SOME PROPERTIES OF 2-AUTO-ENGEL GROUPS

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Abstract. Given a group G, an element $x \in G$ and automorphism $\alpha \in \mathrm{Aut}(G)$, the nth autocommutator $[x,_n \alpha]$ is defined recursively by $[x, \alpha] = x^{-1}x^{\alpha}$ and $[x,_n \alpha] = [[x,_{n-1} \alpha], \alpha]$ for all $n > 1$. The group G is said to be n-auto-Engel if $[x,_n \alpha] = [\alpha,_n x] = 1$, for all $x \in G$ and all $\alpha \in \mathrm{Aut}(G)$, where $[\alpha, x] = [x, \alpha]^{-1}$. We study the structure of 2-auto-Engel groups and show that 2-auto-Engel groups indeed satisfy the equation $\alpha(x)\alpha^{-1}(x) = x^2$, for all $x \in G$ and $\alpha \in \mathrm{Aut}(G)$. Also, we give a precise description of all abelian 2-auto-Engel groups of finite 2-rank as well as 2-auto-Engel 2-groups, which are not purely non-abelian, and construct an infinite family of purely non-abelian 2-auto-Engel 2-groups.

1. Introduction

Let x_1 and x_2 be the elements of a given group G, then $x_1^{x_2} = x_2^{-1}x_1x_2$ and $[x_1, x_2] = x_1^{-1}x_1^{x_2}$ denote the *conjugate* of x_1 by x_2 and the *commutator* of x_1 and x_2, respectively. As in Hegarty [8], the *autocommutator* of an element $x \in G$ and automoprhism $\alpha \in \mathrm{Aut}(G)$ is defined by $[x, \alpha] = x^{-1}x^{\alpha}$. The same as for the commutator subgroup, one may define the *autocommutator subgroup* of G in an analogous way as follows:

$$K(G) = \langle [x, \alpha] : x \in G, \alpha \in \mathrm{Aut}(G) \rangle .$$

The notion of autocommutator subgroups have been already studied in [5, 8, 15]. For each element $x \in G$ and automorphisms $\alpha_1, \ldots, \alpha_n \in \mathrm{Aut}(G)$, we define the autocommutator of $x, \alpha_1, \ldots, \alpha_n$ of weight $n + 1$ ($n \geq 1$), recursively by

$$[x, \alpha_1, \ldots, \alpha_n] = [[x, \alpha_1, \ldots, \alpha_{n-1}], \alpha_n].$$

Clearly the $(n + 1)$st term of the lower central series of G, can be considered as

$$\gamma_{n+1}(G) = \langle [x, \alpha_1, \ldots, \alpha_n] : x \in G, \alpha_1, \ldots, \alpha_n \in \mathrm{Inn}(G) \rangle .$$

So we may define the nth *autocommutator subgroup* of G, as

$$K_n(G) = \langle [x, \alpha_1, \alpha_2, \ldots, \alpha_n] : x \in G, \alpha_1, \ldots, \alpha_n \in \mathrm{Aut}(G) \rangle .$$

One notes that, the nth autocommutator subgroup is a characteristic subgroup of G containing $\gamma_{n+1}(G)$, for all $n \geq 1$. The following series of subgroups

$$G = K_0(G) \geq K(G) = K_1(G) \geq K_2(G) \geq \cdots \geq K_n(G) \geq \cdots ,$$

is called the *lower autocentral series* of G (see [15] for more details).

Let x be an element of G and $\alpha \in \mathrm{Aut}(G)$. Then the autocommutator $[x,_n \alpha]$ ($n \geq 1$) is defined inductively by $[x,_1 \alpha] = [x, \alpha]$ and $[x,_n \alpha] = [[x,_{n-1} \alpha], \alpha]$, for

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$n \geq 2$. The element x is called a *right auto-Engel* element, if for every $\alpha \in \mathrm{Aut}(G)$, there exists a natural number $n = n(x, \alpha)$ such that $[x,_n \alpha] = 1$. If n can be chosen independent of α, then x is called a *right n-auto-Engel* element or simply a *bounded right auto-Engel element*. We denote the sets of right auto-Engel elements and bounded right auto-Engel elements of G by $AR(G)$ and $\overline{AR}(G)$, respectively.

An element x is called a *left auto-Engel* element, if for every $\alpha \in \mathrm{Aut}(G)$, there exists a natural number $n = n(x, \alpha)$ such that $[\alpha,_n x] = 1$. Note that, the automorphism α appears on the left and $[\alpha, x] = [x, \alpha]^{-1}$. If n can be chosen independent of α, then x is called a *left n-auto-Engel* element or simply a *bounded left auto-Engel element*. We denote the sets of left auto-Engel elements and bounded left auto-Engel elements of G by $AL(G)$ and $\overline{AL}(G)$, respectively.

Clearly, if α runs over the set of all inner automorphisms of G, then a right or left auto-Engel element is a right or left Engel element, respectively [17].

The properties of Engel elements and Engel groups are studied by many authors, see for instance [9, 13]. Kappe [13] proved that the set of all right 2-Engel elements of a group forms a characteristic subgroup. Also, Levi [14] has shown that a 2-Engel group is nilpotent of class at most 3. Moreover, it is known that for any 2-Engel group G with finite exponent, we have $\exp(G') = \exp(G/Z(G))$ $(= \exp(\mathrm{Inn}(G)))$.

2. Auto-Engel elements and n-auto-Engel groups

For a given group G and automorphisms α and β of G, we set $x^{\alpha\beta} = (x^\alpha)^\beta$, for all $x \in G$. Using the above notation, we have the following identities.

(a) $[xy, \alpha] = [x, \alpha]^y\ [y, \alpha]$;
(b) $[x, \alpha^{-1}] = ([x, \alpha]^{-1})^{\alpha^{-1}}$;
(c) $[x^{-1}, \alpha] = ([x, \alpha]^{-1})^{x^{-1}}$;
(d) $[x, \alpha\beta] = [x, \beta][x, \alpha]^\beta = [x, \beta][x, \alpha][x, \alpha, \beta]$;
(e) $[x, \alpha]^\beta = [x^\beta, \alpha^\beta]$,
(f) $[x, \alpha^{-1}, \beta]^\alpha [\alpha, \beta^{-1}, x]^\beta [\beta, x^{-1}, \alpha]^x = 1$.

The following relation holds between right and left auto-Engel elements.

Proposition 2.1. *In any group G the inverse of a right n-auto-Engel element is a left $(n+1)$-auto-Engel element, that is $AR(G)^{-1} \subseteq AL(G)$ and $\overline{AR}(G)^{-1} \subseteq \overline{AL}(G)$.*

Proof. Let x be a right n-auto-Engel element and α be any automorphism of G. Then by the above identities, $[x^\alpha,_n \beta] = [x,_n \beta^{\alpha^{-1}}]^\alpha = 1$. So x^α is also a right n-auto-Engel element and hence a right n-Engel element of G. Therefore

$$\begin{aligned} 1 = [x^\alpha,_n x^{-1}]^{x^{-1}} &= [[x^\alpha, x^{-1}],_{n-1} x^{-1}]^{x^{-1}} = [[x[x, \alpha], x^{-1}],_{n-1} x^{-1}]^{x^{-1}} \\ &= [[[x, \alpha], x^{-1}],_{n-1} x^{-1}]^{x^{-1}} = [[x, \alpha],_n x^{-1}]^{x^{-1}} \\ &= [[x, \alpha]^{x^{-1}},_n x^{-1}] = [[\alpha, x^{-1}],_n x^{-1}] \\ &= [\alpha,_{n+1} x^{-1}]. \end{aligned}$$

Hence x^{-1} is a left $(n+1)$-auto-Engel element of G. This argument also shows that $AR(G)^{-1} \subseteq AL(G)$. □

For a group G, if $G = AR(G)$ then Proposition 2.1 implies that $G = AL(G)$, while the converse of the latter statement is not true in general. For example, consider the cyclic group $\mathbb{Z}_6$ of order 6. Clearly, $\mathbb{Z}_6 = AL(\mathbb{Z}_6)$. On the other hand, if

α is the inverting automorphism of $\mathbb{Z}_6$, then one can easily see that $[x,_n \alpha] = x^4$, for each $n \geq 1$. Therefore $AR(\mathbb{Z}_6) \subset \mathbb{Z}_6$. Hence, the following definition is meaningful.

Definition. A group G is called an *n-auto-Engel group* if $[x,_n \alpha] = [\alpha,_n x] = 1$, for all $x \in G$ and $\alpha \in \mathrm{Aut}(G)$. Also, G is called an *auto-Engel group* if $G = AR(G)$.

By the above discussion, the cyclic group $\mathbb{Z}_6$ is not an auto-Engel group. Now, we give some examples of auto-Engel groups.

Example 1. (a) If $\mathbb{Z}_{2^n} = \langle x : x^{2^n} = 1 \rangle$ is the cyclic group of order 2^n and α is an automorphism of $\mathbb{Z}_{2^n}$, given by $\alpha : x \longmapsto x^r$ (r odd), then it is easy to see that $[x^k,_n \alpha] = x^{k(r-1)^n} = 1$ for each $k \in \{1, \ldots, 2^n\}$. Thus $\mathbb{Z}_{2^n}$ is an n-auto-Engel group.

(b) Let $\mathbb{Z}_{2^\infty} = \langle x_1, x_2, \ldots : x_1^2 = 1, x_{i+1}^2 = x_i, i \geq 2 \rangle$ be the Prüfer 2-group. Then $\mathrm{Aut}(\mathbb{Z}_{2^\infty}) = \{\alpha : x_i \longrightarrow x_i^{n_i}, n_{i+1} \equiv n_i \pmod{2^i}, i \geq 1\}$. Hence $[x_i,_i \alpha] = 1$ for all $\alpha \in \mathrm{Aut}(\mathbb{Z}_{2^\infty})$ and $i \geq 1$, which implies that $\mathbb{Z}_{2^\infty}$ is an auto-Engel group. However, $\mathbb{Z}_{2^\infty}$ is not n-auto-Engel group for all $n \geq 1$ since $[x_{n+1},_n \alpha] = x_{n+1}^{(-2)^n} = x_1^{(-1)^n} \neq 1$ if α is the inverting automorphism of $\mathbb{Z}_{2^\infty}$.

(c) If $D_{2^n} = \left\langle x, y : x^{2^{n-1}} = y^2 = 1, x^y = x^{-1} \right\rangle$, is the dihedral group of order 2^n ($n \geq 3$), then

$$\mathrm{Aut}(D_{2^n}) = \left\{ \alpha_{ij} : \begin{array}{c} x \mapsto x^i \\ y \mapsto x^j y \end{array}, i \text{ is odd and } i, j \in \{1, \ldots, 2^{n-1}\} \right\},$$

is the automorphism group of D_{2^n}. Since i is odd, we can see that $[x,_n \alpha_{ij}] = x^{(i-1)^n} = 1$ and $[y,_n \alpha_{ij}] = x^{-j(i-1)^{n-1}} = 1$. Also $[\alpha_{ij},_n x] = [\alpha_{ij},_n y] = 1$, from which it follows that $[x^k y,_n \alpha_{ij}] = [\alpha_{ij},_n x^k y] = 1$, for all $i, j, k \in \{1, \ldots, 2^{n-1}\}$ (i is odd). Therefore the dihedral group of order 2^n ($n \geq 3$) is an n-auto-Engel group but it is not an $(n-1)$-auto-Engel group. We note that D_8 is a 2-Engel group but it is not a 2-auto-Engel group.

The following theorem gives a description of abelian auto-Engel groups.

Theorem 2.2. *Let G be an abelian group.*

(1) *If G is an auto-Engel group, then G is a 2-group.*

(2) *If G has finite 2-rank, then G is an auto-Engel group if and only if G is the direct product of cyclic or quasi-cyclic groups of different cardinalities. Moreover, if G is a finite auto-Engel group, then G is an mn-auto-Engel group, where $m = r_2(G)$ is the 2-rank of G and $n = \log_2 \exp(G)$.*

Proof. (1) If α is the inverting automorphism of G and $x \in G$, then $[x,_n \alpha] = x^{(-2)^n}$. By the assumption, there exists $n = n(x, \alpha)$ such that $[x,_n \alpha] = 1$, which implies that x is a 2-element. Therefore G is a 2-group.

(2) If G has finite 2-rank, then by [18, 4.3.13], G is a direct product of finitely many cyclic or quasi-cyclic groups. If G has a homocyclic direct factor $\langle a \rangle \times \langle b \rangle$ and α is the automorphism of G, which sends a, b to ab^{-1} and a, respectively, then $[a,_3 \alpha] = a$ and hence $[a,_n \alpha] \neq 1$ for all $n \geq 1$, which is a contradiction. Similarly, G has no direct factor of type $\mathbb{Z}_{2^\infty} \times \mathbb{Z}_{2^\infty}$. Hence, G is a direct product of finitely many cyclic or quasi-cyclic groups with different cardinalities.

Conversely, suppose that G is a direct product of finitely many cyclic or quasicyclic groups with different cardinalities. Then $G = Z \times H$, where $Z = 1$ or

$Z = \langle x_1, x_2, \ldots : x_1^2 = 1, x_{i+1}^2 = x_i, i \geq 1\rangle$ is the Prüfer 2-group, and H is a finite abelian 2-group. First we show that Z is a characteristic subgroup of G. If $Z = 1$, then there is nothing to prove. Hence, we assume that $Z \neq 1$. Let $\exp(H) = 2^n$ and $\alpha \in \mathrm{Aut}(G)$. Then $\alpha(x_{i+n}) = x_{i+n}^{l_{i+n}} h_i$ for some $h_i \in H$ and integer l_{i+n}, for all $i \geq 1$. Thus $\alpha(x_i) = \alpha(x_{i+n}^{2^n}) = x_{i+n}^{2^n l_{i+n}} \in Z$, which implies that $\alpha(Z) \subseteq Z$ and Z is a characteristic subgroup of G. If $H = 1$, then by Example 1(b), we are done. Now, suppose that $H \neq 1$ and let $K = \Omega_{n+1}(Z)H = \langle y_1\rangle \times \cdots \times \langle y_m\rangle$, where $|y_i| = 2^{n_i}$ and $n_1 > \cdots > n_m$. If $\alpha \in \mathrm{Aut}(G)$, then

$$\alpha(y_i) = y_1^{2^{n_1-n_i} a_{i,1}} \ldots y_{i-1}^{2^{n_{i-1}-n_i} a_{i,i-1}} y_i^{a_{i,i}} t_i,$$

where $a_{i,j}$ is integer, $a_{i,i}$ is odd and $t_i \in \langle y_{i+1}, \ldots, y_m\rangle$. Thus $|[y_1, \alpha]| < |y_1|$ and if $|[y_j,_j \alpha]| < |y_j|$ for $j = 1, \ldots, i-1$, then $|[y_i,_i \alpha]| < |y_i|$. Because

$$[y_i,_i \alpha] = \prod_{j=1}^{i-1} [y_j,_{i-1}]^{2^{n_j-n_i} a_{i,j}} [y_i^{a_{i,i}-1} t_i,_{i-1} \alpha],$$

and we have $|[y_i^{a_{i,i}-1} t_i,_{i-1} \alpha]| < |y_i|$ and for $j < i$,

$$\left|[y_j,_{i-1} \alpha]^{2^{n_j-n_i} a_{i,j}}\right| \leq \left|[y_j,_j \alpha]^{2^{n_j-n_i} a_{i,j}}\right| < \left|y_j^{2^{n_j-n_i} a_{i,j}}\right| \leq |y_i|.$$

Thus $|[k,_m \alpha]| < |y|$ for all $k \in K$, from which it follows that $[k,_{mn} \alpha] = 1$. Therefore G is an auto-Engel group. □

In the following we shall obtain a sharp bound $n = n(G)$ for a finite auto-Engel abelian group G to be an n-auto-Engel group.

Lemma 2.3. *If $G = \mathbb{Z}_{2^n} \oplus \cdots \oplus \mathbb{Z}_2$, then G is $(2n-1)$-auto-Engel. Furthermore, G is not $(2n-2)$-auto-Engel.*

Proof. If $\alpha \in \mathrm{Aut}(G)$, then by using the heights and orders of elements of G, it follows that

$$\begin{aligned}
\alpha(1,0,0,0,\ldots,0) &= (a_{1,1}, a_{1,2}, \ldots, a_{1,n}),\\
\alpha(0,1,0,0,\ldots,0) &= (2a_{2,1}, a_{2,2}, a_{2,3}, \ldots, a_{2,n}),\\
\alpha(0,0,1,0,\ldots,0) &= (2^2 a_{3,1}, 2a_{3,2}, a_{3,3}, \ldots, a_{3,n}),\\
&\ \vdots\\
\alpha(0,0,0,0,\ldots,1) &= (2^{n-1} a_{n,1}, 2^{n-2} a_{n,2}, \ldots, 2a_{n,n-1}, a_{n,n})
\end{aligned}$$

for some integers $a_{i,j}$ $(1 \leq i, j \leq n)$, where $a_{i,i}$ is odd for $i = 1, \ldots, n$. Moreover, all automorphisms of G do arise in this manner. As an aside, this does allow one to compute that $|\mathrm{Aut}(G)| = 2^{\frac{1}{6}(n-1)n(2n+5)}$.

Now, assuming that for some $x \in G$ and $\alpha \in \mathrm{Aut}(G)$ that

$$[x,_k \alpha] = (2^{m_{k,1}} b_1, 2^{m_{k,2}} b_2, \ldots, 2^{m_{k,n}} b_n)$$

we can compute

$$\begin{aligned}[][x,_{k+1}\alpha] &= (2^{m_{k,1}}b_1(a_{1,1}-1) + 2^{m_{k,2}+1}b_2a_{2,1} + 2^{m_{k,2}+2}b_3a_{3,1} + \cdots + 2^{m_{k,n}+n-1}b_na_{n,1},\\ &= 2^{m_{k,1}}b_1a_{1,1} + 2^{m_{k,2}}b_2(a_{2,2}-1) + 2^{m_{k,3}+1}b_3a_{3,2} + \cdots + 2^{m_{k,n}+n-2}b_na_{n,2},\\ &\vdots\\ &= 2^{m_{k,1}}b_1a_{1,n} + 2^{m_{k,2}}b_2a_{2,n} + \cdots + 2^{m_{k,n-1}}b_{n-1}a_{n-1,n} + 2^{m_{k,n}}b_n(a_{n,n}-1).\end{aligned}$$

Thus, it easily follows that

$$m_{k+1,1} \geq \min\{m_{k,1}+1, m_{k,2}+1, m_{k,3}+2, \ldots, m_{k,n}+n-1\}$$

and for $j \geq 2$

$$m_{k+1,j} \geq \min\{m_{k,1}, \ldots, m_{k,j-1}, m_{k,j}+1, m_{k,j+1}+1, \ldots, m_{k,n}+n-j\}.$$

Therefore $m_{k,j} \geq h_{k,j}$ for all integers k and $j = 1, \ldots, n$, in which h is a function defined by

$$\begin{aligned} h_{0,j} &= 0, \text{ for } j = 1, \ldots, n,\\ h_{k+1,1} &= \min\{h_{k,1}+1, h_{k,2}+1, \ldots, h_{k,n}+n-1\}, \text{ and for } j \geq 2\\ h_{k+1,j} &= \min\{h_{k,1}, \ldots, h_{k,j-1}, h_{k,j}+1, h_{k,j+1}+1, \ldots, h_{k,n}+n-j\}.\end{aligned}$$

Using mathematical induction it is straightforward to show that

$$h_{k,j} = \max\left\{\left\lfloor \frac{k-j+2}{2} \right\rfloor, 0\right\}$$

for all positive integers k and $j = 1, \ldots, n$. Hence,

$$m_{2n-1,j} \geq h_{2n-1,j} \geq \left\lfloor \frac{(2n-1)-j+2}{2} \right\rfloor \geq n-j+1,$$

which implies that $[x,_{2n-1}\alpha] = (0, \ldots, 0)$, as required.

Finally, let $x = (1, 0, 0, \ldots, 0)$ and define the automorphism α of G by

$$\begin{aligned} \alpha(1,0,0,0,\ldots,0) &= (1,1,0,0,\ldots,0),\\ \alpha(0,1,0,0,\ldots,0) &= (2,1,0,0,\ldots,0),\\ \alpha(0,0,1,0,\ldots,0) &= (0,0,1,0,\ldots,0),\\ &\vdots\\ \alpha(0,0,0,0,\ldots,1) &= (0,0,0,0,\ldots,1).\end{aligned}$$

A simple computation shows that

$$[(1,0,\ldots,0),_{2n-2}\alpha] = (2^{n-1},0,\ldots,0) \neq (0,0,\ldots,0),$$

from which the result follows. □

Theorem 2.4. *Let G be a finite abelian group which is the direct product of cyclic 2-groups having distinct orders so that $\exp(G) = 2^n$. Then G is $(2n-1)$-auto-Engel.*

Proof. Since G is isomorphic to a direct factor of the group $\mathbb{Z}_{2^n} \oplus \cdots \oplus \mathbb{Z}_2$, the result follows by Lemma 2.3. □

3. 2-auto-Engel elements and 2-auto-Engel groups

If G is a 1-auto-Engel group, then $[x,\alpha]=[\alpha,x]=1$, for all $x\in G$ and $\alpha\in \mathrm{Aut}(G)$. Hence $\mathrm{Aut}(G)=1$ and consequently $G\cong 1$ or $\mathbb{Z}_2$. Observe that the class of 1-Engel groups coincides with the class of abelian groups.

It is unsolved problem that whether the four subsets $R(G)$, $\overline{R}(G)$, $L(G)$ and $\overline{L}(G)$ are subgroups of G? (see [17]). The same problem also appears in the case of auto-Engel elements.

The properties of 2-Engel groups have been already studied (see for example [3, 10, 13]). In this section, we concentrate on the same properties for 2-auto-Engel groups.

Remark. Let G be a finite 2-auto-Engel abelian group. If α is the inverting automorphism of G, then $x^4=[x,\alpha,\alpha]=1$, for every $x\in G$. Hence $\exp(G)$ divides 4 so that G is the direct sum of cyclic groups of order 2 and 4, whence by Theorem 2.2, $G\cong 1,\mathbb{Z}_2,\mathbb{Z}_4$ or $\mathbb{Z}_4\times\mathbb{Z}_2$. If $G=\langle x\rangle\times\langle y\rangle\cong\mathbb{Z}_4\times\mathbb{Z}_2$ and α is the automorphism of G which sends x and y to xy and x^2y, respectively, then $[x,\alpha,\alpha]=x^2\neq 1$, which is a contradiction. Thus $G\cong 1,\mathbb{Z}_2$ or $\mathbb{Z}_4$.

The next lemma will be used frequently in the proof of our main theorems.

Lemma 3.1. *Let x be a right 2-auto-Engel element and α, β and γ be arbitrary automorphisms of a group G. Then*

(a) *x is a left 2-auto-Engel element;*
(b) *$x^{\mathrm{Aut}(G)}=\langle x^\alpha : \alpha\in\mathrm{Aut}(G)\rangle$ is abelian and its elements are right (so left) 2-auto-Engel elements;*
(c) *$[x,\alpha,\beta]=[x,\beta,\alpha]^{-1}$;*
(d) *$[x,[\alpha,\beta]]=[x,\alpha,\beta]^2$;*
(e) *$[x,\alpha,\beta,\gamma]^2=1$;*
(f) *$[x,[\alpha,\beta],\gamma]=1$.*

Proof. The proof is straightforward (see [15, Theorem 7.13]). □

Theorem 3.2. *The set of all right 2-auto-Engel elements of a group G forms a characteristic subgroup.*

Proof. Let x and y be right 2-auto-Engel elements of a group G, α be any automorphism and φ_y be the inner automorphism induced by y. Then

$$\begin{aligned}[xy^{-1},\alpha,\alpha] &= [[x,\alpha]^{y^{-1}}[y^{-1},\alpha],\alpha]\\ &= [[x,\alpha][y,\alpha]^{-1},\varphi_y\alpha\varphi_{y^{-1}}]^{y^{-1}}\\ &= [[x,\alpha][y,\alpha]^{-1},[\varphi_{y^{-1}},\alpha^{-1}]\alpha]^{y^{-1}}\\ &= ([[x,\alpha],[\varphi_{y^{-1}},\alpha^{-1}]\alpha]^{[y,\alpha]^{-1}}[[y,\alpha]^{-1},[\varphi_{y^{-1}},\alpha^{-1}]\alpha])^{y^{-1}}.\end{aligned}$$

Now, we show that $[[x,\alpha],[\varphi_{y^{-1}},\alpha^{-1}]\alpha]=1$. By Lemma 3.1(c,f) and the fact that x is a right 2-auto-Engel element, we have

$$[[x,\alpha],[\varphi_{y^{-1}},\alpha^{-1}]\alpha]=[[x,\alpha],[\varphi_{y^{-1}},\alpha^{-1}]]^\alpha=[x,[\varphi_{y^{-1}},\alpha^{-1}],\alpha]^{-\alpha}=1.$$

Similarly $[[y,\alpha]^{-1},[\varphi_{y^{-1}},\alpha^{-1}]\alpha]=1$ and hence $[xy^{-1},\alpha,\alpha]=1$. Also, one may easily see that the subgroup of right 2-auto-Engel elements is characteristic. This completes the proof. □

The following lemma is needed in proving our next result (Theorem 3.5, below).

Lemma 3.3. *Let G be a 2-auto-Engel group. Then for every $x, y \in G$, $\alpha \in \mathrm{Aut}(G)$ and $n \in \mathbb{Z}$ the following properties hold:*

(a) $[x, x^\alpha] = 1$;
(b) $[x, \alpha^n] = [x, \alpha]^n = [x^n, \alpha]$;
(c) $[x^\alpha, y] = [x, y^\alpha]$;
(d) $[\alpha, x, y] = [\alpha, y, x]^{-1}$.

Proof. (a) As x is a left 2-auto-Engel element of G, we get $[\alpha, x, x] = 1$ and so that $[x, x^\alpha] = 1$.

(b) Since G is a 2-auto-Engel group, $[x, \alpha^2] = [x, \alpha]^2$ for every $x \in G$ and $\alpha \in \mathrm{Aut}(G)$. On the other hand, as in the proof of Lemma 3.1(c), $[x, \alpha]^{-1} = [x, \alpha^{-1}]$ and by using induction, we get $[x, \alpha^n] = [x, \alpha]^n$ for every $n \in \mathbb{Z}$. Also, part (a) implies that $[x^{-1}, \alpha] = [x, \alpha]^{-1}$ and $[x, \alpha]^n = (x^{-1}\alpha(x))^n = x^{-n}\alpha(x^n) = [x^n, \alpha]$.

(c) By part (a), $[xy^{-1}, (xy^{-1})^\alpha] = 1$. Therefore $xy^{-1}x^\alpha y^{-\alpha} = x^\alpha y^{-\alpha} xy^{-1}$. Clearly $x^{-\alpha}y^{-1}x^\alpha y = x^{-1}y^{-\alpha}xy^\alpha$. Thus $[x^\alpha, y] = [x, y^\alpha]$.

(d) Since G is a 2-auto-Engel (and so 2-Engel) group, we have $[x^\alpha, y, y] = 1$. Therefore by part (c), $[x, y^\alpha, y] = 1$ and similarly $[y, x^\alpha, x] = 1$. As the derived subgroup of a 2-Engel group is abelian, we have $[x^{-\alpha}, y]^x[x, y] = ([y^{-\alpha}, x]^y[y, x])^{-1}$ and hence $[\alpha, x, y] = [\alpha, y, x]^{-1}$. □

Corollary 3.4. *A group G is 2-auto-Engel if and only if G satisfies the equation $\alpha(x)\alpha^{-1}(x) = x^2$, for all $x \in G$ and $\alpha \in \mathrm{Aut}(G)$.*

Proof. By the above lemma, 2-auto-Engel groups satisfy the identity $[x^2, \alpha] = [x, \alpha^2]$, which is clearly equal to the equation $\alpha(x)\alpha^{-1}(x) = x^2$, for all $x \in G$ and $\alpha \in \mathrm{Aut}(G)$. Conversely, suppose that G satisfies the latter identity which implies that $([x, \alpha]^x[x, \alpha])^{\alpha^{-1}} = ([x, \alpha][x, \alpha]^\alpha)^{\alpha^{-1}}$. Hence $([x, \alpha]^{\alpha^{-1}})^x = [x, \alpha]$ and so $[x, \alpha, \alpha^{-1}\varphi_x] = 1$, where φ_x is the inner automorphism defined by x. If we replace the automorphism α by $\varphi_x\alpha^{-1}$, then we have $[[x, \alpha^{-1}][x, \varphi_x]^{\alpha^{-1}}, \alpha] = 1$. Hence, $[x, \alpha, \alpha] = 1$ and since a right 2-auto-Engel element is also a left one, G is a 2-auto-Engel group. □

As we have proved in Lemma 3.3(a), the automorphisms of a 2-auto-Engel group G are all commuting automorphisms (see [4] for more details). Observe that the set $A(G) = \{\alpha \in \mathrm{Aut}(G) : xx^\alpha = x^\alpha x, x \in G\}$ of commuting automorphisms coinsides the full automorphism group $\mathrm{Aut}(G)$ of G if and only if $[\alpha, x, x] = 1$, for every $x \in G$ and $\alpha \in \mathrm{Aut}(G)$. Also, $[G, \alpha] \leq C_G(\alpha)$ for all $\alpha \in \mathrm{Aut}(G)$, implies that G is a 2-auto-Engel group (Lemma 3.1(a)). Hence, our work may be considered as a special case of [4].

Recall that Levi [14] proved that a 2-Engel group is nilpotent of class at most 3 or equivalently $\gamma_3(G) \leq Z(G)$. Moreover, he proved that if G is a 2-Engel group, then $[G', G]^3 = 1$. Now, we prove the following theorem.

Theorem 3.5. *Let G be a 2-auto-Engel group. Then the following statements hold.*

(a) *$K_2(G)$, the second autocommutator subgroup of G, is central.*
(b) *$\mathrm{Aut}(G)^2\mathrm{Inn}(G)$ fixes $K_2(G)$ element-wise.*
(c) $[K(G), G]^3 = 1$.
(d) *If G has finite exponent, then $\exp(K(G)) = \exp(\mathrm{Aut}(G))$. Moreover, if every automorphism of G is central, then $\exp(\mathrm{Aut}(G))$ divides $\exp(Z(G))$.*

Proof. (a) Let $x, y \in G$, $\alpha, \beta \in \mathrm{Aut}(G)$ and φ_x be the inner automorphism induced by x. As G is a 2-auto-Engel group, by Lemma 3.1(f), $[y, [\varphi_x, \alpha], \beta] = 1$. Clearly, $[y, [\varphi_x, \alpha]] = [y, [x, \alpha]]$ and so $[x, \alpha, y, \beta] = 1$. Hence, by Lemma 3.1(c), $[x, \alpha, \beta, y] = 1$. This shows that $K_2(G) \leq Z(G)$.

(b) By the above part, every inner automorphism of G fixes $K_2(G)$ element-wise. Also, by Lemma 3.1(e), for every $g \in K_2(G)$ and each $\gamma \in \mathrm{Aut}(G)$, we have $[g, \gamma]^2 = 1$ and so $g^{\gamma^2} = g$.

(c) Suppose that $x, y \in G$, $\alpha \in \mathrm{Aut}(G)$. Clearly,

$$[x, \alpha^{-1}, y]^\alpha [\alpha, y^{-1}, x]^y [y, x^{-1}, \alpha]^x = 1.$$

By Lemma 3.1(c), $[x, \alpha^{-1}, y, \alpha] = [x, \alpha^{-1}, \alpha, y]^{-1} = 1$. Therefore by part (a), $[x, \alpha^{-1}, y][\alpha, y^{-1}, x][y, x^{-1}, \alpha] = 1$. It is easy to see that $[x, \alpha^{-1}, y] = [y, x^{-1}, \alpha] = [x, \alpha, y]^{-1}$. On the other hand, $[\alpha, y^{-1}, x] = [\alpha, x, y] = [x, \alpha, y]^{-1}$. Hence $[x, \alpha, y]^3 = 1$ and consequently $[K(G), G]^3 = 1$.

(d) From Lemma 3.3(b), it follows immediately that $\exp(\mathrm{Aut}(G)) | \exp(K(G))$. Now, assume that $\exp(\mathrm{Aut}(G)) = n$ $(n \geq 2)$. By Lemma 3.3(b), the generators of $K(G)$ and hence the generators of $K_2(G)$ have order dividing n. By the first part of the theorem, $K_2(G) \leq Z(G)$ and hence $\exp(K_2(G))$ divides n. Now observe that $[[x, \alpha], [y, \beta]] \in K(G)' \leq [K(G), \mathrm{Aut}(G)] = K_2(G)$. On the other hand, by Lemma 3.1(d), $[[x, \alpha], [y, \beta]] = [x, \alpha, [y, \beta]] = [x, \alpha, y, \beta]^2$. Therefore by Lemma 3.3(a),

$$\begin{aligned}([x, \alpha][y, \beta])^n &= [[y, \beta], [x, \alpha]]\ [[y, \beta]^2, [x, \alpha]] \ldots [[y, \beta]^{n-1}, [x, \alpha]] \\ &= [x, \alpha, y, \beta]^{-n(n-1)} = 1.\end{aligned}$$

Hence every element of $K(G)$ has order dividing n so that $\exp(K(G))$ divides $\exp(\mathrm{Aut}(G))$. In particular, if every automorphism of G is central, then $K(G) \leq Z(G)$ and hence $\exp(\mathrm{Aut}(G)) = \exp(K(G))$ divides $\exp(Z(G))$. □

Clearly, if G is a 2-auto-Engel group such that $\mathrm{Aut}(G) = \mathrm{Aut}(G)^2 \mathrm{Inn}(G)$, then $K_3(G) = 1$.

The following corollary is an immediate consequence of Theorem 3.6.

Corollary 3.6. *Let G be a 2-auto-Engel group such that $3 \nmid |G|$. Then*

(a) *$K(G) \leq Z(G)$ and hence the group G is nilpotent of class at most 2;*
(b) *$\mathrm{Aut}_c(G) = \mathrm{Aut}(G)$ so that $\exp(\mathrm{Aut}(G))$ divides $\exp(Z(G))$;*
(c) *$G' \leq L(G)$, where $L(G) = \{g \in G : [g, \alpha] = 1, \alpha \in \mathrm{Aut}(G)\}$ is the absolute center of G.*

The above corollary may be considered as a partial answer to the last question of [6]. In fact, the structure of a 2-auto-Engel group may give some information about the structure of its automorphism group.

For the next lemma, consider the following subgroups of $\mathrm{Aut}(G)$.

$$C_{\mathrm{Aut}(G)}(K(G)) = \{\alpha \in \mathrm{Aut}(G) : [x, \alpha] = 1, x \in K(G)\}$$

and

$$\mathrm{Var}(G) = \{\alpha \in \mathrm{Aut}(G) : [x, \alpha] \in L(G), x \in G\},$$

which is called the *autocentral automorphism group* of G (see [16] for more detail).

Lemma 3.7. *Let G be a 2-auto-Engel group. Then $\mathrm{Aut}(G)' \leq C_{\mathrm{Aut}(G)}(K(G)) \cap \mathrm{Var}(G)$. Moreover, if $K(G) = Z(G)$, then $\mathrm{Aut}(G)'$ is isomorphic to a subgroup of $\mathrm{Hom}(G/Z(G), L(G))$.*

Proof. Let x be an arbitrary element of G and $\alpha, \beta, \gamma \in \mathrm{Aut}(G)$. By Lemma 3.1(d,e), we have $[x, \alpha, [\beta, \gamma]] = 1$. Hence, $\mathrm{Aut}(G)'$ fixes $K(G)$ element-wise. Also, Lemma 3.1(f) implies that $\mathrm{Aut}(G)' \leq \mathrm{Var}(G)$. Finally, if $K(G) = Z(G)$, then by [16, Proposition 2], $\mathrm{Aut}(G)'$ is isomorphic to a subgroup of $\mathrm{Hom}(G/Z(G), L(G))$, as required. □

Theorem 3.8. *Let G be a 2-auto-Engel group. Then $\mathrm{Aut}(G)$ is nilpotent of class at most 2.*

Proof. Lemma 3.1(d,f) implies that $[x, [\alpha, \beta, \gamma]] = [x, [\alpha, \beta], \gamma]^2 = 1$, for all $x \in G$ and $\alpha, \beta, \gamma \in \mathrm{Aut}(G)$. Therefore $[\alpha, \beta, \gamma] = id_G$ and hence $\mathrm{Aut}(G)$ is nilpotent of class at most 2. □

Kappe [13] proved that G is a 2-Engel group if and only if every maximal abelian subgroup of G is normal. We prove that if G is a 2-auto-Engel group, then every maximal abelian subgroup M of G is characteristic. We mention that the converse is not true in general and the cyclic group of order 9 is a counterexample.

Theorem 3.9. *Let G be a 2-auto-Engel group. Then every maximal abelian subgroup of G is characteristic.*

Proof. Let M be a maximal abelian subgroup of the non-abelian group G. Clearly $M = C_G(M)$. Now, we show that for every $\alpha \in \mathrm{Aut}(G)$, the centralizer of α in G, $C_G(\alpha) = \{g \in G : [g, \alpha] = 1\}$ is a characteristic subgroup of G. Let β be an arbitrary automorphism of G and $c \in C_G(\alpha)$. Clearly,

$$[\beta(c), \alpha]^{\beta^{-1}} = [c, \alpha[\alpha, \beta^{-1}]] = [c, [\alpha, \beta^{-1}]] = [c, \alpha, \beta^{-1}]^2 = 1.$$

Therefore, $\beta(c) \in C_G(\alpha)$. Hence $C_G(\alpha)$ is a characteristic subgroup of G. Now, let φ_x be the inner automorphism induced by x. Then $M = C_G(M) = \bigcap_{x \in M} C_G(x) = \bigcap_{x \in M} C_G(\varphi_x)$. Therefore M is a characteristic subgroup of G. □

4. The structure of 2-auto-Engel 2-groups

In this section, we discuss some results about 2-auto-Engel 2-groups. First, we consider non-abelian 2-auto-Engel 2-groups, which are not purely non-abelian.

Theorem 4.1. *Let G be a finite non-abelian 2-auto-Engel 2-group, which is not purely non-abelian. Then $G \cong H \times \mathbb{Z}_4$ such that H is a purely non-abelian 2-auto-Engel 2-group with the following properties:*

(a) $L(H) = Z(H) = H' = K(H) = \Phi(H)$.
(b) $\exp(Z(H)) = \exp(\mathrm{Aut}(H)) = \exp(H)/2 = 2$.
(c) *H/N is elementary abelian for each normal subgroup N of H such that $[H : N] = 4$. Equivalently, every normal second maximal subgroup of H is the intersection of two maximal subgroups.*

Moreover, a group G with the given properties is a 2-auto-Engel group.

Proof. Suppose G is a finite non-abelian 2-auto-Engel 2-group, which is not purely non-abelian. Hence $G = H \times A$ for some non-trivial abelian group A. By the remark in section 3, H is a purely non-abelian 2-auto-Engel 2-group and $A = \langle a \rangle$ is a cyclic group of order 2 or 4. Also, by [2], for each automorphism φ of G, there exist automorphisms $\alpha \in \mathrm{Aut}(H)$ and $\beta \in \mathrm{Aut}(A)$ and homomorphisms $\gamma \in \mathrm{Hom}(H, A)$ and $\delta \in \mathrm{Hom}(A, Z(H))$ such that $\varphi(h, a^i) = (h^\alpha a^{i\delta}, a^{i\beta} h^\gamma)$. Hence

$$[(h, a^i), \varphi] = ([h, \alpha] a^{i\delta}, [a^i, \beta] h^\gamma)$$

so that

$$\begin{aligned}1=[(h,a^i),\varphi,\varphi]&=([[h,\alpha]a^{i\delta},\alpha]([a^i,\beta]h^\gamma)^\delta,[[a^i,\beta]h^\gamma,\beta]([h,\alpha]a^{i\delta})^\gamma)\\&=([a^{i\delta},\alpha]([a^i,\beta]h^\gamma)^\delta,[h^\gamma,\beta]([h,\alpha]a^{i\delta})^\gamma).\end{aligned}$$

Now, assume that $\beta=id$ or the inverting automorphism. If $\beta=id$, then

$$1=[(h,a^i),\varphi,\varphi]=([a^{i\delta},\alpha]h^{\gamma\delta},([h,\alpha]a^{i\delta})^\gamma).$$

If $\alpha=id$, then $h^{\gamma\delta}=a^{i\delta\gamma}=1$. If $|a|=2$ and $h^\gamma=a^j$, then $a^{j\delta}=1$, and by assuming $\delta\neq 1$ we get $j=2$ and so $h^\gamma=1$. Hence $\gamma=1$, which a contradiction. Thus we should have $|a|=4$.

Since $a^\delta\in \mathrm{Ker}\gamma$ and γ varies over all homomorphisms in $\mathrm{Hom}(H,\langle a^2\rangle)$, it follows that $a^\delta\in\Phi(H)$. In particular, $\Omega_2(Z(H))\subseteq L(H)$. Hence

$$\Omega_2(Z(H))\subseteq\Phi(H)\cap L(H).$$

On the other hand, $[H,\alpha]^\gamma=1$ for each α and γ, which implies that

$$K(H)\subseteq\bigcap_\gamma \mathrm{Ker}(\gamma)\subseteq\Phi(H).$$

Hence $K(H)\subseteq\Phi(H)$.

If β is the inverting automorphism, then

$$[(h,a^i),\varphi,\varphi]=([a^{i\delta},\alpha](a^{2i}h^\gamma)^\delta,h^{-2\gamma}([h,\alpha]a^{i\delta})^\gamma),$$

which implies that $h^{2\gamma}=a^{2\delta}=1$. Hence $\mathrm{Im}\gamma\subseteq\langle a^2\rangle$ and either $\gamma=1$ or $\mathrm{Ker}\gamma$ is a maximal subgroup of H so that $\bigcap\mathrm{Ker}\gamma=\Phi(H)$ (for each normal subgroup N of H of index 4, H/N should be non-cyclic so that $\Phi(H)\subseteq\bigcap_{N\trianglelefteq H,[H:N]=4}N$). Also, $Z(H)$ is an elementary abelian 2-group for $(a^\delta)^2=1$ that is $\exp(Z(H))=2$. This also shows that

$$Z(H)=L(H)\subseteq\Phi(H).$$

Hence the first assertion follows.

Conversely, it is evident that a finite group G with the above properties is also a 2-auto-Engel group. This completes the proof. □

The following propositions gives criterions for recognition of some non-abelian 2-auto-Engel groups.

Proposition 4.2. *Let G be a group whose center and automorphisms group are elementary abelian 2-groups. Then G is a 2-auto-Engel group of nilpotency class 2.*

Proof. Since $\mathrm{Aut}(G)$ is abelian, we have $\mathrm{Aut}_c(G)=C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))=\mathrm{Aut}(G)$, where $\mathrm{Aut}_c(G)$ is the group of central automorphisms of G. Hence for every $x\in G$ and every $\alpha\in\mathrm{Aut}(G)$, $[x,\alpha]\in Z(G)$. As $\mathrm{Aut}(G)$ and $Z(G)$ are elementary abelian 2-groups, we have $[x,\alpha,\alpha]=[x,\alpha]^{-2}=1$. Therefore G is a 2-auto-Engel group. Moreover, since $\mathrm{Inn}(G)$ is abelian, G is nilpotent of class 2 and the proof is complete. □

Proposition 4.3. *Let G be a purely non-abelian 2-group such that $G'=\Phi(G)$ and $\mathrm{Aut}(G)$ is abelian. Then G is a 2-auto-Engel group.*

Proof. Since $\mathrm{Aut}(G)$ is abelian, $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$. Hence, by a result of Adney and Yen [1], there is one-to-one correspondence

$$\begin{array}{rcl}\theta : \mathrm{Aut}(G) & \longrightarrow & \mathrm{Hom}\left(\frac{G}{G'}, Z(G)\right) \\ \alpha & \longmapsto & \overline{\alpha},\end{array}$$

where $(gG')^{\overline{\alpha}} = g^{-1}g^{\alpha}$ for all $g \in G$. Now since $G' = \Phi(G)$, we know that G/G' is an elementary abelian 2-group. Hence, for each $\alpha \in \mathrm{Aut}(G)$, the image of $\overline{\alpha}$ must be an elementary abelian 2-group, as well. This follows that $\mathrm{Aut}(G)$ is an elementary abelian 2-group for $Z(G) \subseteq G'$ and hence

$$g^{\alpha^2} = g(gG')^{2\overline{\alpha}}(gG')^{\overline{\alpha}^2} = g$$

for all $g \in G$ and $\alpha \in \mathrm{Aut}(G)$.

Now, $[g, \alpha, \alpha] = [(gG')^{\overline{\alpha}}, \alpha]$. Let $z = (gG')^{\overline{\alpha}}$. Then $g = (g^{\alpha})^{\alpha} = (xz)^{\alpha} = xzz^{\alpha}$, which implies that $z^{\alpha} = z^{-1} = z$ as z has order ≤ 2 being an image of $\overline{\alpha}$. Therefore, $[g, \alpha, \alpha] = [z, \alpha] = 1$, as required, □

We conclude this section by giving some examples of purely non-abelian 2-auto-Engel groups.

Example 2. The following family of 2-groups is constructed in [12]. Let $G(n)$ $(n \geq 3)$ be the finite 2-group with the following presentation:

$$\begin{aligned} G(n) = \langle a_1, \ldots, a_n, b : a_1^2 = a_2^4 = \cdots = a_n^4 = 1, a_{n-1}^2 = b^2, \\ [a_1, b] = 1, [a_n, b] = a_1, [a_{i-1}, b] = a_i^2, \\ [a_j, a_k] = 1, 3 \leq i \leq n, 1 \leq j < k \leq n\rangle. \end{aligned}$$

The group $G(n)$ is of order 2^{2n} with exponent 4 whose center and automorphism groups are isomorphic to $\mathbb{Z}_2^n$ and $\mathbb{Z}_2^{n^2}$, respectively. Therefore, by Proposition 4.2, $G(n)$ is a purely non-abelian 2-auto-Engel group for each $n \geq 3$.

Note that $G(3) \cong (\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$ is the group $(64, 68)$ in the GAP [7] small groups library. There is yet another group of order 64, which is a non-abelian 2-auto-Engel group, and has the following presentation (see [12]).

$$\begin{aligned} \langle a, b, c, d : a^2 = b^4 = c^4 = d^2 = [a, b] = [a, c] = [a, d] = [b, c] = [c^2, d] = 1, \\ [b, d] = c^2, [c, d] = a\rangle. \end{aligned}$$

In the next example, we construct yet another family of infinitely many non-abelian 2-auto-Engel 2-groups, which includes the above group (The group $(64, 69)$ of GAP small groups library which is isomorphic with $(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$ and has shorter presentation than the one given in [12].

Example 3. Let G_n $(n \geq 2)$ be the finite 2-group with the following presentation:

$$\begin{aligned} G_n = \langle x, x_1, \ldots, x_n : x^4 = x_i^4 = [x_i, x_j] = x^2x_1^2 \ldots x_n^2 = [x, x_n]^2 = 1, 1 \leq i, j \leq n, \\ [x, x_i] = x_{i+1}^2, 1 \leq i < n\rangle. \end{aligned}$$

Then, by Theorem 4.5, G_n is a 2-auto-Engle group.

The proof of the following lemma is straightforward and we omit its proof.

Lemma 4.4. *Consider the above group G_n $(n \geq 2)$, then*

(1) $|G_n| = 2^{2(n+1)}$;

(2) *every element of* G_n *can be uniquely written in the form* $x^i x_1^{i_1} \dots x_n^{i_n} [x, x_n]^j$, *where* $i, j = 0, 1$ *and* $i_1, \dots, i_n \in \{0, 1, 2, 3\}$;
(3) $G_n' = \langle x_2^2, \dots, x_n^2, [x, x_n] \rangle \cong \mathbb{Z}_2^n$;
(4) $Z(G_n) = \Phi(G_n) = \Omega_1(G_n) = G_n' \times \langle x_1^2 \rangle \cong \mathbb{Z}_2^{n+1}$;
(5) $G_n/G_n' \cong \mathbb{Z}_4 \times \mathbb{Z}_2^n$ *and* $G_n/Z(G_n) \cong \mathbb{Z}_2^{n+1}$.

Theorem 4.5. *The automorphism group of* G_n *(*$n \geq 2$*) is an elementary abelian* 2*-group of order* $2^{(n+1)^2}$.

Proof. Clearly, G_n is a purely non-abelian group and $H := \langle x_1, \dots, x_n, [x, x_n] \rangle \cong \mathbb{Z}_4^n \times \mathbb{Z}_2$ is the unique (characteristic) abelian maximal subgroup of G_n. We use the method of [12] to obtain the structure of $\mathrm{Aut}(G_n)$. First we construct two descending sequences of characteristic subgroups of G_n as follows:

$$K_0 = H, \quad K_i/K_{i-1}^2 = Z(G_n/K_{i-1}^2) \quad (1 \leq i \leq n),$$
$$L_0 = H, \quad L_i = \{h \in H : h^2 \in [G_n, L_{i-1}]\} \quad (1 \leq i \leq n).$$

A simple calculation shows that

$$K_i = \langle x_1, \dots, x_{n-i}, x_{n-i+1}^2, \dots, x_n^2, [x, x_n] \rangle,$$
$$L_i = \langle x_1^2, \dots, x_i^2, x_{i+1}, \dots, x_n, [x, x_n] \rangle.$$

Obviously,

$$K_{n-i} \cap L_{i-1} = \langle x_1^2, \dots, x_{i-1}^2, x_i, x_{i+1}^2, \dots, x_n^2, [x, x_n] \rangle = \langle x_i, Z(G_n) \rangle.$$

This shows that $x_i^{-1}\alpha(x_i) \in Z(G_n)$, for all $\alpha \in \mathrm{Aut}(G_n)$ and $1 \leq i \leq n$. Now, observe that $G_n = H \langle x \rangle$ and $\alpha(x) = x x_1^{i_1} \dots x_n^{i_n} [x, x_n]^j$, where $j = 0, 1$ and $i_1, \dots, i_n \in \{0, 1, 2, 3\}$. Hence

$$\alpha(x^2) = x^2 x_1^{2i_1} x_2^{2(i_2 - i_1)} \dots x_n^{2(i_n - i_{n-1})} [x, x_n]^{-i_n}.$$

Clearly, $\alpha(x_i^2) = x_i^2$ for $i = 1, \dots, n$, from which it follows that

$$\alpha(x)^2 x_1^2 \cdots x_n^2 = 1.$$

Thus

$$x_1^{2i_1} x_2^{2(i_2 - i_1)} \dots x_n^{2(i_n - i_{n-1})} [x, x_n]^{-i_n} = 0,$$

which implies that $i_1, \dots, i_n$ are even. Therefore $x^{-1}\alpha(x) \in Z(G_n)$ and consequently every automorphism of G_n is central. Also, it is not difficult to see that $L(G_n) = Z(G_n)$ and $\exp(\mathrm{Aut}(G_n)) = 2$. Hence, G_n is a 2-auto-Engel group. Moreover, $|\mathrm{Aut}(G_n)| = |\mathrm{Hom}(G_n/G_n', Z(G_n))| = 2^{(n+1)^2}$, as required. □

Example 4. The following group of [11] gives an instance of a group satisfying the conditions of Proposition 4.3 and that it is not of the type described in Proposition 4.2. Let

$$\begin{aligned} G = \langle x_1, x_2, x_3, x_4, x_5 : {} & x_1^4 = x_2^4 = x_3^4 = x_4^4 = x_5^2 = 1, \\ & [x_1, x_2] = x_1^2, [x_1, x_3] = x_3^2, [x_1, x_4] = 1, [x_1, x_5] = x_1^2, \\ & [x_2, x_3] = x_2^2, [x_2, x_4] = 1, [x_2, x_5] = x_4^2, \\ & [x_3, x_4] = x_4^2, [x_3, x_5] = z_4^2, [x_4, x_5] = 1 \rangle. \end{aligned}$$

A simple verification shows that G is a purely non-abelian group of order 2^9, $\mathrm{Aut}(G)$ is elementary abelian of order 2^{20}, $G' = \Phi(G)$ is elementary abelian of order 2^4 and $Z(G)$ is a group of order 2^5 and exponent 4. Hence, by Proposition 4.3, G is a 2-auto-Engel group.

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References

[1] J. E. Adney and T. Yen, Automorphisms of p-group, *Illinois J. Math.* **9**(1) (1965), 137–143.

[2] J. N. S. Bidwell, M. J. Curran and D. J. McCaughan, Automorphisms of direct products of finite groups, *Arch. Math.* **86** (2006), 481–489.

[3] W. Burnside, On groups in which every two conjugate operations are permutable, *Proc. London Math. Soc.* **35** (1902), 28–37.

[4] M. Deaconescu, G. Silberberg and G. L. Walls, On commuting automorphisms of groups, *Arch. Math.* **79** (2002), 423–429.

[5] M. Deaconescu and G. L. Walls, Cyclic groups as autocommutator groups, *Comm. Algebra* **35** (2007), 215–219.

[6] M. Farrokhi D. G. and M. R. R. Moghaddam, On the center of automorphism group of a group, Submitted.

[7] The GAP Group, *GAP-Groups, Algorithms and Programming, Version 4.4.12, 2008*, (http://www.gap-system.org/).

[8] P. V. Hegarty, The absolute center of a group, *J. Algebra* **169** (1994), 929–935.

[9] H. Heineken, A class of three-Engel groups, *J. Algebra* **17** (1971), 341–345.

[10] C. Hopkins, Finite groups in which conjugate operations are commutative, *Amer. J. Math.* **51** (1929), 35–41.

[11] V. K. Jain, R. K. Rai and M. K. Yadav, On finite p-groups with abelian automorphism group, arXiv:1304.1974v1.

[12] A. R. Jamali, Some new non-abelian 2-groups with abelian automorphism groups, *J. Group Theory* **5** (2002), 53–57.

[13] W. P. Kappe, Die a-norm einer gruppe, *Illinois J. Math.* **5** (1961), 187–197.

[14] F. W. Levi, Groups in which the commutator operation satisfies certain algebraic conditions, *J. Indian Math. Soc.* **6** (1942), 87–97.

[15] M. R. R. Moghaddam, F. Parvaneh and M. Naghshineh, The lower autocentral series of abelian groups, *Bull. Korean Math. Soc.* **48** (2011), 79–83.

[16] M. R. R. Moghaddam and H. Safa, Some properties of autocentral automorphisms of a group, *Ricerche Mat.* **59** (2010), 257–264.

[17] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Part 2. Springer-Verlag, New York, 1972.

[18] D. J. S. Robinson, *A Course in the Theory of Groups*, Second Ed. Springer-Verlag, New York, 1995.

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