# PCF THEORY AND CARDINAL INVARIANTS OF THE REALS

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Abstract. The *additivity spectrum* $\mathrm{ADD}(\mathcal{I})$ of an ideal $\mathcal{I} \subset \mathcal{P}(I)$ is the set of all regular cardinals $\kappa$ such that there is an increasing chain $\{A_\alpha : \alpha < \kappa\} \subset \mathcal{I}$ with $\cup_{\alpha<\kappa} A_\alpha \notin \mathcal{I}$.

We investigate which set $A$ of regular cardinals can be the additivity spectrum of certain ideals.

Assume that $\mathcal{I} = \mathcal{B}$ or $\mathcal{I} = \mathcal{N}$, where $\mathcal{B}$ denotes the $\sigma$-ideal generated by the compact subsets of the Baire space $\omega^\omega$, and $\mathcal{N}$ is the ideal of the null sets.

We show that if $A$ is a non-empty progressive set of uncountable regular cardinals and $\mathrm{pcf}(A) = A$, then $\mathrm{ADD}(\mathcal{I}) = A$ in some c.c.c generic extension of the ground model. On the other hand, we also show that if $A$ is a countable subset of $\mathrm{ADD}(\mathcal{I})$, then $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{I})$.

For countable sets these results give a full characterization of the additivity spectrum of $\mathcal{I}$: a non-empty countable set $A$ of uncountable regular cardinals can be $\mathrm{ADD}(\mathcal{I})$ in some c.c.c generic extension iff $A = \mathrm{pcf}(A)$.

## 1. Introduction

Many cardinal invariants are defined in the following way: we consider a family $\mathfrak{X} \subset \mathcal{P}(\big[\omega\big]^\omega)$ and define our cardinal invariant $\mathfrak{x}$ as $\mathfrak{x} = \min\{|X| : X \in \mathfrak{X}\}$ or $\mathfrak{x} = \sup\{|X| : X \in \mathfrak{X}\}$. The set $\{|X| : X \in \mathfrak{X}\}$ is called the *spectrum of* $\mathfrak{x}$.

For example, consider the family $\mathfrak{A} = \{\mathcal{A} \subset \big[\omega\big]^\omega : \mathcal{A} \text{ is a MAD family}\}$. Then $\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \in \mathfrak{A}\}$, so we can say that the *spectrum of* $\mathfrak{a}$ is the cardinalities of the maximal almost disjoint subfamilies of $\big[\omega\big]^\omega$.

The value of many cardinal invariants can be modified almost freely by using a suitable forcing, but their spectrums should satisfy more requirements.

In [8] Shelah and Thomas investigated the cofinality spectrum of certain groups. Denote $\mathrm{CF}(Sym(\omega))$ the *cofinality spectrum* of the group of all permutation of natural numbers, i.e. the set of regular cardinals $\lambda$ such that $Sym(\omega)$ is the union of an increasing chain of $\lambda$ proper subgroups. Shelah and Thomas showed that $CF(Sym(\omega))$ cannot be an arbitrarily prescribed set of regular uncountable cardinals: if $A = \langle \lambda_n : n \in \omega\rangle$ is a strictly increasing sequence of elements of $\mathrm{CF}(Sym(\omega))$, then $\mathrm{pcf}(A) \subseteq \mathrm{CF}(Sym(\omega))$. On the other hand, they also showed that if $K$ is a set of regular cardinals which satisfies certain natural requirements (see [8, Theorem 1.3]), then $CF(Sym(\omega)) = K$ in a certain c.c.c generic extension.

In this paper we investigate the *additivity spectrum* of certain ideals in a similar style. Denote $\mathfrak{Reg}$ the class of all infinite regular cardinals. Given any ideal $\mathcal{I} \subset$

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$\mathcal{P}(I)$ for each $A \in \mathcal{I}^+$ put

$$\mathrm{ADD}(\mathcal{I}, A) = \{\kappa \in \mathfrak{Reg} : \exists \text{ increasing } \{A_\alpha : \alpha < \kappa\} \subset \mathcal{I} \text{ s.t. } \cup_{\alpha<\kappa} A_\alpha = A\},$$

and let

$$\mathrm{ADD}(\mathcal{I}) = \cup\{\mathrm{ADD}(\mathcal{I}, A) : A \in \mathcal{I}^+\}.$$

Clearly $\mathrm{add}(\mathcal{I}) = \min \mathrm{ADD}(\mathcal{I})$. We will say that $\mathrm{ADD}(\mathcal{I})$ is the *additivity spectrum of* $\mathcal{I}$.

As usual, $\mathcal{M}$ and $\mathcal{N}$ denote the null and the meager ideals, respectively. Let $\mathcal{B}$ denote the $\sigma$-ideal generated by the compact subsets of $\omega^\omega$. Clearly we have

$$\mathcal{B} = \{F \subset \left[\omega\right]^\omega : F \text{ is } \le^*\text{-bounded } \}.$$

So the poset $\langle \omega^\omega, \le^* \rangle$ has a natural, cofinal, order preserving embedding $\Phi$ into $\langle \mathcal{B}, \subset \rangle$ defined by the formula $\Phi(b) = \{x : x \le^* b\}$. Denote by $\mathrm{ADD}(\langle \omega^\omega, \le^* \rangle)$ the set of all regular cardinals $\kappa$ such that there is an unbounded $\le^*$-increasing chain $\{b_\alpha : \alpha < \kappa\} \subset \omega^\omega$. Clearly $\mathrm{ADD}(\mathcal{B}) \supseteq \mathrm{ADD}(\langle \omega^\omega, \le^* \rangle)$ and $\mathfrak{b} = \min \mathrm{ADD}(\mathcal{B}) = \min \mathrm{ADD}(\langle \omega^\omega, \le^* \rangle)$. Farah, [4], proved that if GCH holds in the ground model then given any non-empty set $A$ of uncountable regular cardinals with $\aleph_1 \in A$ we have $\mathrm{ADD}(\langle \omega^\omega, \le^* \rangle = A$ in some c.c.c extension of the ground model. So $\mathrm{ADD}(\langle \omega^\omega, \le^* \rangle)$ does not have any closedness property. Moreover, standard forcing arguments show that $\mathrm{ADD}(\mathcal{I}) \cap \{\aleph_n : 1 \le n < \omega\}$ can also be arbitrary, where $\mathcal{I} \in \{\mathcal{B}, \mathcal{M}, \mathcal{B}\}$.

However, the situation change dramatically if we consider the whole spectrum $\mathrm{ADD}(\mathcal{I})$. Let $\mathcal{I} = \mathcal{B}$ or $\mathcal{I} = \mathcal{N}$. On one hand, we show that $\mathrm{ADD}(\mathcal{I})$ should be closed under certain pcf operations: if $A$ is a countable subset of $\mathrm{ADD}(\mathcal{I})$, then $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{I})$ (see Theorems 3.10 and 3.6).

On the other hand, we show that if $A$ is a non-empty set of uncountable regular cardinals, $|A| < \min(A)^{+n}$ for some $n \in \omega$ (especially, if $A$ is progressive), and $\mathrm{pcf}(A) = A$, then $\mathrm{ADD}(\mathcal{I}) = A$ in some c.c.c generic extension of the ground model (see Theorem 2.3).

For countable sets these results give a full characterization of the additivity spectrum of $\mathcal{I}$: a non-empty countable set $A$ of uncountable regular cardinals can be $\mathrm{ADD}(\mathcal{I})$ in some c.c.c generic extension iff $A = \mathrm{pcf}(A)$.

## 2. Construction of additivity spectrums

To start with we recall some results from pcf-theory. We will use the notation and terminology of [1]. A set $A \subset \mathfrak{Reg}$ is *progressive* iff $|A| < \min(A)$.

The proofs of the next two propositions are standard applications of pcf theory, they could be known, but the author was unable to find them in the literature. Proposition 2.2 is similar to [8, Theorem 3.20], but we do not use any assumption concerning the cardinal arithmetic.

**Proposition 2.1.** *Assume that* $A = \mathrm{pcf}(A) \subset \mathfrak{Reg}$ *is a progressive set, and* $\lambda \in \mathfrak{Reg}$*. Then there is a family* $\mathcal{F} \subset \prod A$ *with* $|\mathcal{F}| < \lambda$ *such that for each* $g, h \in \prod A$

$$\textit{if } g <_{J_{<\lambda}[A]} h \textit{ then there is } f \in \mathcal{F} \textit{ such that } g < \max(f, h).$$

*Proof.* For each $\mu \in \mathrm{pcf}(A) = A$ let $B_\mu \subset A$ be a generator of $J_{<\mu^+}[A]$, i.e.

$$J_{<\mu^+}[A] = \langle J_{<\mu}[A] \cup \{B_\mu\} \rangle_{gen}.$$

Since $cf(\langle \prod B_\mu, \le \rangle) = \max \mathrm{pcf}(B_\mu) = \mu$ by [1, Theorem 4.4], we can fix a family $\mathcal{F}_\mu \subset \prod B_\mu$ with $|B_\mu| = \mu$ such that $\mathcal{F}_\mu$ is cofinal in $\langle \prod B_\mu, \le \rangle$.

We claim that

$$\mathcal{F} = \{\max(f_1^{\mu_1}, \dots, f_n^{\mu_n}) : \mu_1 < \dots < \mu_n < \lambda, f_i^{\mu_i} \in \mathcal{F}_{\mu_i}\}$$

satisfies the requirements.

Since $A$ is progressive, $|\mathcal{F}| \le \sup(A \cap \lambda) < \lambda$.

Assume that $g <_{J_{<\lambda}[A]} h$ for some $g, h \in \prod A$. Let $X = \{a \in A : g(a) \ge h(a)\}$. Then $X \in J_{<\lambda}[A]$, so there are $\mu_1, \dots \mu_n \in \mathrm{pcf}(A) \cap \lambda = A \cap \lambda$ such that $X \subset B_{\mu_1} \cup \dots \cup B_{\mu_n}$. For each $1 \le i \le n$ choose $f_i^{\mu_i} \in \mathcal{F}_{\mu_i}$ with $g \restriction B_{\mu_i} < f_i^{\mu_i}$.

Then $g < \max(h, f_1^{\mu_1}, \dots, f_n^{\mu_n})$ and $\max(f_1^{\mu_1}, \dots, f_n^{\mu_n}) \in \mathcal{F}$. $\square$

**Proposition 2.2.** *Assume that $A = \mathrm{pcf}(A) \subset \mathfrak{Reg}$ is a progressive set, and $\lambda \in \mathfrak{Reg} \setminus A$. If $\langle g_\alpha : \alpha < \lambda \rangle \subset \prod A$ then there are $K \in \left[\lambda\right]^\lambda$ and $s \in \prod A$ such that $g_\alpha < s$ for each $\alpha \in K$.*

*Proof.* If $\lambda > \max \mathrm{pcf}(A)$, then the equality $cf \langle \prod A, < \rangle = \max \mathrm{pcf}(A)$ yields the result. So we can assume $\lambda < \max \mathrm{pcf}(A)$.

The poset $\left\langle \prod A, <_{J_{<\lambda^+}[A]} \right\rangle$ is $\lambda^+$-directed. by [1, Theorem 3.4] Since $\lambda \notin \mathrm{pcf}(A)$, we have $J_{<\lambda}[A] = J_{<\lambda^+}[A]$, and so the poset $\langle \prod A, <_{J_{<\lambda}[A]} \rangle$ is $\lambda^+$-directed, as well. Thus there is $h \in \prod A$ such that $g_\alpha <_{J_{<\lambda}[A]} h$ for each $\alpha < \lambda$.

By Proposition 2.1 there is a family $\mathcal{F} \subset \prod A$ with $|\mathcal{F}| < \lambda$ such that for each $\alpha < \lambda$ there is $f_\alpha \in \mathcal{F}$ such that $g_\alpha < \max(h, f_\alpha)$. Since $|\mathcal{F}| < \lambda = \mathrm{cf}(\lambda)$, there are $K \in \left[\lambda\right]^\lambda$ and $f \in \mathcal{F}$ such that $f_\alpha = f$ for each $\alpha \in K$.

Then $s = \max(h, f) \in \prod A$ and $K \in \left[\lambda\right]^\lambda$ satisfy the requirements. $\square$

**Theorem 2.3.** *Assume that $\mathcal{I}$ is one of the ideals $\mathcal{B}, \mathcal{M}$ and $\mathcal{N}$. If $A = \mathrm{pcf}(A)$ is a non-empty set of uncountable regular cardinals, $|A| < \min(A)^{+n}$ for some $n \in \omega$, then $A = \mathrm{ADD}(\mathcal{I})$ in some c.c.c generic extension $V^P$.*

*Especially, if $\emptyset \ne Y \subset \mathrm{pcf}(\{\aleph_n : 1 \le n < \omega\})$ then $\mathrm{pcf}(Y) = \mathrm{ADD}(\mathcal{I})$ in some c.c.c generic extension $V^P$.*

The proof is based on Theorem 2.5 below. To formulate it we need the following definition.

**Definition 2.4.** Let $\varphi$ be a formula with one free variable, and assume that

$$ZFC \vdash \text{"}I_\varphi = \{x : \varphi(x)\} \textit{ is an ideal"}.$$

We say that the ideal $\mathcal{I}_\varphi$ has the *Hechler property* iff given any $\sigma$-directed poset $Q$ there is a c.c.c poset $P$ such that

$$V^P \models Q \text{ is order isomorphic to some cofinal subset of } \langle \mathcal{I}, \subset \rangle.$$

If "$ZFC \vdash \mathcal{I}_\varphi = \mathcal{I}_\psi$", then clearly $\mathcal{I}_\varphi$ is Hechler iff $\mathcal{I}_\psi$ is. So for well-known ideals, i.e. for $\mathcal{B}$ and for $\mathcal{N}$, we will speak about the *Hechler property of $\mathcal{I}$* instead of the Hechler property of $\mathcal{I}_\phi$, where $\phi$ is one of the many equivalent definitions of $\mathcal{I}$.

**Theorem 2.5.** *Assume that the ideal $\mathcal{I}$ has the Hechler property. If $A = \mathrm{pcf}(A)$ is a non-empty set of uncountable regular cardinals, $|A| < \min(A)^{+n}$ for some $n \in \omega$, then in some c.c.c generic extension $V^P$ we have $A = \mathrm{ADD}(\mathcal{I})$.*

*Proof of theorem 2.3 from Theorem 2.5.* To prove the first part of the theorem, it is enough to show that $\mathcal{I}$ has the Hechler property. However,

- Hechler proved in [6] that $\mathcal{B}$ has the Hechler property,
- Bartoszynski and Kada showed in [2] that $\mathcal{M}$ has the Hechler property,
- Burke and Kada proved in [3] that $\mathcal{N}$ has the Hechler property.

This proves the first part of the theorem.

Assume now that $\emptyset \neq Y \subset \mathrm{pcf}(\{\aleph_n : 1 \leq n < \omega\})$. Then $A = \mathrm{pcf}(Y)$ has cardinality $< \omega_4$ by the celebrated theorem of Shelah. Thus $|A| < \min(A)^{+4}$, so we can apply the first part of the present Theorem for the set $A$. □

*Remark.* The problem whether $\mathcal{N}$ and $\mathcal{M}$ have the Hecler property was raised in a preliminary version of the present paper.

**Corollary 2.6.** *If the ideal $\mathcal{I}$ has the Hechler property and* $\mathrm{cf}(\left[\aleph_\omega\right]^\omega, \subset) > \aleph_{\omega+1}$, *then in some c.c.c generic extension* $\mathrm{ADD}(\mathcal{I}) \cap \aleph_\omega$ *is infinite, but* $\aleph_{\omega+1} \notin \mathrm{ADD}(\mathcal{I})$.

*Proof of the corollary.* If $\max \mathrm{pcf}(\{\aleph_n : 1 \leq n < \omega\}) = \mathrm{cf}(\left[\aleph_\omega\right]^\omega, \subset) > \aleph_{\omega+1}$, then there is an infinite set $X \subset \{\aleph_n : n \in \omega\}$ such that $\mathrm{pcf}(X) = X \cup \{\aleph_{\omega+2}\}$ . Now we can apply theorem 2.5 for $A = X \cup \{\aleph_{\omega+2}\}$ to obtain the desired extension. □

*Proof of theorem 2.5.* Since $|A| < \min(A)^{+n}$, there is a partition $F \cup^* Y$ of $A$ such that $F$ is finite, $Y$ is progressive, and $\max(F) < \min(Y)$. Observe that $Y = pcf(Y)$, and clearly $F = \mathrm{pcf}(F)$.

Let $Q = \langle \prod A, \leq \rangle$, where $f \leq f'$ iff $f(\kappa) \leq g(\kappa)$ for each $\kappa \in A$. Then $Q$ is $\sigma$-directed because $\aleph_0 \notin A$. Since $\mathcal{I}$ has Hechler property, there is a c.c.c poset $P$ such that in $V^P$ the ideal $\mathcal{I}$ has a cofinal subset $\{I_q : q \in Q\}$ which is order-isomorphic to $Q$, i.e. $I_q \subset I_{q'}$ iff $q <_Q q'$.

We are going to show that the model $V^P$ satisfies our requirement.

**Claim 2.7.** $A \subset \mathrm{ADD}(\mathcal{I})$.

*Proof.* Fix $\kappa \in A$. For each $\alpha < \kappa$ consider the function $g_\alpha \in \prod A$ defined by the formula

$$g_\alpha(a) = \begin{cases} \alpha & \text{if } a = \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{g_\alpha : \alpha < \aleph_n\}$ is $\leq$-increasing and unbounded in $Q$, so $\{I_{g_\alpha} : \alpha < \kappa\}$ is increasing and unbounded in $\langle \mathcal{I}, \subset \rangle$. Hence $\kappa \in \mathrm{ADD}(\mathcal{I})$. □

**Claim 2.8.** $\mathrm{ADD}(\mathcal{I}) \subset A$.

*Proof of the claim.* Assume that $\lambda \in \mathfrak{Reg} \setminus A$. We show that $\lambda \notin \mathrm{ADD}(\mathcal{I})$.

Let $\mathfrak{J} = \{J_\alpha : \alpha < \lambda\} \subset \mathcal{I}$ be increasing.

For each $\alpha < \lambda$ pick $g_\alpha \in \prod A$ such that $J_\alpha \subset I_{g_\alpha}$.

Since $\lambda \notin \mathrm{pcf}(A)$, applying Proposition 2.2 twice, first for $Y$, then for $F$, we obtain $K \in \left[\lambda\right]^\lambda$ and $s \in \prod A$ such that $g_\alpha < s$ for each $\alpha \in K$.

Thus $J_\alpha \subset I_s$ for $\alpha \in K$. Since the sequence $\mathfrak{J} = \{J_\alpha : \alpha < \lambda\}$ is increasing, and $K$ is cofinal in $\lambda$, we have

$$\cup\{J_\alpha : \alpha < \lambda\} = \cup\{J_\alpha : \alpha \in K\} \subset I_s.$$

So the sequence $\mathfrak{J} = \{J_\alpha : \alpha < \lambda\}$ does not witness that $\lambda \in \mathrm{ADD}(\mathcal{I})$.

Since $\mathfrak{J}$ was arbitrary, we proved the Claim. □

The two claims complete the proof of the theorem. □

## 3. Restrictions on the additivity spectrum

The first theorem we prove here resembles to [8, Theorem 2.1].

**Theorem 3.1.** *Assume that $\mathcal{I} \subset \mathcal{P}(I)$ is a $\sigma$-complete ideal, $Y \in \mathcal{I}^+$, and $A \subset \mathrm{ADD}(\mathcal{I}, Y)$ is countable. Then $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{I}, Y)$.*

*Proof.* For each $a \in A$ fix an increasing sequence $\mathfrak{F}_a = \{F^a_\alpha : \alpha < a\} \subset \mathcal{I}$ such that $\bigcup \mathfrak{F}_a = Y$.

Let $\kappa \in \mathrm{pcf}(A)$. Fix an ultrafilter $\mathcal{U}$ on $A$ such that $\mathrm{cf}(\prod A/\mathcal{U}) = \kappa$ and fix an $\le_{\mathcal{U}}$-increasing, $\le_{\mathcal{U}}$-cofinal sequence $\{g_\alpha : \alpha < \kappa\} \subset \prod A$. For $g \in \prod A$ let

$$U(g) = \big\{x \in I : \{a \in A : x \in F^a_{g(a)}\} \in \mathcal{U}\big\}.$$

In the next three claims we show that the sequence $\{U(g_\alpha) : \alpha < \kappa\}$ witnesses $\kappa \in \mathrm{ADD}(\mathcal{I}, Y)$.

**Claim 3.2.** *$U(g) \in \mathcal{I}$ for each $g \in \prod A$.*

Indeed, $U(g) \subset \bigcup\{F^a_{g(a)} : a \in A\} \in \mathcal{I}$ because $\mathcal{I}$ is $\sigma$-complete.

**Claim 3.3.** *If $g_1, g_2 \in \prod A$, $g_1 \le_{\mathcal{I}} g_2$, then $U(g_1) \subset U(g_2)$.*

Indeed, fix $x \in I$. Since

$$\{a \in A : x \in F^a_{g_2(a)}\} \supset \{a \in A : x \in F^a_{g_1(a)}\} \cap \{a \in A : g_1(a) \le g_2(a)\}$$

and $\{a \in A : g_1(a) \le g_2(a)\} \in \mathcal{U}$, we have that $\{a \in A : x \in F^a_{g_1(a)}\} \in \mathcal{U}$ implies $\{a \in A : x \in F^a_{g_2(a)}\} \in \mathcal{U}$, i.e., if $x \in U(g_1)$, then $x \in U(g_2)$, too.

**Claim 3.4.** *$\bigcup\{U(g_\alpha) : \alpha < \kappa\} = Y$.*

Indeed, fix $y \in Y$. For each $a \in A$ choose $g(a) < a$ such that $y \in F^a_{g(a)}$. Then $y \in U(g)$. Pick $\alpha < \kappa$ such that $g \le_{\mathcal{U}} g_\alpha$. Then $U(g) \subset U(g_\alpha)$ and so $y \in U(g_\alpha)$.

The three claims together give that sequence $\langle U(g_\alpha) : \alpha < \kappa\rangle \subset \mathcal{I}$ really witnesses that $\kappa \in \mathrm{ADD}(\mathcal{I}, Y)$. □

**Corollary 3.5.** *If $\mathcal{I} \in \{\mathcal{B}, \mathcal{N}, \mathcal{M}\}$, $Y \in \mathcal{I}^+$, and $A \subset \mathrm{ADD}(\mathcal{I}, Y)$ is countable, then $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{I}, Y)$.*

As we will see in the next two subsections, for the ideals $\mathcal{B}$ and $\mathcal{N}$ we can prove stronger closedness properties.

### 3.1. The ideal $\mathcal{B}$.

If $F \subset \omega^\omega$ and $h \in \omega^\omega$, we write $F \le^* h$ iff $f \le^* h$ for each $f \in F$.

**Theorem 3.6.** *If $A \subset \mathrm{ADD}(\mathcal{B})$ is progressive and $|A| < \mathfrak{h}$, then $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{B})$.*

*Proof.* For each $a \in A$ fix an increasing sequence $\mathfrak{F}_a = \{F^a_\alpha : \alpha < a\} \subset \mathcal{B}$ with $\cup\mathfrak{F}_a \notin \mathcal{B}$. We can assume that the functions in the families $F^a_\alpha$ are all monotone increasing.

Let $\kappa \in \mathrm{pcf}(A)$. Pick an ultrafilter $\mathcal{U}$ on $A$ such that $\mathrm{cf}(\prod A/\mathcal{U}) = \kappa$, and fix an $\le_{\mathcal{U}}$-increasing, $\le_{\mathcal{U}}$-cofinal sequence $\{g_\alpha : \alpha < \kappa\} \subset \prod A$.

For $g \in \prod A$ let

$$\mathrm{Bd}(g) = \big\{h \in \omega^\omega : \{a \in A : F^a_{g(a)} \le^* h\} \in \mathcal{U}\big\},$$

and

$$\mathrm{In}(g) = \{x \in \omega^\omega : x \le^* h \text{ for each } h \in \mathrm{Bd}(g)\}.$$

**Claim 3.7.** *For $g_1, g_2 \in \prod A$, if $g_1 \le_{\mathcal{U}} g_2$, then we have* $\mathrm{Bd}(g_1) \supset \mathrm{Bd}(g_2)$ *and* $\mathrm{In}(g_1) \subset \mathrm{In}(g_2)$.

*Proof of the claim.* For each $h \in \omega^\omega$,

$$\{a \in A : F^a_{g_1(a)} \le^* h\} \supset \{a \in A : F^a_{g_2(a)} \le^* h\} \cap \{a \in A : g_1(a) \le g_2(a)\}.$$

Since $\{a \in A : g_1(a) \le g_2(a)\} \in \mathcal{U}$, we have that $\{a \in A : F^a_{g_2(a)} \le^* h\} \in \mathcal{U}$ implies $\{a \in A : F^a_{g_1(a)} \le^* h\} \in \mathcal{U}$, i.e., if $h \in \mathrm{Bd}(g_2)$, then $h \in \mathrm{Bd}(g_1)$, too.

From the relation $\mathrm{Bd}(g_1) \supset \mathrm{Bd}(g_2)$ the inclusion $\mathrm{In}(g_1) \subset \mathrm{In}(g_2)$ is straightforward by the definition of the operator In. $\square$

**Claim 3.8.** $\mathrm{Bd}(g) \ne \emptyset$ *for each $g \in \prod A$.*

Indeed, for each $a \in A$ let $h_a \in \omega^\omega$ such that $F^a_{g(a)} \le^* h_\alpha$. Since $|A| < \mathfrak{h} \le \mathfrak{b}$, there is $h \in \omega^\omega$ such that $h_a \le^* h$ for each $a \in A$. Then $h \in \mathrm{Bd}(g)$.

**Claim 3.9.** *The sequence $\mathfrak{F} = \langle \mathrm{In}(g_\alpha) : \alpha < \kappa \rangle$ witnesses that $\kappa \in \mathrm{ADD}(\mathcal{B})$.*

By claim 3.7, we have $\mathrm{In}(g_\alpha) \subset \mathrm{In}(g_\beta)$ for $\alpha < \beta < \kappa$, and each $\mathrm{In}(g_\alpha)$ is in $\mathcal{B}$ by claim 3.8.

So all we need is to show that $F = \bigcup\{\mathrm{In}(g_\alpha) : \alpha < \kappa\} \notin \mathcal{B}$, i.e. $F$ is not $\le^*$-bounded. Let $x \in \omega^\omega$ be arbitrary. We will find $y \in F$ such that $y \not\le^* x$.

For each $a \in A$ let $F^a = \cup\{F^a_\alpha : \alpha < a\}$, and put

$$\mathcal{J}(a) = \{E \subset \omega : \exists f \in F^a\ x \restriction E <^* f \restriction E\}.$$

Since the functions in $F^a$ are all monotone increasing and $F^a$ is unbounded in $\langle \omega^\omega, \le^* \rangle$, for each $B \in \left[\omega\right]^\omega$ the family $\{f \restriction B : f \in F^a\}$ is unbounded in $\left\langle \omega^B, \le^* \right\rangle$, so $B$ contains some element of $\mathcal{J}(a)$. In other words, $\mathcal{J}(a)$ is dense in $\langle \omega^\omega, \subset^* \rangle$. Since every $\mathcal{J}(a)$ is clearly open and $|A| < \mathfrak{h}$,

$$\mathcal{J} = \bigcap\{\mathcal{J}(a) : a \in A\}$$

is also dense in $\langle \omega^\omega, \subset^* \rangle$. Fix an arbitrary $E \in \mathcal{J}$. For each $a \in A$ pick $f^a \in F^a$ which witnesses that $E \in \mathcal{J}(a)$, i.e. $x \restriction E <^* f^a$. Choose $g(a) < a$ with $f^a \in F^a_{g(a)}$.

Define the function $y \in \omega^\omega$ as follows:

$$y(n) = \begin{cases} x(n) + 1 & \text{if } n \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Then $y \le^* f^a \in F^a_{g(a)}$, and so if $F^a_{g(a)} \le^* h$, then $y \le^* h$. Thus $y \in \mathrm{In}(g)$. Fix $\alpha < \kappa$ such that $g \le_{\mathcal{U}} g_\alpha$. By Claim 3.7, $\mathrm{In}(g) \subset \mathrm{In}(g_\alpha)$, hence $y \in \mathrm{In}(g_\alpha) \subset F$ and clearly $y \not\le^* x$, and so $F \not\le^* x$. Since $x$ was an arbitrary elements of $\omega^\omega$, we are done. $\square$

## 3.2. **The ideal $\mathcal{N}$.**

**Theorem 3.10.** *If $A \subset \mathrm{ADD}(\mathcal{N}) \cap Reg$ is countable, then* $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{N})$.

To prove the theorem above we need some preparation. Denote $\lambda$ the product measure on $2^\omega$, and $\lambda_\omega$ the product measure of countable many copies of $\langle 2^\omega, \lambda \rangle$. By [5, 417J] the products of measures are associative. Since $\omega \times \omega = \omega$, and $\langle 2^\omega, \lambda \rangle$ itself is the product of countable many copies of the natural measure space on 2 elements, we have the following fact.

**Fact 3.11.** *There is a bijection $f : 2^\omega \to (2^\omega)^\omega$ such that $\lambda(X) = \lambda_\omega(f[X])$ for each $\lambda$-measurable set $X \subset 2^\omega$. So*

$$(\dagger) \qquad \mathrm{ADD}(\mathcal{N}) = \mathrm{ADD}(\mathcal{N}_\omega),$$

*where $\mathcal{N}_\omega = \{X \subset (2^\omega)^\omega : \lambda_\omega(X) = 0\}$.*

Denote $\lambda^*$ the outer measure on $2^\omega$. Clearly for some $X \subset 2^\omega$ we have $\lambda^*(X) > 0$ iff $X \notin \mathcal{N}$.

As we will see soon, Theorem 3.10 follows easily from the next result.

**Theorem 3.12.** *If $A \subset \mathrm{ADD}(\mathcal{N})$ is countable, then there is $Y \subset 2^\omega$ such that $\lambda^*(Y) = 1$ and $A \subset \mathrm{ADD}(\mathcal{N}, Y)$.*

*Proof of theorem 3.10 from Theorem 3.12.* By Theorem 3.12 there is $Y \subset 2^\omega$ such that $A \subset \mathrm{ADD}(\mathcal{N}, Y)$ and $\lambda^*(Y) = 1$. Now apply theorem 3.1 for $Y$ and $A$ to obtain $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{N}, Y) \subset \mathrm{ADD}(\mathcal{N})$. $\square$

*Proof of Theorem 3.12.* First we prove some easy claims.

**Claim 3.13.** *If $X \subset 2^\omega$ is measurable, $1 > \lambda(X) > 0$, then there is $x \in 2^\omega$ such that $\lambda(X \cup (X + x)) > \lambda(X)$, where $X + x = \{x' + x : x' \in X\}$.*

*Proof of the claim.* By the Lebesgue density theorem, there are $y, z \in 2^\omega$ and $\varepsilon > 0$ such that for each $0 < \delta < \varepsilon$ we have $\lambda(X \cap [y-\delta, y+\delta]) > \delta$ and $\lambda(X \cap [z-\delta, z+\delta]) < \delta$. Let $x = z - y$. Then $\lambda((X \cup (X + x)) \cap [z - \delta, z + \delta]) \geq \lambda(X \cap [y - \delta, y + \delta]) > \delta > \lambda(X \cap [z - \delta, z + \delta])$. So $\lambda(X \cup (X + x)) > \lambda(X)$. $\square$

**Claim 3.14.** *If $X \subset 2^\omega$ is Lebesgue-measurable, $\lambda(X) > 0$, then there is a set $\{x_n : n < \omega\} \subset 2^\omega$ such that $\lambda(\bigcup\{X + x_n : n \in \omega\}) = 1$.*

*Proof of the claim.* Apply claim 3.13 as long as you can increase the measure. We should stop after countable many steps. $\square$

**Claim 3.15.** *If $X \subset 2^\omega$, $\lambda^*(X) > 0$, then there are real numbers $\{x_n : n < \omega\}$ such that $\lambda^*(\bigcup\{X + x_n : n \in \omega\}) = 1$.*

*Proof of the claim.* Fix a Lebesgue measurable set $Y$ such that $X \subset Y$ and for each measurable set $Z$ with $Z \subset Y \setminus X$ we have $\lambda(Z) = 0$. Apply claim 3.14 for $Y$: we obtain a set $\{x_n : n < \omega\} \subset 2^\omega$ such that taking $Y^* = \bigcup\{Y + x_n : n < \omega\}$ we have $\lambda(Y^*) = 1$. Let $X^* = \bigcup\{X + x_n : n < \omega\}$. Then $\lambda^*(X^*) = 1$. Indeed, if $Z \subset Y^*$ is measurable with $\lambda(Z) > 0$, then there is $n$ such that $\lambda(Z \cap (Y + x_n)) > 0$. Let $T = (Z - x_n) \cap Y$. Then $T \subset Y$ is measurable with $\lambda(T) > 0$, so there is $t \in T \cap X$. Then $t + x_n \in Z \cap X^*$, i.e. $Z \not\subset Y^* \setminus X^*$. $\square$

**Lemma 3.16.** *If $0 < \lambda^*(X)$, then there is $X^* \subset 2^\omega$ such that $\lambda^*(X^*) = 1$ and $\mathrm{ADD}(\mathcal{N}, X^*) = \mathrm{ADD}(\mathcal{N}, X)$.*

*Proof.* Fix $\{x_n : n < \omega\} \subset 2^\omega$ such that $\lambda(X^*) = 1$, where $X^* = \bigcup\{X + x_n : n < \omega\}$. If $\kappa \in \mathrm{ADD}(\mathcal{N}, X)$, then there is an increasing sequence $\langle I_\nu : \nu < \kappa\rangle \subset \mathcal{N}$ such that $\bigcup_{\zeta<\kappa} I_\nu = X$. Let $J_\nu = \bigcup\{I_\nu + x_n : n < \omega\}$. Then the sequence $\langle J_\nu : \nu < \kappa\rangle$ witnesses $\kappa \in \mathrm{ADD}(\mathcal{N}, X^*)$.

If $\langle J_\nu : \nu < \kappa\rangle$ witnesses that $\kappa \in \mathrm{ADD}(\mathcal{N}, X^*)$, then $I_\nu = J_\nu \cap X$ witnesses that $\kappa \in \mathrm{ADD}(\mathcal{N}, X)$. $\square$

Denote $\lambda^*_\omega$ the outer measure generated by $\lambda_\omega$ on $(2^\omega)^\omega$.

**Lemma 3.17.** *If $\{Y_n : n < \omega\} \subset \mathcal{P}(2^\omega)$ with $\lambda^*(Y_n) = 1$, then $\lambda^*_\omega(\prod Y_n) = 1$.*

*Proof.* Write $Y^* = \prod Y_n$.

Assume on the contrary that there is $Z \subset (2^\omega)^\omega \setminus Y^*$ with $\lambda_\omega(Z) > 0$. Since the measure $\lambda_\omega$ is regular, we can assume that $Z$ is compact. By induction, we pick elements $y_0 \in Y_0, \dots, y_n \in Y_n, \dots$ such that $\lambda_\omega(Z_n) > 0$, where

$$Z_n = \{z \in (2^\omega)^\omega : \langle z_0, \dots, z_{n-1}\rangle^\frown z \in Z\}.$$

Especially $Z_0 = Z$.

If $Z_n$ is defined, let

$$T_n = \{t \in 2^\omega : \lambda(\{z : \langle t\rangle^\frown z \in Z_n\}) > 0\}$$

By Fubini theorem, $\lambda(T_n) > 0$, so we can pick $y_n \in T_n \cap Y_n$.

Let $y = \langle y_n : n < \omega\rangle \in \prod Y_n$. Then for each $n \in \omega$ there is some $z$ such that $(y \restriction n)^\frown z \in Z$, and so $y \in Z$ because $Z$ is compact. $\square$

We are ready to conclude the proof of Theorem 3.12.

Enumerate first $A$ as $\{\kappa_n : n < \omega\}$. For each $n < \omega$ apply lemma 3.16 to get $X_n \subset 2^\omega$ such that $\lambda^*(X_n) = 1$ and $\kappa_n \in \mathrm{ADD}(\mathcal{N}, X_n)$, and fix an increasing sequence $\langle T^n_\nu : \nu < \kappa_n\rangle \subset \mathcal{N}$ with $\cup_{\nu<\kappa_n} T^n_\nu = X_n$.

Let $X^* = \prod_{n\in\omega} X_n \subset (2^\omega)^\omega$. Then $\lambda^*(X^*) = 1$ by 3.17, and so the increasing sequence

$$\big\langle \big(\prod_{m<n} X_m\big) \times T^n_\nu \times \big(\prod_{m>n} X_m\big) : \nu < \kappa_n \big\rangle \subset \mathcal{N}_\omega$$

witnesses that $\kappa_n \in \mathrm{ADD}(\mathcal{N}_\omega, X^*)$ for $n < \omega$. But $A \subset \mathrm{ADD}(\mathcal{N}_\omega, X^*)$ implies $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{N}_\omega, X^*) \subset \mathrm{ADD}(\mathcal{N}_\omega)$ by Theorem 3.1. Finally, $\mathrm{ADD}(\mathcal{N}_\omega) = \mathrm{ADD}(\mathcal{N})$ by (†) from Fact 3.11, so we are done. $\square$

**Corollary 3.18.** *Let $\mathcal{I}$ be either the ideal $\mathcal{B}$ or the ideal $\mathcal{N}$. Assume that $A$ is a non-empty set of uncountable regular cardinals. If $A$ is countable, or $\max A \le cf(\big[\aleph_\omega\big]^\omega, \subset)$, then the following statements are equivalent:*

*(1) $A = \mathrm{ADD}(\mathcal{I})$ in some c.c.c extension of the ground model,*

*(2) $A = \mathrm{pcf}(A)$.*

*Proof.* (2) $\Longrightarrow$ (1): if $A$ is countable, then $A$ is progressive.

If $\sup(A) \le \mathrm{cf}(\big[\aleph_\omega\big]^\omega, \subset)$, then we have $A \subset \mathrm{pcf}(\aleph_n : 1 \le n < \omega)$, and so $|A| < \omega_4 \le \min(A)^{+4}$ by the celebrated theorem of Shelah [7].

So in both case we can apply Theorem 2.5 to get (1).

(1) $\Longrightarrow$ (2): By Theorems 3.6 and 3.10, we have that

$$(\star) \qquad A = \cup\{\mathrm{pcf}(A') : A' \in \big[A\big]^\omega\}.$$

If $A$ is countable, $(\star)$ gives immediately that $A = \mathrm{pcf}(A)$.

If $\sup(A) \le \mathrm{cf}(\big[\aleph_\omega\big]^\omega, \subset)$, then $A \subset \mathrm{pcf}(\aleph_n : 1 \le n < \omega)$, so by the Localization Theorem (see [1, Theorem 6.6.]) we have $\mathrm{pcf}(A) = \cup\{\mathrm{pcf}(A') : A' \in \big[A\big]^\omega\}$. Thus even in this case, $(\star)$ gives $A = \mathrm{pcf}(A)$. $\square$

Finally we mention a problem. We could not prove that if $A \subset \mathrm{ADD}(\mathcal{M})$ is countable, then $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{M})$ because the following question is open:

**Problem 3.19.** *Is it true that if $A \subset \mathrm{ADD}(\mathcal{M})$ is countable, then $A \subset \mathrm{ADD}(\mathcal{M}, Y)$ for some $Y \notin \mathcal{M}$?*

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