_k^{(n)}\geqq\sigma_{k-1}^{(n)}(\sigma_1^{(n)}-k+1) \qquad (k=1, 2, \ldots, n)$$

is valid.

To prove (3. 2) we first remark that it is a quadratic inequality of type (2.1). Further it is easy to see that (3. 2) is an exact inequality. As a matter of fact if l among the events $A_1, \ldots, A_n$ are equal to Ω and the other $n-l$ to $\emptyset$, then three cases are possible:

a) either $l\leqq k-2$, in which case $\sigma_k^{(n)}=\sigma_{k-1}^{(n)}=0$ and thus both sides of (3. 2) are equal to 0,

b) or $l=k-1$ in which case $\sigma_k^{(n)}=0$ and $\sigma_1^{(n)}-k+1=0$ and thus again both sides of (3. 2) are equal to 0,

c) or $l\geqq k$, in which case $\sigma_k^{(n)}=\binom{l}{k}$, $\sigma_{k-1}^{(n)}=\binom{l}{k-1}$ and $\sigma_1^{(n)}=l$. As however

$$k\binom{l}{k}=\binom{l}{k-1}(l-k+1)$$

we have equality in (3. 2) in this case too. Thus (3. 2) is exact. Now let us check that (3. 2) holds for $S_2(\frac{1}{2}, \frac{1}{2})$. Suppose that among the events $A_1, \ldots, A_n$ l_1 are equal to Ω, l_2 to α, l_3 to β $(l_1+l_2+l_3 \leqq n)$ and the remaining $n-l_1-l_2-l_3$ to $\emptyset$. In this case

$$\sigma_j^{(n)} = \frac{1}{2}\left[\binom{l_1+l_2}{j}+\binom{l_1+l_3}{j}\right] \quad \text{for} \quad 1 \leqq j \leqq n$$

and thus

$$k\sigma_k^{(n)} - \sigma_{k-1}^{(n)}(\sigma_1^{(n)}-k+1) = \tfrac{1}{4}(l_2-l_3)\left[\binom{l_1+l_2}{k-1}-\binom{l_1+l_3}{k-1}\right] \geqq 0 \tag{3. 3}$$

Thus by Theorem 2 (3. 2) holds on every probability space S for any choice of the events $A_1, \ldots, A_n$.

It is interesting to compare (3. 2) with GUMBEL's inequality (1. 9). The fact that (3. 2) is exact, while in GUMBEL's inequality we have equality (as seen from the proof) on $S_1(1)$ only if $l=n$ or $l = n-1$, shows that, (3. 2) gives sometimes a better estimate than (1. 9). Another such instance is when the events all have probability $\frac{1}{2}$, and $k=2$. In this case (1. 9) gives for $\sigma_2^{(n)}$ only the trivial lower estimate 0, while (3. 2) gives the non-trivial (in fact, asymptotically best possible) lower estimate $\sigma_2^{(n)} \geqq \dfrac{n(n-2)}{8}$.

For $k=2$ we obtain as a special case of (3. 2) the well known inequality

$$\sigma_2^{(n)} \geqq \binom{\sigma_1^{(n)}}{2}. \tag{3. 4}$$

It follows from (3. 2) by induction that

$$\sigma_k^{(n)} \geqq \binom{\sigma_1^{(n)}}{k}. \tag{3. 5}$$

It should be noted that one can deduce from (3. 4) the following inequality:

$$\text{If} \quad \sigma_2^{(n)} \leqq \binom{n}{2}p^2 \quad \text{then} \quad \sigma_1^{(n)} \leqq np+\frac{1}{2}(1-p)+\frac{1-p^2}{4p(2n-1)}$$

As a matter of fact, it follows from (3. 4) and the inequality $\sqrt{1+x} \leqq 1+\dfrac{x}{2}$ that

$$\sigma_1^{(n)} \leqq \frac{1+\sqrt{1+8\sigma_2^{(n)}}}{2} \leqq \frac{1}{2}+\frac{1}{2}(2np-p)\sqrt{1+\frac{1-p^2}{(2np-p)^2}}$$

and thus that

$$\sigma_1^{(n)} \leqq np+\frac{1-p}{2}+\frac{1-p^2}{4p(2n-1)}$$

REMARK. The exact maximum of $\sigma_1^{(n)}$ under condition $\sigma_2^{(n)} \leqq \binom{n}{p}p^2$ was determined in [4].

EXAMPLE 2. Let us consider the quadratic relation

$$\mathrm{P}^2(A+B)+\mathrm{P}^2(AB) = \mathrm{P}^2(A)+\mathrm{P}^2(B)+2\mathrm{P}(A\bar{B})\mathrm{P}(\bar{A}B) \tag{3. 6}$$

It is evidently valid on $S_1(1)$ and also on $S_2(\frac{1}{2}, \frac{1}{2})$, thus it holds identically.

§ 4. Cubic Inequalities

Theorem 2 can be generalized for cubic inequalities

$$\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{i_3=1}^{N} c_{i_1,i_2,i_3}\mathsf{P}(F_{i_1})\mathsf{P}(F_{i_2})\mathsf{P}(F_{i_3}) \geqq 0 \tag{4.1}$$

where $F_1, \dots, F_N$ are Boolean functions of the variable events $A_1, \dots, A_n$. The inequality (4.1) is called *exact of order* 2 if for every p $(0 \leqq p \leqq 1)$ equality stands in (4.1), if $A_1, \dots, A_n$ are all events of S_2 $(p, 1-p)$. (Clearly an inequality which is exact of order 2 is exact.)

We prove the following

THEOREM 3. *Let* (4.1) *be an inequality which is exact of order* 2. *If* (4.1) *holds on* $S_3(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, *it holds on every probability space.*

PROOF. If (1. 2) is the canonical form of F_j we have

$$\begin{aligned} &\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{i_3=1}^{N} c_{i_1,i_2,i_3}\mathsf{P}(F_{i_1})\mathsf{P}(F_{i_2})\mathsf{P}(F_{i_3}) = \\ &= \sum_{r_1=0}^{2^n-1}\sum_{r_2=0}^{2^n-1}\sum_{r_3=0}^{2^n-1} d(r_1, r_2, r_3)\mathsf{P}(B_n(r_1))\mathsf{P}(B_n(r_2))\mathsf{P}(B_n(r_3)) \end{aligned} \tag{4.2}$$

where

$$d(r_1, r_2, r_3) = \sum_{r_h \in E_{i_h} (h=1,2,3)} c_{i_1,i_2,i_3} (h=1, 2, 3) \tag{4.3}$$

Clearly (4.1) being exact implies that

$$d(r, r, r) = 0 \qquad (0 \leqq r \leqq 2^n - 1).$$

Let us put for $r \neq s$

$$D(r, s) = d(r, r, s) + d(r, s, r) + d(s, r, r).$$

Now from the supposition that in (4.1) equality holds on $S_2(p, q)$ $(q = 1-p)$ it follows that for any pair of numbers r, s $(r \neq s)$

$$D(r, s)p + D(s, r)q = 0. \tag{4.4}$$

By supposition (4. 4) holds for $p = \frac{1}{2}$ and also for some p for which $0 < p < \frac{1}{2}$; it follows that

$$D(r, s) = 0 \quad \text{if} \quad s \neq r. \tag{4.5}$$

Thus we obtain, putting

$$\begin{aligned} D(r_1, r_2, r_3) = d(r_1, r_2, r_3) + d(r_1, r_3, r_2) + d(r_2, r_1, r_3) + \\ + d(r_2, r_3, r_1) + d(r_3, r_1, r_2) + d(r_3, r_2, r_1) \end{aligned}$$

that

$$\begin{aligned} &\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{i_3=1}^{N} c_{i_1,i_2,i_3}\mathsf{P}(F_{i_1})\mathsf{P}(F_{i_2})\mathsf{P}(F_{i_3}) = \\ &= \sum_{0 \leqq r_1 < r_2 < r_3 \leqq 2^n-1} D(r_1, r_2, r_3)\mathsf{P}(B_n(r_1))\mathsf{P}(B_n(r_2))\mathsf{P}(B_n(r_3)) \end{aligned} \tag{4.6}$$

Now let r_1, r_2, r_3 be any three different numbers, $0 \leqq r_1 < r_2 < r_3 \leqq 2^n - 1$. Let us denote the atoms of $S_3(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ by α_1, α_2 and α_3. Let us put

$$A_k = \sum_{\varepsilon_k(r_i)=1} \alpha_i$$

It is easy to show that for this choice of the values of the variable events A_k we have

$$B_n(r_i) = \alpha_i \qquad (i = 1, 2, 3). \tag{4.7}$$

As a matter of fact $A_k^{\varepsilon_k(r_i)} \supseteqq \alpha_i$ $(k = 1, 2, \ldots, n)$ thus

$$B_n(r_i) = \prod_{k=1}^{n} A_k^{\varepsilon_k(r_i)} \supseteqq \alpha_i \tag{4.8}$$

As however the events $B_n(r_1)$, $B_n(r_2)$, $B_n(r_3)$ are disjoint, (4. 8) implies (4. 7).

Clearly (4. 7) implies that for any s, different from each of r_1, r_2, r_3, one has $B_n(s) = \emptyset$. Thus for the above choice of the values of the variables $A_1, \ldots, A_n$ we have

$$\sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_3=1}^{N} c_{i_1, i_2, i_3} \mathsf{P}(F_{i_1}) \mathsf{P}(F_{i_2}) \mathsf{P}(F_{i_3}) = \frac{1}{27} D(r_1, r_2, r_3) \tag{4.9}$$

As by supposition (4.1) holds on $S_3(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we obtain from (4. 9)

$$D(r_1, r_2, r_3) \geqq 0 \quad \text{for} \quad 0 \leqq r_1 < r_2 < r_3 \leqq 2^n - 1. \tag{4.10}$$

In view of (4. 6) it follows that (4.1) holds for every probability space S.

As an example consider the following cubic inequality

$$\mathsf{P}(AB)\mathsf{P}(BC)\mathsf{P}(AC) \geqq \mathsf{P}^2(ABC)[\mathsf{P}(AB) + \mathsf{P}(AC) + \mathsf{P}(BC) - 2\mathsf{P}(ABC)] \tag{4.11}$$

Clearly (4.11) is exact of order two. Thus we have to check only that (4.11) holds on $S_3(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which is easily done.

Theorem 3 could be generalized also for polynomial inequalities of degree greater than 3.

REFERENCES

[1] RÉNYI, A.: Quelques remarques sur les probabilités d'événements dépendantes, *Journal de Math.* **37** (1958) 393—398.

[2] RÉNYI, A.: *Wahrscheinlichkeitsrechnung mit einer Anhang über Informationstheorie*, Deutscher Verlag der Wissenschaften, Berlin, 1962.

[3] FRÉCHET, M.: *Les probabilités associées a un systeme d'événements compatibles et dépendants, I, II*, Hermann, Paris, 1940, 1943.

[4] RÉNYI, A., ERDŐS, P., NEVEU, J.: An elementary inequality between the probability of events, *Math. Scandinavica* **13** (1963) 99—104.

University of Ghana, Legon (Accra) and Eötvös Loránd University, Budapest

(Received August 10, 1967.)