with the sup norm. This space will be denoted by Q_R^C. It is easily seen that Q_R^C is isometrically isomorphic to that closed subspace of $Q_{R^\#}^C$ whose elements are continuous at plus and minus infinity. Since $R^\#$ is a generalized interval one may readily show:

THEOREM 4. *A continuous linear transformation $\mathcal{K}$ on a normed linear space V into Q_R^C is compact if and only if $M_{\mathcal{K}}$ is uniformly quasi-continuous. Furthermore every such transformation is uniformly approximatable by transformations of finite range.*

5. With obvious and quite small modifications the results of Sections 3 and 4 may be applied to those subspaces of the uniformly quasi-continuous functions whose elements are left continuous, right continuous or continuous, respectively.

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MEASURES IN DENUMERABLE SPACES

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1. Introduction. The purpose of this paper is essentially didactic: to call attention to and make explicit certain properties of measures on denumerable spaces (i.e., spaces with a denumerable number of points) that justify the conventional treatment of such spaces in probability texts. After our results were obtained we discovered that the main one could be derived easily from some theorems of Boolean algebra. Since our theorems have not, to our knowledge, been previously pointed out, and since our proofs are in a measure-theoretic setting that requires no knowledge of Boolean algebra, we have been motivated to contribute this note.

By a *ring* $\mathcal{R}$ of subsets of a space Ω is meant a nonempty class of subsets that is closed under finite union and difference. If $\mathcal{R}$ is closed under countable union, it is called a *σ-ring*. The terms *algebra* and *σ-algebra* refer, respectively, to rings and σ-rings containing the space Ω. The sets in a ring $\mathcal{R}$ are called *$\mathcal{R}$-measurable.* We use the term *measure* to mean an extended real-valued, nonnegative, countably-additive set function defined on a ring.

The *power set*, $\mathcal{P}(\Omega)$, is the σ-algebra of all subsets of Ω. A measure μ is called *continuous* if each singleton set A is measurable, and $\mu(A)=0$ for all singletons. The *trivial measure* is the measure on $\mathcal{P}(\Omega)$ that vanishes identically. The symbol "$\varnothing$" denotes the empty set.

Our results fill a small gap in the literature on measure theory concerning the possibility of extending a measure from a given σ-ring to the power set.

Let $\mathfrak{M}$ denote the family of all measures on all σ-rings of an arbitrary space Ω, and let $\mathfrak{N}$ denote the family of all measures on $\mathcal{P}(\Omega)$. It may happen that each measure in $\mathfrak{M}$ is the restriction of at least one measure in $\mathfrak{N}$, or loosely speaking, that the class of all measures on all σ-rings is no richer than the class of all measures on the power set. In this case we shall say that Ω, or more precisely the cardinal number of Ω, has the *full extension property*; it is clear that either all sets of the same cardinality have the full extension property or none do. It is evident also that if a cardinal $\aleph$ fails to have the full extension property, so does every cardinal $\aleph^*$ with $\aleph^*>\aleph$. (This is also true if "σ-ring" is replaced by "σ-algebra" in the formulation of the notion of full extension property.)

Ulam [7] has shown that the only real-valued continuous measure on the power set of a space whose cardinality is less than the first weakly inaccessible cardinal is the trivial one. (A cardinal number $\aleph_\alpha$ is called weakly inaccessible if (a) $\aleph_\alpha>\aleph_0$, (b) α is a limit ordinal, (c) $\aleph_\alpha$ is not the sum of fewer than $\aleph_\alpha$ numbers each of which is less than $\aleph_\alpha$.) Hence, in particular, if the cardinality of Ω is $\aleph_1$, $\mathcal{P}(\Omega)$ cannot carry a nontrivial continuous measure. On the other hand, an example in Sect. 2 demonstrates the existence of a space Ω of cardinality $\aleph_1$ which has a σ-algebra $\mathfrak{F}$ of subsets containing all singletons and a nontrivial continuous probability measure μ on $\mathfrak{F}$. By Ulam's Theorem μ cannot be extended to $\mathcal{P}(\Omega)$, so that $\aleph_1$ and hence all greater cardinals fail to have the full extension property. Intuitively it seems clear that if card $\Omega\leqq\aleph_0$, all measures on sub-σ-rings of $\mathcal{P}(\Omega)$ should be compatible with the assignments of masses to points, and thus that $\aleph_0$ should have the full extension property. We shall show that this is so in Section 3. The connection between our results and Boolean algebra is described in Section 4. The implications for probability theory are discussed in Section 5.

2. An example of a nonextendible measure. Let α be any ordinal greater than zero, and let Ω_α denote the set of all ordinal numbers less than α. Let α^* denote the smallest ordinal α such that Ω_α is uncountable. Then the cardinal number of Ω_{α^*} is $\aleph_1$.

We shall call a set $A\subset\Omega_{\alpha^*}$ a cosection if there exists a number $\beta\in\Omega_{\alpha^*}$ such that $\alpha>\beta$ implies $\alpha\in A$. It is not difficult to verify from well-known properties of Ω_{α^*} ([2], p. 29) that the class of sets $\mathfrak{F}$ consisting of all cosections and their complements is a σ-algebra. Moreover, the function

$$\mu\colon\ \mathfrak{F}\to[0,1]$$

defined by setting

$$\mu(A) = \begin{cases} 1, & \text{if } A \text{ is a cosection} \\ 0, & \text{if } A^c \text{ is a cosection} \end{cases}$$

is a probability measure which, by Ulam's Theorem, cannot be extended to $\mathcal{P}(\Omega_{\alpha^*})$. We therefore have the following result:

THEOREM 2.1. *If card $\Omega \geqq \aleph_1$, Ω does not have the full extension property.*

3. Structure of σ-rings and measures in a countable space. As is shown in Theorem 3.1, every σ-ring of subsets of a countable space has a particularly simple structure, an important feature of which is implied by the following lemma:

LEMMA 3.1. *Let Ω be a space containing a countable number of points. If $\mathfrak{F}$ is a nonempty class of subsets of Ω closed under countable union or countable intersection, then $\mathfrak{F}$ is closed, respectively, under arbitrary union or intersection.*

Proof. We give the proof only for the case of intersections; the proof for unions is similar.

Suppose that $\mathfrak{F}$ is closed under countable intersection. Let $\mathcal{C} \subset \mathfrak{F}$ be an arbitrary class of sets and put

$$D = \bigcap_{A \in \mathcal{C}} A.$$

Consider the complementary set D^c. If D^c is empty, then $\mathcal{C}$ is the class whose only member is Ω, so that $D \in \mathfrak{F}$.

If D^c is not empty, to each $\omega \in D^c$ there corresponds a set A_ω^c such that $\omega \in A_\omega^c$ and $A_\omega \in \mathcal{C}$. Let

$$E = \bigcap_{\omega \in D^c} A_\omega;$$

plainly $E \supset D$. Since D^c is countable and $A_\omega \in \mathcal{C}$, we have $E \in \mathfrak{F}$. Noting that

$$\omega \in D^c \Rightarrow \omega \in A_\omega^c \Rightarrow \omega \in E^c,$$

that is, $D \supset E$, we conclude that $D = E \in \mathfrak{F}$.

Now let $\mathcal{R}$ denote a σ-ring of subsets of a countable space Ω. We define a binary relation "$\sim$" on Ω by setting

$$\omega \sim \omega'$$

if and only if every set in $\mathcal{R}$ that contains ω' also contains ω. It is clear from the definition that this relation is reflexive and transitive; it follows from the properties of $\mathcal{R}$ that the relation is also symmetric. For suppose that $\omega \sim \omega'$ but

$$\omega' \nsim \omega.$$

Then there exists a set $A \in \mathcal{R}$ such that $\omega \in A$ and $\omega' \notin A$. Setting $\Omega_{\mathcal{R}} = \bigcup_{E \in \mathcal{R}} E$, it follows from Lemma 3.1 that $\Omega_{\mathcal{R}} \in \mathcal{R}$. Hence

$$\Omega_{\mathcal{R}} - A \in \mathcal{R},$$
$$\omega' \in \Omega_{\mathcal{R}} - A,$$

and $\omega \notin \Omega_{\mathcal{R}} - A$, which contradicts the relation $\omega \sim \omega'$.

Thus "$\sim$" is an equivalence relation on Ω. Plainly $\omega \sim \omega'$ if and only if $\omega \in \bigcap_{\omega' \in A \in \mathcal{R}} A$. Hence we have proved:

LEMMA 3.2. *The equivalence class,* $[\omega']$, *of* ω' *is given by* $[\omega'] = \bigcap_{\omega' \in A \in \mathcal{R}} A$.

It follows from Lemmas 3.1 and 3.2 that the class of sets $\{[\omega'] : \omega' \in \Omega_{\mathcal{R}}\}$ is a measurable partition of $\Omega_{\mathcal{R}}$, i.e., one in which each set is in $\mathcal{R}$.

An *atom* of a ring $\mathcal{R}$ of sets is a nonempty, $\mathcal{R}$-measurable set A whose only $\mathcal{R}$-measurable subsets are A and $\varnothing$. The theorem below shows that a σ-ring in a countable space is generated by its atoms.

THEOREM 3.1. *If* Ω *is countable and* $\mathcal{R}$ *is an arbitrary* σ*-ring of subsets of* Ω, *there exists a countable, measurable partition of* $\Omega_{\mathcal{R}}$ *into atoms. The atoms are just the equivalence classes* $[\omega]$, $\omega \in \Omega_{\mathcal{R}}$, *and each* $\mathcal{R}$*-measurable set* A *is the union of the atoms contained in* A.

Proof. Suppose there is a point $\omega \in \Omega_{\mathcal{R}}$ and a set $B \in \mathcal{R}$ such that B is a nonempty, proper subset of $[\omega]$. Then there are equivalent points $\omega' \in B$ and $\omega'' \in [\omega] - B$, which contradicts the definition of equivalence.

Thus the sets $[\omega]$, $\omega \in \Omega_{\mathcal{R}}$, are atoms which form a measurable partition of $\Omega_{\mathcal{R}}$; plainly there are no other atoms. To complete the proof we note that $B \in \mathcal{R}$ implies

$$B = B \cap \Omega_{\mathcal{R}} = B \cap \bigcup_{\omega \in \Omega_{\mathcal{R}}} [\omega] = \bigcup_{\omega \in \Omega_{\mathcal{R}}} B \cap [\omega] = \bigcup_{[\omega] \subset B} [\omega],$$

since $B \cap [\omega] = [\omega]$ if $[\omega] \subset B$ and is empty otherwise.

Let $\mathcal{F}$ and $\mathcal{F}^*$ be σ-algebras of subsets of spaces Ω and Ω^*, respectively. We shall say that $\mathcal{F}$ and $\mathcal{F}^*$ are isomorphic if there exists a $1:1$ mapping ϕ of $\mathcal{F}$ onto $\mathcal{F}^*$ that preserves countable unions and complements, i.e.,

$$\text{(a)} \quad \phi\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} \phi(A_i),$$
$$\text{(b)} \quad \phi(A^c) = (\phi(A))^c.$$

It follows easily from (a) and (b) that ϕ also preserves countable intersections, differences, and set inclusion. We can now state the following corollary to Theorem 3.1:

COROLLARY. *If* Ω *is countable, any* σ*-algebra* $\mathcal{F}$ *of subsets of* Ω *is isomorphic to the* σ*-algebra* $\mathcal{F}^*$ *of all subsets of the set* Ω^* *of all atoms of* $\mathcal{F}$.

Proof. We should like to emphasize that each point $\omega^* \in \Omega^*$ is a subset of Ω. Thus if A is a subset of Ω and A^* is a subset of Ω^*, ω^* can bear the relation of inclusion to A and the relation of membership to A^*. With this in mind, we define a mapping

$$\phi: \mathfrak{F} \rightarrow \mathfrak{F}^*$$

by setting, for $A \in \mathfrak{F}$, $\phi(A) = \{\omega^* \in \Omega^*: \omega^* \subset A\}$.

The mapping ϕ is "onto" since, if $A^* \in \mathfrak{F}^*$, then the set A defined by

$$A = \left(\bigcup_{\omega^* \in A^*} \omega^* \right) \in \mathfrak{F}$$

has the property

$$\phi(A) = A^*.$$

Moreover, ϕ is 1:1; for if $A \in \mathfrak{F}$, $B \in \mathfrak{F}$, $A \neq B$, there exists a point ω in one of the sets that is not in the other, say $\omega \in A$, $\omega \notin B$. Then $[\omega] \subset A$ and $[\omega] \subset B^c$, so that $\phi(A) \neq \phi(B)$.

To show that ϕ preserves countable unions, it suffices to note that for any sequence $A_1, A_2, \cdots$ of $\mathfrak{F}$-measurable sets,

$$\phi(A_i) = \{[\omega]: \omega \in A_i\}$$

and

$$\phi\left(\bigcup_{i=1}^{\infty} A_i \right) = \left\{ [\omega]: \omega \in \bigcup_{i=1}^{\infty} A_i \right\} = \bigcup_{i=1}^{\infty} \{[\omega]: \omega \in A_i\} = \bigcup_{i=1}^{\infty} \phi(A_i).$$

Similarly the relation

$$\begin{aligned} \phi(A^c) &= \{[\omega]: \omega \in A^c\} = \{[\omega]: \omega \in \Omega\} - \{[\omega]: \omega \in A\} \\ &= \phi(\Omega) - \phi(A) = \Omega^* - \phi(A) = (\phi(A))^c \end{aligned}$$

shows that ϕ preserves complements. Thus $\mathfrak{F}$ and $\mathfrak{F}^*$ are isomorphic.

It follows easily from Theorem 3.1 that $\aleph_0$ has the full extension property. For completeness the argument is given below.

THEOREM 3.2. *Let Ω be a countable set, $\mathfrak{R}$ a σ-ring of subsets, and μ an arbitrary measure on $\mathfrak{R}$. There exists a measure μ^* on the class of all subsets of Ω whose restriction to $\mathfrak{R}$ is μ.*

Proof. For each equivalence class $[\omega]$, $\omega \in \Omega_{\mathfrak{R}}$, let $p_{[\omega]}: [\omega] \rightarrow \overline{R}_+$ be any function such that

$$\sum_{\alpha \in [\omega]} p_{[\omega]}(\alpha) = \mu([\omega]).$$

Assume that $\Omega_{\mathfrak{R}} = \Omega$. Let $p: \Omega \rightarrow \overline{R}_+$ be the mapping whose restriction to $[\omega]$ is $p_{[\omega]}$ for all $\omega \in \Omega$. For each set $A \subset \Omega$ define

$$\mu^*(A) = \sum_{\omega \in A} p(\omega).$$

Clearly $\mu^*\colon \mathcal{P}(\Omega)\to\overline{R}_+$ is a measure and $\mu^*([\omega])=\mu([\omega])$, $\omega\in\Omega$. More generally, for $B\in\mathcal{R}$ we have from Theorem 3.1,

$$\mu^*(B) = \mu^*\left(\bigcup_{[\omega]\subset B}[\omega]\right) = \sum_{[\omega]\subset B}\mu^*([\omega]) = \sum_{[\omega]\subset B}\mu([\omega]) = \mu(B).$$

If $\Omega_{\mathcal{R}}\neq\Omega$, we define p on $\Omega_{\mathcal{R}}$ as before and define it arbitrarily on $\Omega-\Omega_{\mathcal{R}}$.

4. Related theorems of Boolean algebra. Let $\mathcal{A}$ be a Boolean algebra and let m be an infinite cardinal. $\mathcal{A}$ is said to be *m-complete* if for every indexed family $\{A_t\}_{t\in T}$, where card $T=m$ and $A_t\in\mathcal{A}$, the join $\bigcup_{t\in T}A_t$ exists in $\mathcal{A}$. This is equivalent to the condition that for every m-indexed family $\{A_t\}_{t\in T}$, the meet $\bigcap_{t\in T}A_t$ exists in $\mathcal{A}$. If $\mathcal{A}$ is an m-complete Boolean algebra for every m, then $\mathcal{A}$ is said to be a *complete* Boolean algebra. (For the definitions of Boolean algebra, infinite joins, and infinite meets, see [4].) A Boolean algebra $\mathcal{A}$ is said to satisfy the *m-chain condition* provided that every set of disjoint elements in $\mathcal{A}$ has cardinality $\leqq m$. (Since an algebra of sets may be viewed as a Boolean algebra, the terminology introduced in this section may also be applied to algebras of sets.)

In this terminology, Lemma 3.1 of the previous section has an obvious corollary.

COROLLARY TO LEMMA 3.1. *A σ-algebra of subsets of a countable space is a complete Boolean algebra.*

It is interesting to note that this corollary is a special case of a much more general theorem of Boolean algebra due to Tarski.

THEOREM (Tarski). *Every m-complete Boolean algebra $\mathcal{A}$ satisfying the m-chain condition is a complete Boolean algebra.*

For a detailed proof of this theorem the reader is referred to [**4**, **5**]. The proof rests on the inductively proved fact that if the join $\bigcup_{t\in T}A_t$ exists for any m-indexed set $\{A_t\}_{t\in T}$ of disjoint elements of a Boolean algebra $\mathcal{A}$, then $\mathcal{A}$ is m-complete. Under the hypothesis of Tarski's Theorem it is evident that the join of any indexed set of disjoint elements exists.

Since a σ-algebra of subsets of a countable space is not only $\aleph_0$-complete but also satisfies the $\aleph_0$-chain condition, the corollary to Lemma 3.1 is a special case of Tarski's Theorem.

Our Theorem 3.1 also has a more general Boolean algebra counterpart. An element $A\neq 0$ of a Boolean algebra $\mathcal{A}$ is said to be an *atom* of $\mathcal{A}$ if for every $B\in\mathcal{A}$ the inclusion

$$B\subset A$$

implies that either $B=0$ or $B=A$. A Boolean algebra is called *atomic* if for every element $A\neq 0$ there exists an atom $B\subset A$. The following theorem, due to

Lindenbaum and Tarski [4, 6], relates completeness to atomicity:

THEOREM (Lindenbaum and Tarski). *A complete Boolean algebra $\mathfrak{A}$ is isomorphic to a complete algebra of sets if and only if $\mathfrak{A}$ is atomic. In this case $\mathfrak{A}$ is isomorphic to the algebra of all subsets of the set of all atoms of $\mathfrak{A}$.*

Two Boolean algebras are called *isomorphic* if there exists a 1:1 mapping of one onto the other that preserves binary join and complement.

Since, as we have seen, a σ-algebra of subsets of a countable space is complete, it follows from the Theorem of Lindenbaum and Tarski that it is atomic. It is evident that the atoms of a Boolean algebra are disjoint. Hence, there are only countably many atoms in a σ-algebra of subsets of a countable space Ω. Clearly the union of these atoms is Ω, so that they form a countable, measurable partition of the space.

5. Implications for probability theory. In the Kolmogorov formulation of the axioms of probability, the mathematical description of a random experiment $\mathcal{E}$ consists of a triple $(\Omega, \mathfrak{F}, P)$, where Ω is a set, $\mathfrak{F}$ is a σ-algebra of subsets of Ω, and $P: \mathfrak{F} \rightarrow [0, 1]$ is a probability measure. In applications of the Kolmogorov model each point in Ω is interpreted as a possible "primary" or "indecomposable" outcome of $\mathcal{E}$, and each set $A \in \mathfrak{F}$ is interpreted as the event that "the outcome of $\mathcal{E}$ is one of the points in A." Thus the intuitive notion of event is formalized as a set and, moreover, it is postulated that the only events of $\mathcal{E}$ to which probabilities are associated are the sets in $\mathfrak{F}$. These probabilities are given, of course, by the measure P.

In this approach any σ-algebra of subsets of Ω is admissible as an event class; in particular, for example, it is not required that the singleton sets themselves be events, despite the fact that each point is intuitively thought of as a possible outcome. Keeping this intuitive interpretation of the points of Ω in mind, it is natural for the student of probability to wonder why an arbitrary subset of Ω should not be regarded as a possible event to which a probability is attached. Indeed, from the point of view of the scientist or engineer interested in applications, the definition of an event as a member of a distinguished σ-algebra of subsets may seem decidedly unnatural. Certainly the problem of justifying it is faced by every teacher of probability.

One argument, which we shall not elaborate here, rests on a distinction between events and observable events [3]. This argument gives an intuitive interpretation of the fact that probability is not defined for all events, only for observable events. However, a purely mathematical argument can also be given, based on the theorems proved in the previous section. Although in every application of probability the governing probability measure must satisfy the axioms of probability, i.e., be $\geqq 0$, countably additive, and have $P(\Omega)=1$, each application typically involves its own additional set of conditions, conditions dictated by a combination of empirical and theoretical considerations peculiar to the particular application; these additional restrictions may, in a certain sense, clash with the general probability axioms.

Consider, for example, the idealized experiment of making an infinite sequence of tosses of a coin. Suppose that each toss is made independently of the others and that the tosses are made under identical conditions. A possible outcome of this experiment is an infinite sequence of heads and tails such as

$$H, T, T, T, H, T, H, H, \cdots,$$

where H stands for head and T for tail. The sample space Ω for this experiment may be taken to be the collection of all infinite sequences of H's and T's. Let p, $0<p<1$, denote the probability of head in a single toss. The set of all sequences having exactly k H's among the first n coordinates corresponds to the event that exactly k heads occurred among the first n tosses, and elementary probabilistic considerations lead to the conclusion that to this set of sequences we should associate the probability $\binom{n}{k}p^k(1-p)^{n-k}$. It can be shown that this assignment of probabilities leads to a unique probability measure on the ring of all subsets of Ω determined by conditions on a finite number of coordinates. (A set A is said to be determined by conditions on a finite number of coordinates if there exists an integer n such that for each point ω in Ω the first n coordinates of ω determine whether or not ω is in A.) It is evident that the singleton sets do not belong to the ring of sets to which we have thus far attached probabilities. Suppose now that we try to extend the probability measure we have defined to some σ-algebra of subsets of Ω containing all the singletons. Consider the singleton set whose only member is the point $(H, H, H, \cdots)$. For every integer n, this event is contained in the event "n heads in the first n tosses." Since probability is monotonic, it follows that the probability of the singleton is, for every n, less than or equal to p^n. This implies that the probability of the singleton is zero. A similar argument leads to the conclusion that each singleton set must have probability zero. Thus, if we could extend the probability measure we originally defined to some σ-algebra containing all of the singleton sets, the extended probability measure would have to be continuous. Since the cardinality of the space under consideration is that of the continuum, it follows from Ulam's Theorem [7], under the continuum hypothesis, that $\mathcal{P}(\Omega)$ cannot carry a nontrivial continuous measure. (This was also shown independently by Banach and Kuratowski [1].) Yet physical considerations dictate that each singleton in Ω must have probability zero. Hence, if the probability model is to be faithful to the physical situation, it is impossible in the present example to define an event to be an arbitrary subset of Ω. (At least this is so if the continuum hypothesis is adopted as an axiom of set theory. Even if it is not, if the question of whether or not $\mathcal{P}(\Omega)$ can carry a nontrivial continuous measure is decidable from the other axioms of set theory, it must be decided in the negative, since this is the conclusion when the continuum hypothesis is used.) It is, of course, possible to define a continuous measure on the σ-algebra generated by the ring with which we started, which is done in the usual treatment of this problem in probability theory. By restricting the notion of event to such a σ-algebra, we obtain a model that fits the physical situation at the price of seeming artificiality in the definition of event. This example illustrates the fact that the class of probability measures on arbitrary σ-algebras in a space whose cardinality is that of the continuum is more useful than measures obtained by restriction from the power set. Hence, the decisive advantage of Kolmogorov's definition of event is that it leads to a larger and more useful class of probability spaces than would result from defining an event to be an arbitrary subset of a sample space.

The treatment of countable spaces in probability is in striking contrast to that of uncountable spaces. In the former case it is assumed invariably that the event class is the power set of the sample space. There appears to be no justification for this in the literature other than the fact that the procedure of assigning probabilities to all singletons leads in a simple way to a measure on the power set. Yet how can we be sure that we shall not encounter an experiment in which the sample space is countable and the conditions associated with the experiment are incompatible with all measures arising from point masses? The assurance is given by Theorem 3.2 which shows that the class of all measures on all σ-algebras of subsets of a countable space is no richer than the class of measures on the power set, i.e., the class arising from point masses.

Another way of viewing the situation in the countable case is suggested by the corollary to Theorem 4.1, which shows that no theory in which events are defined as elements of an arbitrary σ-algebra can be more general than one in which events are defined as members of the power set; for each σ-algebra $\mathfrak{F}$ of subsets of a countable space is isomorphic to the power set $\mathcal{P}$ of some space, so that a mathematical model involving $\mathfrak{F}$ can be replaced by one involving $\mathcal{P}$. In the countable case, therefore, the definition of an event as an arbitrary subset of the sample space is not only natural but correct.

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