UNIFORM FLOWS IN CASCADE GRAPHS

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§ 1. Cascade graphs

We shall call a directed graph G, having a finite or denumerably infinite number of points, a cascade graph, if it has the following properties:

a) There is in G a point $\underline{a}_0$--called the source--such that for any other point $\underline{a}$ of G there is in G a directed path from $\underline{a}_0$ to $\underline{a}$.

b) For any point $\underline{a}$ of G each directed path from $\underline{a}_0$ to $\underline{a}$ has the same length $r(\underline{a})$--called the rank of $\underline{a}$.

c) The number of points of G having rank k is finite for every $k \geq 1$.

We shall denote the set of all points of G by V and the set of all points of G having rank k by V_k $(k=0,1,2,\ldots)$. The source has clearly the rank 0 and it is the only point with this property: Thus V_0 is a one-element set containing the element $\underline{a}_0$ only, i.e., $V_0 = \{\underline{a}_0\}$. As the supposition b) has to hold for $\underline{a} = \underline{a}_0$ also, it follows that there is no directed cycle in G containing $\underline{a}_0$. It follows further that if there is in G an edge from the point $\underline{a}$ to the point $\underline{b}$ then $r(\underline{b}) - r(\underline{a}) = 1$, i.e. every edge starting from a point in V_k leads to a point in V_{k+1} $(k=0,1,\ldots)$. Let us denote by $d(\underline{a})$ the number of edges of G ending at the point $\underline{a}$, i.e. the indegree of $\underline{a}$, and by $D(\underline{a})$ the number of edges of G starting from $\underline{a}$, i.e. the outdegree of $\underline{a}$. Clearly both $d(\underline{a})$ and $D(\underline{a})$ are finite for every $a \in V$. We call a point $\underline{a} \in V$ an endpointof G if $D(\underline{a}) = 0$. We shall denote the set of endpoints of G by E and the set of endpoints of rank k of G by E_k. For any finite set A let $|A|$ denote the number of elements of A. We put $N_k = |V_k|$ and $M_k = |V_k - E_k|$, i.e. N_k

denotes the total number of points of rank k, and M_k the number of those points of rank k, which are not endpoints. We shall denote further by R the maximum of $r(\underline{a})$ for $\underline{a} \in V$ if V is finite and put $R = +\infty$ if V is infinite. Clearly if $M_k = 0$ for some value of k then $N_m = 0$ for $m > k$ and R is finite, and conversely.

Let us consider now some examples of cascade graphs.

Example 1. If G is a rooted directed tree in which all edges are directed away from the root, and $D(\underline{a})$ is finite for every point $\underline{a}$ of G, then G is a cascade graph, its source being the root of the tree. Conversely if in a cascade graph G, $d(\underline{a}) = 1$ for all points $\underline{a}$ different from $\underline{a}_0$ (for which of course $d(\underline{a}_0) = 0$) then G is a rooted directed tree in which all edges are directed away from the root.

Example 2. Let S be a finite set. Let the points of the graph G be all subsets of S and connect $a \subseteq S$ with $b \subseteq S$ by an edge (directed from a to b) if and only if b is obtained from a by omitting one of its elements. The graph G thus obtained is a cascade graph and for every $a \subseteq S$ one has $r(a) = |S| - |a|$.

Example 3. Let S be a finite set and let the points of the graph G be all non-negative integral valued functions defined on S. If f and g are two such functions, draw an edge from f to g if and only if $g(x) \geq f(x)$ for all $x \in S$ and $\sum_{x \in S} (g(x)-f(x)) = 1$. In this way we get a cascade graph and the rank of a function f is $r(f) = \sum_{x \in S} f(x)$.

Let us call a cascade graph simple, if it does not contain any infinite directed path. In a simple cascade graph to every point $\underline{a}$ there corresponds a nonempty set $T(\underline{a})$ consisting of those endpoints of the graph which can be reached from $\underline{a}$ by a directed path. We call $T(\underline{a})$ the target-set of the point $\underline{a}$. If there is a directed path from $\underline{a} \in V$ to $\underline{b} \in V$ then $T(\underline{b}) \subseteq T(\underline{a})$.

A subset A of the vertices of a cascade graph G is called an antichain, if for any two points a ε A and b ε A there does not exist in G a directed path from a to b. An antichain is called saturated, if it is not a proper subset of another antichain. An antichain is called a blocking antichain, if any directed path from the source to an endpoint, and every infinite directed path starting at the source, passes through a (unique) point of the antichain. Clearly every blocking antichain is saturated but a saturated antichain is not necessarily blocking. For instance in Example 2, let S be the set S = {1, 2 ...n} where n $\geq$ 3, and let the antichain A consist of the two sets {1} and {2, 3, ...n}. Then A is saturated, as every subset of S not containing the set {1} is a subset of the set {2, 3, ...,n} but it is not blocking. If the cascade graph G is a rooted tree (see Example 1) then a saturated antichain is always blocking if G is finite, but not necessarily if G is infinite. (See the following Example.)

Example 4. Let the points of the graph G be all finite sequences, each term of which is one of the numbers 0, 1, ..., q-1 where q $\geq$ 2; the empty sequence is also a point of G. Let there be in G an edge from the point a to the point b if the sequence b is obtained from the sequence a by adding to the end of the sequence a one more digit, i.e. one of the numbers 0, 1, ..., q-1. In this way we obtain a cascade graph, which is a tree, and has no endpoints. Let us take q = 2 and let the antichain A consist of the sequences 0, 10, 110, 1110, ... A is clearly a saturated antichain, but it is not blocking as it does not block the infinite path from the source (this being the empty sequence) leading through the points 1, 11, 111, ...

In a simple cascade graph the set of all endpoints is a blocking antichain. In any cascade graph the set V_k is an antichain and if there are no endpoints of rank < k, then V_k is a blocking antichain.

Let us write, for any two pointa a and b of a cascade graph, a < b if there is a directed path from a to b. G is a partially ordered set* with respect to the order relation < , but not necessarily a lattice. The following example shows that cascade graphs which are lattices have remarkable properties.

Example 5. Let $\mathcal{G}$ be a finite combinatorial geometry (see H. Crapo and G. C. Rota [1]). Let G be the graph, the points of which are the flats of $\mathcal{G}$ and there is an edge in G from the flat a to the flat b if and only if b covers a. Then G is a cascade graph, which is a lattice, and the rank of any flat a in G is the same as in $\mathcal{G}$. Let us remove from the cascade graph G its unique element with maximal rank: We obtain again a cascade graph, the endpoints of which are the copoints of $\mathcal{G}$. This cascade graph has among others the following remarkable property: The target sets corresponding to different points are different, and the target sets of points having the same rank form a Sperner-system (i.e. none of them contains any other as a subset.)

If a is any point of the cascade graph G we denote by Γa the set of all endpoints of edges starting at a. If A is any set of points of A we denote by ΓA the set of those points which are the endpoints of at least one edge starting at a point in A, i.e. we put $\Gamma A = \bigcup_{a \in A} \Gamma a$. If b is any point of G we denote by $\Gamma^{-1} b$ the set of those points a for which $b \in \Gamma a$.

§ 2. Random walks on a cascade graph

Let us assign to each edge ab (from a to b) of a cascade graph G a non-negative number w(a, b) such that

*What we call a cascade graph is, considered as a partially ordered set, a graded partially ordered sets: See Birkhoff [2] and Klarner [3], where the graded partially ordered sets with a given maximal rank and given number of points are counted.

(2.1.) $$\sum_{\underline{b} \in \Gamma \underline{a}} w(\underline{a}, \underline{b}) = 1 ,$$

for all vertices $\underline{a}$ of G which are not endpoints. Such a function w(a, b) defines a (Markovian) random walk on the edges of G as follows: The random walk starts always from the source $\underline{a}_0$ and proceeds to a point $\underline{a}$ of rank 1 with probability $w(\underline{a}_0, \underline{a})$; after arriving to the point $\underline{a}$, the walk continues with probability $w(\underline{a}, \underline{b})$ to a point $\underline{b}$, etc. Thus the random walk proceeds always along a directed path of G, until it reaches an endpoint, while if the path is infinite, the walk continues indefinitely. Such a random walk defines uniquely a probability measure P on the power set of the set of all paths starting from the source (this set being finite or denumerable). Let $B_{\underline{a}}$ denote the event that the random walk arrives eventually to the point $\underline{a}$. (In other words let $B_{\underline{a}}$ denote the set of all paths containing the point $\underline{a}$.) Let us put

(2.2.) $$w(a) = P(B_{\underline{a}}).$$

Let A be any antichain, then by definition the events $B_{\underline{a}}$ $(\underline{a} \in A)$ are mutually exclusive. Thus we have for every antichain A

(2.3.) $$\sum_{a \in A} w(a) \leq 1 .$$

If A is a blocking antichain, then the events $B_{\underline{a}}$ $(\underline{a} \in A)$ form a complete set of events (i.e. the sets $B_{\underline{a}}$ of paths are disjoint and their union is the set of all paths), and thus we have

(2.4.) $$\sum_{a \in A} w(a) = 1 .$$

§ 3. Normal cascade graphs

We shall call a cascade graph G normal if the transition-probabilities $w(\underline{a}, \underline{b})$ can be chosen in such a way that $w(\underline{a})$ depends only on the rank $r(\underline{a})$ of $\underline{a}$, i.e.

(3.1.) $$w(\underline{a}) = f(r(a))$$

where f(x) is a function defined on the set of non-negative integers.

Let B_k denote the event that the random walk does not stop before arriving to a point of rank k, and let C_k denote the event that the random walk does not stop at an endpoint of rank k. Then we have evidently*

(3.2.) $$P(B_{k+1}) = P(B_k)P(C_k | B_k) .$$

Now let G be a normal cascade graph, and suppose that the transition probabilities have been chosen so that (3.1) holds: In this case we call the random walk a uniform flow. In case we have a uniform flow on G, clearly

(3.3.) $$P(B_k) = N_k f(k)$$

and

(3.4.) $$P(C_k | B_k) = \frac{M_k}{N_k} .$$

It follows

(3.5.) $$f(k+1) = f(k)\frac{M_k}{N_{k+1}}$$

and thus, as f(0) = 1, we get

(3.6.) $$f(k) = \frac{1}{N_k} \prod_{j<k} \frac{M_j}{N_j} \quad \text{for } k \geqq 1$$

where an empty product is by definition equal to 1. Especially if the cascade graph is finite and it has no endpoits of less than maximal rank, (i.e. for every endpoint e one has r(e) = R), or if the

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* $P(C_k | B_k)$ denotes the conditional probability of C_k under condition B_k.

cascade graph is infinite and it does not contain any endpoints, then we have $M_j = N_j$ for $j < R$ and thus

(3.7.) $$f(k) = \frac{1}{N_k} .$$

The following theorem is an immediate consequence of (2.3) and (3.6):

THEOREM 1. Let G be a normal cascade graph. Let N_k denote the total number of points of rank k of G and M_k the number of those points of rank k which are not endpoints. Let A be an antichain in G and let n_k denote the number of points of rank k in A. Then the inequality

(3.8.) $$\sum_{k=0}^{R} \frac{n_k}{N_k} \prod_{j<k} \frac{M_j}{N_j} \leqq 1.$$

holds; if A is a blocking antichain there is equality in (3.8.).

COROLLARY: If $M_k = N_k$ for $k < R$, where R is the maximal rank of points of G, then for every antichain A we have

(3.9.) $$\sum_{k=0}^{R} \frac{n_k}{N_k} \leqq 1$$

with equality standing in (3.9.) if A is a blocking antichain.

Remark: Notice that (3.9.) can be written also in the equivalent form

(3.10.) $$\sum_{a \varepsilon A} \frac{1}{N_{r(a)}} \leqq 1.$$

We shall refer to Theorem 1. as the uniform flow theorem. In the next § we shall give a necessary and sufficient condition for the normality of a cascade graph.

§ 4. A necessary and sufficient condition for the normality of a cascade graph

THEOREM 2. A cascade graph G is normal if and only if it satisfies the following condition: For every $k \geq 1$ and for every subset A of the set $V_k - E_k$ of points of rank k which are not endpoits, one has

(4.1.) $$M_k |\Gamma A| \geq N_{k+1} |A|$$

where N_{k+1} is the number of points of rank $k+1$, and M_k the number of points of rank k which are not endpoints.

Proof* of Theorem 2. Clearly for every random walk (i.e. probability flow) on G and for every set of points $A \subseteq V-E$ one has

(4.2.) $$\sum_{\underline{a} \varepsilon A} w(\underline{a}) \leqq \sum_{\underline{b} \varepsilon \Gamma A} w(\underline{b})$$

Thus if $A \subset V_k - E_k$ and the flow is uniform then (4.1) holds, i.e. the condition is necessary. Let us prove now its sufficiency, i.e. that if (4.1) holds, one can choose the transition probabilities $w(\underline{a}, \underline{b})$ so that the flow should be uniform. We shall prove the existence of such transition probabilities $w(\underline{a}, \underline{b})$ step by step, i.e. by induction on the rank k of $\underline{a}$. Clearly if we put for every point $\underline{b}$ of rank 1, $w(\underline{a}_0, \underline{b}) = \frac{1}{N_1}$, then we have $w(\underline{b}) = \frac{1}{N_1}$ for every point $\underline{b}$ of rank 1. Let us suppose that we have already determined $w(\underline{a}, \underline{b})$ for all points $\underline{a}$ of rank $< k$ so that (3.1) holds for all $\underline{a} \varepsilon V_k$. We have to show that one can choose the values of $w(\underline{a}, \underline{b})$ for all $a \varepsilon V_k$ so that (3.1) holds with $k + 1$ instead of k too.

In other words, we have to choose the transition probabilities $w(\underline{a}, \underline{b})$ for $a \varepsilon V_k - E_k$ in such a way that they should be non-negative

* The proof given here is due to Dr. G. Katona.

and should satisfy the following two sets of equations:

$$\sum_{b \varepsilon \Gamma a} w(a, b) = 1 \quad \text{for all } a \varepsilon V_k - E_k \tag{4.3.}$$

and

$$\sum_{a \varepsilon \Gamma^{-1} b} w(a, b) = \frac{M_k}{N_{k+1}} \quad \text{for all } b \varepsilon V_{k+1} \tag{4.4.}$$

Let $a_1, a_2, \dots a_{M_k}$ and $b_1, b_2, \dots, n_{N_{k+1}}$ denote the elements of the sets $V_k - E_k$ and V_{k+1} respectively. We consider the auxiliary graph G* defined as follows: G* has $2M_kN_{k+1}$ points which we denote by $a_{i,j}$ $(1 \leqq i \leqq M_k ; 1 \leqq j \leqq N_{k+1})$ and $b_{u,v}$ $(1 \leqq u \leqq N_{k+1} ; 1 \leqq v \leqq M_k)$. We connect the points $a_{i,j}$ and $b_{u,v}$ in G* if and only if there is in G an edge from a_i to b_u. Thus G* is a bipartite graph with the two classes of points $\mathcal{A} = \{a_{i,j}\}$ and $\mathcal{B} = \{b_{u,v}\}$. Let for any subset A* of the set $\mathcal{A}$, Γ^*A^* denote the subset of those $b_{u,v}$ which are connected by at least one $a_{i,j}$ in G*. Let A denote the set of those $a_i \varepsilon V_k - E_k$ for which $a_{i,j}$ is for at least one j contained in A*. Then we have

$$|A^*| \leqq N_{k+1}|A| \tag{4.5.}$$

and

$$|\Gamma^*A^*| = M_k|\Gamma A| . \tag{4.6.}$$

Thus (4.1) implies

$$|\Gamma^*A^*| = M_k|\Gamma A| \geqq N_{k+1}|A| \geqq |A^*| . \tag{4.7.}$$

But (4.7) means that the conditions of the <u>marriage-theorem</u> (see e.g.

Harper, L. and Rota, G. C. [4] for the existence of a one-to one matching between the sets $\mathcal{A}$ and $\mathcal{B}$ so that each $a_{i,j}$ is matched with such a $b_{u,v}$ with which it is connected in G*, are fulfilled, and thus such a matching exists. Let us take such a matching and let s(i, u) denote the number of $a_{i,j}$ $(1 \leq j \leq N_{k+1})$ which are matched to a $b_{u,v}$ $(1 \leq v \leq M_k)$. Then we have evidently

$$\sum_{u=1}^{N_{k+1}} s(i, u) = N_{k+1} \quad \text{for } 1 \leq i \leq M_k \tag{4.8.}$$

and

$$\sum_{i=1}^{M_k} s(i, u) = M_k. \tag{4.9.}$$

Further s(i, u) = 0 if a_i and b_u are not connected in G. Thus putting

$$W(a_i, b_u) = \frac{s(i, u)}{N_{k+1}} \quad (1 \leq i \leq M_k,\ b_u \in \Gamma a_i) \tag{4.10.}$$

the equations (4.3) and (4.4) hold. Thus Theorem 2 is proved.

From Theorem 2 one can easily deduce the following:

<u>COROLLARY</u>: <u>If in a cascade graph</u> G <u>the outdegree</u> D(<u>a</u>) <u>of a point</u> <u>a</u>, <u>which is not an endpoint, depends only on the rank</u> r(<u>a</u>) <u>of</u> <u>a</u>, <u>and</u> <u>the indegree</u> d(b) <u>of any point</u> <u>b</u> <u>depends only on the rank</u> r(b) <u>of</u> <u>b</u> <u>then</u> G <u>is normal</u>.

<u>Proof</u>: Let us denote the outdegree of a point of rank k which is not an endpoint by D_k, and the indegree of a point of rank k by d_j. As the total number of edges starting from some point of rank k is equal to the total number of edges leading to a point of rank k + 1, we have

$$M_k \cdot D_k = N_{k+1} \cdot d_{k+1} . \tag{4.11.}$$

Now let A be any subset of $V_k - E_k$. As the number of edges going out from one of the points in A cannot be larger than the number of edges arriving to a point in ΓA, we have

$$d_{k+1}|\Gamma A| \geqq D_k|A| \quad . \tag{4.12.}$$

Multiplying both sides of (4.12) by N_{k+1}, and using (4.11) we get that (4.1) holds, i.e. that G is normal.

Remark. Instead of deducing the above corollary from Theorem 2 one can prove its statement by constructing effectively the uniform flow on the cascade graph. As a matter of fact, if for every a which is not an endpoint we put $w(a, b) = |\Gamma a|^{-1}$, then we get a uniform flow on the cascade graph G satisfying the conditions of the corollary.

The cascade graphs satisfying the conditions of the above Corollary of Theorem 4.1 are called semiregular cascade graphs. As by the Corollary every semiregular cascade graph is normal, it follows that the statement of Theorem 1 holds for every antichain of a semiregular cascade graph; this special case of Theorem 1 is due to Kirby A. Baker [5], who has formulated this result in a slightly different terminology, but the result expressed in our terminology is just the statement of Theorem 1 for semiregular cascade graphs.

§ 5. Kraft's inequality and Sperner's theorem as special cases of the uniform flow theorem

In spite of the extreme simplicity of its proof, Theorem 1 is a common source of several known results, thus for example Kraft's inequality (see e.g. Feinstein [6] and Sperner's theorem (see e.g. Lubell [7] . In this § we shall deduce these theorems and some of their generalizations as special instances of the uniform flow theorem.*

*I want to mention that I obtained the uniform flow theorem by analysing Lubell's elegant proof of Sperner's theorem [7].

Let us deal first with the Kraft inequality. Let us consider the cascade graph described in Example 4: Let the points of G be all finite sequences which can be formed from the digits 0, 1, ..., q-1 ($q \geq 2$) and let us draw a directed edge from the sequence $\underline{a}$ to the sequence $\underline{b}$ if and only if $\underline{b}$ is obtained by adding one more digit to the end of $\underline{a}$. Let the source be the empty sequence. It is easy to see that the cascade graph thus obtained is a tree, moreover a regular search tree in which there are q edges going out from every point and no endpoints. Let A be an antichain in G; thus A is a finite or denumerable family of finite sequences formed from the digits 0, 1, ..., q-1 such that no sequence in A is an initial segment (prefix) of another sequence in A. Such families of sequences are called in information theory q-ary prefix codes, and its elements codewords. Evidently G is a normal cascade graph; to show this we do not need Theorem 2, because it is easy to see that putting $w(\underline{a}, \underline{b}) = q^{-1}$ for every edge $\underline{a}\,\underline{b}$ of G we get $w(\underline{a}) = q^{-r(\underline{a})}$, i.e. a uniform flow. Applying Theorem 1 we obtain the following result, called Kraft's inequality:

If A is a q-ary prefix code ($q \geq 2$) and n_k denotes the number of codewords of length k in the code, then the inequality

$$\sum_{k=0}^{\infty} \frac{n_k}{q^k} \leq 1 \tag{5.1.}$$

holds; if A has the property that every infinite sequence consisting of the digits 0, 1, ..., q-1 contains one of the codewords from A as an initial segment, then there is equality in (5.1).

Clearly (5.1) can be formulated also in another form, speaking about trees instead of codes. It is natural to ask in general which rooted trees, considered as cascade graphs, are normal. The answer is very simple: A rooted tree is a normal cascade graph if and only if it is semiregular, i.e. the outdegree of any point which is not an end-point depends only on the rank of the point. If $D(a) = q_k$ for all $a \in V_k - E_k$ then $N_{k+1} = M_k \cdot q_k$ and thus we get from the uniform flow theorem the following result, which reduces to the Kraft inequality in the special case when $q_k = q$ for all k.

If A is an antichain in a semiregular rooted tree such that $D(a) = q_k$ if $a \in V_k - E_k$ and if n_k denotes the number of elements of the antichain A having rank r, then the inequality

$$\sum_{k=0}^{\infty} \frac{n_k}{q_1 q_2 \cdots q_k} \leq 1 \tag{5.2.}$$

holds.

Of course Kraft's inequality (as well as its generalization) can be proved quite easily directly, but it is instructive to consider this inequality as a particular instance of the uniform flow theorem.

Now let us consider how Sperner's theorem is obtained from Theorem 1. Let G be the cascade graph of Example 2 of § 1. Let the points of G be all subsets of an n-element set S, and let there be an edge in G from the set $\underline{a} \subseteq S$ to the set $\underline{b} \subseteq S$ if and only if $\underline{b}$ is obtained from $\underline{a}$ by omitting one of its elements. To show that the cascade graph G is normal it is sufficient to point out that if we put

$$w(\underline{a}, \underline{b}) = \frac{1}{n - r(a)} \quad \text{for every } \underline{a} \text{ with } r(\underline{a}) < n \tag{5.3.}$$

(notice that $r(a) = n$ only if $\underline{a}$ is the empty set and this is the (unique) endpoint of G), then we get, taking into account that there are $(n-k)!$ paths from the source to a point a for which $r(\underline{a}) = n - k$ i.e. to set $\underline{a}$ having k elements, it follows that

(5.4.) $$w(\underline{a}) = \frac{1}{\binom{n}{r(\underline{a})}} \quad \text{for all points } a,$$

i.e. there exists a uniform flow in G. (This follows also from the corollary of Theorem 2. It is easy to see further that a subset A of the points of G, i.e. a family of subsets of S in an antichain if for any two different sets $\underline{a} \subseteq S$ and $\underline{b} \subseteq S$ belonging to A, $\underline{a}$ is not a subset of $\underline{b}$, i.e. if A is a Sperner-system of subsets S. Thus the uniform flow theorem yields the following result:

If A is a Sperner system of subsets of an n-element set, and A contains n_k sets having k elements (k=0,1,...) then the inequality

(5.5.) $$\sum_{k=0}^{n} \frac{n_k}{\binom{n}{k}} \leq 1$$

holds.

As

(5.6.) $$\max_k \binom{n}{k} = \binom{n}{[\frac{n}{2}]},$$

we obtain from (5.6) the usual (though slightly weaker) form of Sperner's theorem:

(5.7.) $$|A| \leq \binom{n}{[\frac{n}{2}]} .$$

Before going further let us add a remark. Comparing the two special cases just discussed, it turns out that the property of a system of sets being a Sperner-system plays the same role in Sperner's theorem as the prefix property of a code in Kraft's inequality. As a matter of fact, there is a real connection between these two concepts, not only a superficial analogy.

As mentioned earlier the prefix property of a code implies that if the codewords of such a code are written one after the other, without indicating where one codeword ends and the next begins, and if the code has the prefix property, the sequence of symbols can be uniquely decoded, i.e. the codewords can be unambiguously separated from another. Now let us impose on the code the auxiliary restriction that a sequence of codewords should be uniquely decodable even in the case when the letters within each codeword are arbitrarily rearranged, i.e. if the codewords are unordered sets of letters, and not ordered sets as usual. If we require further that the same letter should not occur more than once in any codeword, then such a code is uniquely decodable if the codewords (considered as unordered sets of letters) form a Sperner-system. Expressed in the language of search theory*, a Sperner-system corresponds to such a strategy of search in which the values of the test functions are obtained simultaneously and one does not know which value comes from which test function.

For example let us consider the 19 numbers

0,1,2,3,4,5,6,7,8,12,13,14,16,17,18,19,23,24,29.

These numbers can be uniquely characterized by their residues mod 2, mod 3, mod 5, even if these three residues are given in a random order. For instance if we are told that the three residues are 0, 0, and 2, it is easy to see that among the 19 numbers enlisted above only 20 has these residues, namely it is congruent to 0 mod 2 and mod 5 and to 2 mod 3.

As a further application of the uniform flow theorem, we prove the following generalization of Sperner's theorem:

THEOREM 3. *Let $\mathcal{A}$ be a family of ordered r-tuples* ($r \geq 1$)

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* As regards the connection of the results of this paper with search theory, see [8] .

of disjoint subsets of an n-element set S, such that if $(A_1, A_2, \ldots, A_r)$ and $(B_1, B_2, \ldots, B_r)$ both belong to the family $\mathcal{A}$ then the relations $A_j \subseteq B_j$ $(j=1,1,\ldots,r)$ cannot hold simultaneously. Then the number of elements of the family $\mathcal{A}$ satisfies the inequality

$$|\mathcal{A}| \leq r^{\left[\frac{r(n+1)}{r+1}\right]} \cdot \binom{n}{\left[\frac{r(n+1)}{(r+1)}\right]} . \tag{5.8.}$$

which is best possible, i.e. equality is possible in (5.8) by a suitable choice of the class $\mathcal{A}$.

Proof of Theorem 3. Let us construct a cascade graph G as follows: The points of G are all possible $(r + 1)^n$ ordered r-tuples $(A_1, A_2, \ldots, A_r)$ of disjoint subsets of S. The source of G is the r-tuple each element of which is the empty set. The rank of an r-tuple $\underline{a} = (A_1, A_2, \ldots, A_r)$ is $r(\underline{a}) = \sum_{j=1}^{r} |A_j|$ and there is an edge from $\underline{a} = (A_1, A_2, \ldots, A_r)$ to $\underline{b} = (B_1, B_2, \ldots, B_r)$ if and only if the following conditions are satisfied: $A_j \subseteq B_j$ for $j=1,2,\ldots,r$ and $r(\underline{b}) = r(\underline{a}) + 1$; in other words there is in G an edge from $\underline{a}$ to $\underline{b}$ if $A_j = B_j$ for all but one value of j $(1 \leq j \leq r)$--say except for i --and B_i is obtained from A_i by adding one more element of S to A_i. Clearly the conditions of Theorem 3 mean that $\mathcal{A}$ should be an antichain in G. Thus if we show that G is normal, we can apply Theorem 1.

We shall prove the normality of G by verifying that the conditions of the corollary to Theorem 1 are fulfilled. Let $a = (A_1, A_2, \ldots, A_r)$ be any point in G which is not an endpoint; then we have $r(\underline{a}) < n$ because the endpoints of G are the points with $r(\underline{a}) = n$. Thus there are $r - r(\underline{a})$ elements of S which can be added to one of the sets

A_1 to get a point $\underline{b}$ to which there leads an edge from $\underline{a}$: The number of possible choices is thus $r\cdot(n-r(\underline{a}))$. Thus $D(a) = r\,(n-r(\underline{a}))$ depends only on the rank of $\underline{a} \in V - E$. Let now $\underline{b}$ be any point of G. By a similar argument we get that $d(\underline{b}) = r(\underline{b})$. Thus all conditions of the Corollary of Theorem 1 are satisfied, and therefore G is normal. Thus we can apply Theorem 1 to the antichain $\mathcal{A}$ and we get, in view of $N_k = r^k \binom{n}{k}$ and $M_k = N_k$ if $k < n$, that

(5.9.)

$$\sum_{k=0}^{n} \frac{n_k}{r^k \binom{n}{k}} \leq 1,$$

where n_k denotes the number of elements of $\mathcal{A}$ having rank k.
As however

$$(5.10) \qquad \max_{0 \leq k \leq n} r^k \binom{n}{k} = r^{\left[\frac{(n+1)r}{r+1}\right]} \binom{n}{\left[\frac{(n+1)r}{(r+1)}\right]}, \text{ and } \sum_{k=0}^{n} n_k = |\mathcal{A}|$$

it follows that (5.8) holds.

Clearly for r = 1 Theorem 3 is nothing else than Sperner's theorem. To show that the inequality (5.8) is best possible take for $\mathcal{A}$ the family of all r-tuples $(A_1, A_2, \ldots, A_r)$ of disjoint subsets of S such that $\sum_{j=1}^{r} |A_j| = \left[\frac{(n+1)r}{r+1}\right]$ (i.e. all points of rank $\left[\frac{(n+1)r}{(r+1)}\right]$ of G); $\mathcal{A}$ clearly satisfies the requirements of the Theorem 3 and $|\mathcal{A}|$ is equal to the right hand side of (5.8).

Other generalizations of Sperner's theorem can also be obtained from Theorem 3. We intend to return to these elsewhere.

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