Antichains in the homomorphism order of graphs *

Dwight Duffus[1] Péter L. Erdős[2] Jaroslav Nešetřil[3]
Lajos Soukup[2]

[1] Mathematics and Computer Science Department,
Emory University, Atlanta, GA USA, 30322
dwight@mathcs.emory.edu
[2] Alfréd Rényi Institute of Mathematics,
Budapest, P.O. Box 127, H-1364 Hungary,
{elp,soukup}@renyi.hu
[3] Dept. Applied Math. and Institute of Theoretical Computer Science,
Charles University, Prague, Czech Republic
nesetril@kam.mff.cuni.cz

August 29, 2010

Abstract

Denote by $\mathbb{G}$ and $\mathbb{D}$, respectively, the the homomorphism poset of the finite undirected and directed graphs, respectively.

A maximal antichain A in a poset P *splits* if A has a partition (B, C) such that for each $p \in P$ either $b \le_P p$ for some $b \in B$ or $p \le_p c$ for some $c \in C$.

We construct both splitting and non-splitting infinite maximal antichains in $\mathbb{G}$ and in $\mathbb{D}$.

A point $y \in P$ is a *cut point* in a poset P if and only if there is $x <_p <_P y < z$ such that $[x, z] = [x, y] \cup [y, z]$. We show if $G \in \mathbb{D}$ contains any directed circle, then G can not be a cut point in $\mathbb{D}$.

1 Introduction

For any fixed type of finite relational structure, homomorphisms induce an ordering of the set of all structures. In particular, given two graphs [respectively, directed

*The second author was partly supported by Hungarian NSF, under contract Nos. T 37758, T37846, NK62321, AT048826. The fourth author was supported by Hungarian NSF, under contract No. 61600

graphs] G and H write $G \leq H$ or $G \to H$ provided that there is a homomorphism from G to H, that is, a map $f : V(G) \to V(H)$ such that for all $\{x, y\} \in E(G)$, $\{f(x), f(y)\} \in E(H)$ [respectively, for all $\langle x, y\rangle \in E(G)$, $\langle f(x), f(y)\rangle \in E(H)$]. Then the relation $\leq$ is a quasi-order and so it induces an equivalence relation: we say that G and H are *homomorphism-equivalent* or *hom-equivalent*, and write $G \sim H$ if and only if $G \leq H$ and $H \leq G$. The *homomorphism posets* $\mathbb{G}$ and $\mathbb{D}$ are the partially ordered sets of all equivalence classes of finite undirected and directed graphs, respectively, ordered by the $\leq$. We will often abuse notation by replacing the classes that comprise $\mathbb{G}$ and $\mathbb{D}$ with their members.

These partially ordered sets are of significant intrinsic interest and are useful tools in the study of graph and digraph properties. For instance, it is easily seen that both are countable distributive lattices: the supremum, or join, of any pair is their disjoint sum, and the infimum, or meet, is their categorical or relational product. Both $\mathbb{G}$ and $\mathbb{D}$ are "predominantly" dense – the former shown by Welzl [19] and the latter, by Nešetřil and Tardif [15]. Both also embed all countable partially ordered sets – see [18] for a presentation.

The maximal chains and antichains of an ordered set are subobjects of interest. In this case, maximal antichains are particularly relevant because of their relationship to the notion of a *homomorphism duality*, introduced by Nešetřil and Pultr [13]: say that an ordered pair $\langle F, D\rangle$ of graphs, or directed graphs, is a *duality pair* if

$$F\!\to\; = \;\not\to\! D \tag{1}$$

where $F\!\to\; = \{G : F \to G\}$ and $\not\to\! D = \{G : G \not\to D\}$. Equivalently, the set of all structures is partitioned by the upset $F\!\to$ and the downset $\to\! D$. [Here we use the other common notation $F^{\uparrow}$ and $D^{\downarrow}$ for upsets and downsets, respectively.]

One important motivation for consideration of duality pairs is that of an "obstruction" to a graph property. For instance, the possibility of a homomorphism of a graph G to K_2, a 2-coloring, is obstructed by the existence of a homomorphism of some odd cycle to G. While there are no nontrivial duality pairs in $\mathbb{G}$, in $\mathbb{D}$, each tree can play the role of F in (1). In fact, in [15], Nešetřil and Tardif obtain a correspondence between duality pairs and gaps in the homomorphism order for general relational structures. They use this to characterize duality pairs and generalize this by describing exactly when the left handside of (1) can be replaced by a finite union of final segments. They further note in [16] that the 2-element maximal antichains in $\mathbb{D}$ are exactly the duality pairs $\langle F, D\rangle$ where F is a tree and D is its dual.

Foniok, Nešetřil and Tardif [10] are concerned with the most general circumstance. Let $\mathcal{F}$ and $\mathcal{D}$ both be finite antichains of structures of fixed type Δ. Call $\langle \mathcal{F}, \mathcal{D}\rangle$ a *generalized duality* if

$$\bigcup_{F\in\mathcal{F}} F\!\to\; = \bigcap_{D\in\mathcal{D}} \not\to\! D \tag{2}$$

Equivalently, with $\mathbb{S}$ denoting the homomorphism poset of Δ-structures, $\mathbb{S}$ is partitioned by

$$\mathbb{S} = \left(\bigcup_{F \in \mathcal{F}} F\!\to \right) \cup \left(\bigcup_{D \in \mathcal{D}} \to\! D \right). \tag{3}$$

The generalized dualities are characterized in [10]. They also show that when Δ consists of one k-ary relation, which contains the graph cases, every maximal antichain in the lattice of Δ-structures is of the form $\mathcal{F} \cup \mathcal{D}$. Conversely, for all but three exceptional cases, the generalized dualities $\langle \mathcal{F}, \mathcal{D} \rangle$ yield a maximal antichain $\mathcal{F} \cup \mathcal{D}$.

It is quite natural to ask, in more general circumstances, if maximal antichains possess these sorts of partitions. Indeed, Ahlswede, Erdős and Graham [1] introduced the notion of "splitting" a maximal antichain. Say that a maximal antichain A of a poset P *splits* if A can be partitioned into two subsets B and C such that $P = B^{\uparrow} \cup C^{\downarrow}$; say that P has the *splitting property* if all of its maximal antichains split. They obtained sufficient conditions for the splitting property, from which they proved, in particular, that all finite Boolean lattices possess it. The property is also a useful tool in combinatorial investigations of posets, particularly distributive lattices; see, for instance [2, 3]. It is also a natural notion for infinite posets; see [4, 8, 9].

The correspondence between generalized dualities and maximal antichains obtained in [10] and the partition in (3) demonstrate that for $\Delta = (k)$, essentially all finite maximal antichains in the lattice $\mathbb{S}$ of Δ-structures split.

This paper is motivated by two goals. First, we would like to obtain general order theoretic conditions on countable posets that ensure antichains split and, thereby, obtain some of the duality results tied to finite maximal antichains described above. See Section 4 for applications to $\mathbb{G}$ and Section 5 for results on $\mathbb{D}$. Second, we obtain splitting and non-splitting results for infinite maximal antichains; in particular, these results underscore differences between the structures $\mathbb{G}$ and $\mathbb{D}$. The necessary results on splitting and related notions are given in Section 3, which is preceded in Section 2 by a directed version of what is known as the Sparse Incomparability Lemma.

In addition to the selected papers cited in this section, we refer the reader to the book [11] by Hell and Nešetřil that is devoted to graph homomorphisms. Chapter 3 gives a thorough introduction and many of the key results on maximal antichains and dualities in $\mathbb{G}$ and $\mathbb{D}$.

2 A Directed Sparse Incomparability Lemma

Recall that the the *girth* of a graph, $\mathrm{girth}(G)$, is the length of a shortest cycle contained in the graph. In case G is directed, its girth is that of the underlying undirected graph, that is, of the symmetric version of G. In one of the first applications of the probabilistic method, in 1959 Paul Erdős [5] showed the existence of graphs with independently prescribed girth and chromatic number. More precisely, for all natural numbers k and ℓ there is a graph G such that $\chi(G) > k$ and $\mathrm{girth}(G) > \ell$.

Based on another probabilistic argument due to Erdős and Hajnal [6], Nešetřil and Rödl [14] obtained an interesting generalization, referred to as the "Sparse Incomparability Lemma": for every pair of graphs H and G such that $G \to H$ but $H \not\to G$, and for every positive integer ℓ there exists a graph H' with $\mathrm{girth}(H') > \ell$ such that $H' \to H$ and $H' \not\to G$.

Here is a formulation from which the Sparse Incomparibilty Lemma follows, itself a special case of a more far-reaching generalization.

Theorem 2.1 (Nešetřil-Zhu [17])**.** *For every graph H and for every positive integers k and ℓ there exists a graph G with the following properties:*

(1) $\mathrm{girth}(G) > \ell$*, and*

(2) *for every graph H_0 with at most k vertices, $G \to H_0$ if and only if $H \to H_0$.*

Here, we require a directed graph version of Theorem 2.1. The following is a special case of a Sparse Incomparability Lemma for finite relational structures [12].

Theorem 2.2 (Directed Sparse Incomparability Lemma)**.** *For each directed graph $H = (W, F)$ and for all integers $m, \ell \in \mathbb{N}$ there is a directed graph H' such that*

(1) $\mathrm{girth}(H') > \ell$*,*

(2) *for each directed graph G with $|V(G)| < m$ we have $H' \to G$ if and only if $H \to G$, and*

(3) *H and H' have the same numbers of connected components. In particular, if H is connected then so is H'.*

Regarding the proof of Theorem 2.2, there are both probabilistic and deterministic arguments available. For instance, it is straightforward to adapt the probabilistic proof of Nešetřil-Rödl. We found an alternative approach based on what appears to be a new graph parameter. Here is a brief outline of the argument.

Given a graph $G = (V, E)$ let the *bipartite stability number* $\alpha_b(G)$ be the maximum integer β such that:

$$\exists A, B \in [V]^{\beta} \text{with } A \cap B = \emptyset \text{ and no edge between } A \text{ and } B.$$

Clearly $\alpha_b(G) \geq \alpha(G)/2$, where $\alpha(G)$ denotes the usual stability or independence number of G. The following result is obtained by adjusting the Erdős-Rényi proof [7] that there are graphs of large girth and small independence number.

Lemma 2.3. *For all $k, \ell \in \mathbb{N}$ and for all but finitely many $n \in \mathbb{N}$ there exists a connected graph $G' = (V, E)$ with $|V| = n$, girth$(G') > \ell$ and $\alpha_b(G') < n/k$.*

Let H, m and ℓ be as in the statement of Theorem 2.2. Let $k = 3m|W|$ and $n = kj$ for sufficiently large j. By Lemma 2.3, there exists a graph $G' = (V, E)$ such that

- $V = W \times [3mj]$

- $\text{girth}(G') > \ell$
- $\alpha_b(G') < n/k = j$

In effect, we “blow up” each vertex of H into a class of $3mj$ vertices.

Define a digraph graph $H^* = (V, E^*)$ as follows: if $\langle h,i\rangle, \langle h',i'\rangle \in V$ then $\langle\langle h,i\rangle, \langle h',i'\rangle\rangle \in E^*$ if and only if $(\langle h,i\rangle, \langle h',i'\rangle) \in E$ and $\langle h,h'\rangle \in F$.

One now argues that if H is connected then H^* has a large enough connected component that satisfies **(1)**, **(2)**, and **(3)** of the theorem.

3 The splitting property

In the forthcoming sections we would like to apply some results from [9] to the posets $\mathbb{G}$ and $\mathbb{D}$ to obtain antichains with certain properties related to dualities and partitions of $\mathbb{G}$ and $\mathbb{D}$. Concerning $\mathbb{G}$ it would be enough just to quote some theorems from [9], but concerning $\mathbb{D}$ we should reformulate them a bit to make them applicable here.

Let $\mathcal{P} = (P, <)$ be a poset. We say that a subset $A \subset P$ is *cut-free in* $\mathcal{P}$ provided there are no $y \in A$ and $x, z \in P$ such that $x <_p y <_p z$ and $A \cap [x,z] = A \cap ([x,y] \cup [y,z])$. An element $y \in P$ is called *cut-point* iff there are $x, z \in P$ such that $x <_p y <_p z$ and $[x,z] = [x,y] \cup [y,z]$. Clearly there is no cut-point in a cut-free set.

If $\mathcal{P} = (P, <)$ is a poset and $A \subset P$ then we define the *upset* $A^{\uparrow}$ and the *downset* $A^{\downarrow}$ of A as follows:

$$A^{\uparrow} = \{p \in P : \exists a \in A \ a \leq_P p\}$$

and

$$A^{\downarrow} = \{p \in P : \exists a \in A \ p \leq_P a\}.$$

An *antichain* in P is a set of pairwise incomparable elements. We say A *split*s if there is a partition (B, C) of A such that $P = B^{\uparrow} \cup C^{\downarrow}$. We say that A *strongly split*s if and only if there is a partition (B, C) of A such that for each $p \in P \setminus A$ either the set $B \cap p^{\downarrow}$ or the set $C \cap p^{\uparrow}$ are infinite.

Definition 3.1. Let $\mathcal{P} = \langle P, \leq\rangle$ be a poset and let $P' \subset P$.

1. P' is an *upward loose kernel* in $\mathcal{P}$ if

   (UL) for all finite subsets $F \subseteq P'$ and $x \in P \setminus F^{\uparrow}$ there is $y \in [x^{\uparrow} \cap P']$, $y \neq x$, such that each element of F is incomparable to y.

2. P' is a *downward loose kernel* in $\mathcal{P}$ if

   (DL) for all finite subsets $F \subseteq P'$ and $x \in P \setminus F^{\downarrow}$ there is $y \in [x^{\downarrow} \cap P']$, $y \neq x$, such that each element of F is incomparable to y.

We say that the poset $\mathcal{P}$ is *upward loose* if P itself is an upward loose kernel in $\mathcal{P}$, and *downward loose* is defined dually. (In [9] the terms *loose* and *dually loose* were used.)

Remarks. An upward loose kernel P' in poset $\mathcal{P}$ does not have maximal elements; in particular, P' is infinite. Clearly the dual statement holds for any downward loose kernel. Also, if $\mathcal{P}$ contains an upward loose kernel then there is an upward loose kernel of $\mathcal{P}$ that is maximal, with respect to containment. This is easily shown using Zorn's Lemma. Again, the dual statement holds for downward loose kernels.

Definition 3.2. Let $\mathcal{P} = \langle P, \leq \rangle$ be a poset and $P' \subset P$. We say that P' has the *finite antichain extension property* provided

(FAE) for all finite antichains $F \subseteq P'$ and $x \notin F$ there is $y \in [x^{\updownarrow} \cap P']$ such that each element of F is incomparable to y.

Observation 3.3. *If $P' \subset P$ is both upward loose kernel and downward loose kernel in $\mathcal{P}$ then P' has the finite antichain extension property.*

The following observation is a sharpening of [9, Theorem 3.9]. We include the straightforward proof to illustrate how the FAE can be applied.

Theorem 3.4. *Let $\mathcal{P} = \langle P, \leq \rangle$ be a countable infinite poset and assume that $P' \subset P$ has the finite antichain extension property. Let $A_1 \subset P'$ be a finite antichain. Then there is a strongly splitting $\mathcal{P}$-maximal antichain $A_1 \subset A \subset P'$.*

Proof. Let $\{p_n : n < \omega\}$ be an ω-abundant enumeration of P, that is, the set $\{n : p_n = p\}$ is infinite for each $p \in P$. By induction on i construct an infinite antichain $A = \{a_i : i < \omega\} \subset P'$ as follows: if $p_i \notin \{a_j : j < i\}$ then let a_i be comparable to p_i. If $p_i \in \{a_j : j < i\}$ then let $n_i = \min\{n : p_n \notin \{a_m : m < i\}\}$ and let a_i be comparable to p_{n_i}. This construction can be carried out because P' has the finite antichain extension property.

Let $p \in P \setminus A$. Then the set $A_p = \{a_i : p_i = p\}$ is infinite and for each $a \in A_p$ the element a and p are comparable. Let (B, C) be a partition of A such that $|B \cap A_p| = |C \cap A_p| = \omega$ for each $p \in P \setminus A$.

Then the partition (B, C) has the required property. □

The following results show that the existence of a loose kernel guarantees an infinite non-splitting maximal antichain. The first is a slight generalization of [9, Theorem 3.6] .

Theorem 3.5. *Assume that $\mathcal{P} = \langle P, \leq \rangle$ is a countable poset and $P' \subset P$ is an upward loose kernel. Then there exists an infinite non-splitting antichain $A \subset P'$ and A is maximal in $\mathcal{P}$.*

Proof. See [9, Theorem 3.6]. □

Theorem 3.6. *Assume that $\mathcal{P} = \langle P, \leq \rangle$ is a countable poset and $P' \subset P$ is an upward loose kernel in P. Assume that $A_1 \subset P'$ is a non-maximal antichain in P. Then there exists an infinite non-splitting antichain A such that $A_1 \subset A \subset P'$ and A is maximal in $\mathcal{P}$.*

Proof. The set $P' \setminus A_1^{\updownarrow}$ is an upward loose kernel in $P \setminus A_1^{\updownarrow}$, and $P \setminus A_1^{\updownarrow} \neq \emptyset$ because A_1 was not a maximal antichain. Hence by Theorem 3.5 there is a $P \setminus {A_1}^{\updownarrow}$-maximal antichain $A' \subset P' \setminus A_1^{\updownarrow}$ which does not split in $P \setminus A_1^{\updownarrow}$. Then $A = A_1 \cup A'$ i s a maximal antichain in P having the required properties. □

4 The homomorphism poset $\mathbb{G}$

The partially ordered set $\mathbb{G}$ of hom-equivalence classes of finite undirected graphs is known to have only two finite maximal antichains — $\{K_1\}$ and $\{K_2\}$. Consequently, there are no nontrivial dualities. However, in studying the ordered set $\mathbb{G}$, it is interesting to know whether maximal antichains split and whether antichains extend to maximal ones that split.

Let $\mathbb{G}' = \mathbb{G} \setminus \{K_1, K_2\}$. For any bipartite graph G, $G \to K_2$, so we know that all graphs $\mathbb{G}'$ are hom-equivalent to graphs all of whose connected components contain odd cycles. The *oddgirth* of a graph G, oddgirth(G), is the length of the shortest odd cycle contained in the graph. As with girth, if graph does not contain any odd cycles, its oddgirth is regarded as infinite.

The notion of oddgirth is useful in dealing with homomorphism questions because of this: for graphs G and H, if oddgirth$(G) <$ oddgirth(H) then $G \nrightarrow H$. Also, it is straightforward to contruct graphs of prescribed oddgirth and chromatic number using shift graphs — for instance, see [11, Theorem 2.23]. Alternatively, the original Erdős result could be used in the first part of the proof below.

Theorem 4.1. *$\mathbb{G}'$ is both upward loose and downward loose.*

Proof. Let $\mathcal{F} \subseteq \mathbb{G}'$ be finite. To see that $\mathbb{G}'$ is upward loose, let $X \in \mathbb{G} \setminus \mathcal{F}^{\uparrow}$, that is, $F \nrightarrow X$ for all $F \in \mathcal{F}$. Let Y' be a graph such that for all $F \in \mathcal{F}$,

(1) oddgirth$(Y') >$ oddgirth(F') for all components F' of F, and

(2) $\chi(Y') > \chi(F)$

Now let $Y = X + Y'$ and let $F \in \mathcal{F}$. By **(1)**, $F \nrightarrow Y$, since no component of F has a homomorphism to Y and $F \nrightarrow X$. By **(2)**, $Y' \nrightarrow F$, so $Y \nrightarrow F$. Hence, (UL) holds and $\mathbb{G}'$ is upward loose.

Now let us show that $\mathbb{G}'$ is downward loose. Let $H \in \mathbb{G} \setminus \mathcal{F}^{\downarrow}$, that is, $H \nrightarrow F$ for all $F \in \mathcal{F}$. Let $k = \max\{|V(H)|, |V(F)| : F \in \mathcal{F}\}$ and $\ell = \max\{\text{oddgirth}(F) : F \in \mathcal{F}\} + 1$. Here ℓ is finite because $\mathcal{F} \subset \mathbb{G}'$.

By Theorem 2.1 there is a graph $G \in \mathbb{G}'$ such that for all $K \in \mathbb{G}$ where $|V(K)| \leq k$ we have $G \to K$ if and only if $H \to K$, and girth$(G) > \ell$. Therefore $G \to H$ but for all $F \in \mathcal{F}$ we have $G \not\to F$. Since girth$(G) >$ oddgirth(F) we have $F \not\to G$ for each $F \in \mathcal{F}$. Furthermore $H \not\to K_2$ therefore $H \in \mathbb{G}'$, therefore $G \not\to K_2$ and so $G \in \mathbb{G}'$. Thus (DL) holds and $\mathbb{G}'$ is downward loose. □

As noted above, it is well-known that $\mathbb{G}'$ has no finite maximal antichains. We include a short proof to illustrate the relationship between upward loose sets and maximal antichains.

Corollary 4.2. *There is no finite maximal antichain in* $\mathbb{G}'$.

Proof. Indeed, let $\mathcal{F} \subset \mathbb{G}'$ be a finite antichain. Then $K_1 < F_i$ (for each i) therefore $K_1 \notin \mathcal{F}^{\uparrow}$. The application of Theorem 4.1 gives us an element of $\mathbb{G}'$, which is incomparable to $\mathcal{F}$. □

Since there are no finite maximal antichains and every finite antichain extends to a maximal one, each finite antichain can be extended to an infinite maximal antichain. The following shows that quite different behavior can be found in the various extensions.

Corollary 4.3. *Let $A \subseteq \mathbb{G}'$ be a finite antichain. Then*

(1) *there is a non-splitting maximal antichain $A_1 \subset \mathbb{G}'$ with $A \subset A_1$, and*

(2) *there is a strongly splitting maximal antichain $A_2 \subset \mathbb{G}'$ such that $A \subset A_2$.*

Proof. **(1)** This is a direct consequence of Theorem 3.6, applied to the poset $\mathbb{G}$ and the upward loose subset $\mathbb{G}'$.

(2) This follows from an application of Theorem 3.4 to $\mathbb{G}$ and the downward loose subset $\mathbb{G}'$. □

The notion of a *cut-point* and a *cut-free* subset are closely tied to the splitting property: see [1] and [9]. It has also been studied independently in the context of homomorphism orders of graphs: see [12]. We provide a short proof that $\mathbb{G}'$ is cut-free, both to illustrate an application of the sparse incomparability lemma and to highlight differences between $\mathbb{G}$ and $\mathbb{D}$ that we shall see again in the next section.

Proposition 4.4. $\mathbb{G}'$ *is cut-free.*

Proof. We need to show that for all triples $F < G < H$, if $G \in \mathbb{G}'$ (and therefore $H \in \mathbb{G}'$) then there is a $G' \in \mathbb{G}'$ such that $F < G' < H$ and G' is incomparable to G. Since oddgirth(G) is finite for $G \in \mathbb{G}'$, we can apply Theorem 2.1 to H with parameters $k = \max(|V(G)|, |V(F)|) + 1$ and $\ell = \text{oddgirth}(G) + 1$ to get a graph H' such that:

- $H' \to H$, since $H \to H$,
- $H' \nrightarrow G$, since $H \nrightarrow G$,
- $H' \nrightarrow F$, since $H \nrightarrow F$, and
- $\text{girth}(H') > \ell$.

Since oddgirth$(G) < \ell$ we have $X \nrightarrow H'$ for each connected component X of G. Therefore the graph $G' = F + H'$ satisfies the requirements.

□

5 Homomorphism poset $\mathbb{D}$

In the study of $\mathbb{G}$, long odd cycles play a crucial role. So, one might hope that the investigation the following two subsets $\mathbb{D}'$ and $\mathbb{D}^*$ of $\mathbb{D}$ would lead to the construction of nice antichains.

Before defining these, it is useful to recall that a finite directed graph X is a *core* if every homomorphism of X to itself is bijective. Every digraph is homomorphically equivalent to a unique core, and, so, every directed graph class contains exactly one core (cf. [11]). For the rest of this section, we shall use "graph" for "directed graph" and, given a directed graph X, $\overline{X}$ denotes its undirected version.

Let $\mathbb{D}'$ be the subset of $\mathbb{D}$ consisting of all directed graph classes whose core X has the property that every connected component of $\overline{X}$ contains an odd cycle. Furthermore, let $\mathbb{D}^*$ be the subset of $\mathbb{D}$ consisting of all graph classes with core Y such that $\overline{Y}$ is connected and contains an odd cycle. Of course, $\mathbb{D}^* \subseteq \mathbb{D}'$, while the graph Z which is the sum of the two orientations of a 3-cycle is in $\mathbb{D}'$ and not in $\mathbb{D}^*$.

One can prove the following:

Theorem 5.1. *In the partially ordered set* $\mathbb{D}$*,*

(1) $\mathbb{D}'$ *is an upwards loose kernel, and*

(2) $\mathbb{D}^*$ *is cut-free.*

Moreover, $\langle \mathbb{D}^*, \leq \rangle$ *is downward loose.*

Unfortunately, we can also prove that

Observation 5.2. $\mathbb{D}'$ *does not have the finite antichain extension property.*

Proof. Let T_3 be the transitive tournament on three vertices and let $P_3 = (W, F)$ be the directed path on four vertices: $W = \{x_0, x_1, x_2, x_3\}$ and $F = \{(x_i, x_{i+1}) : i = 0, 1, 2\}$. Now let $H = T_3 + P_3$, the disjoint union of T and P_3. Then H is a core in $\mathbb{D}$. Also, $T \in \mathbb{D}'$, is a core and $T < H$. Regard $\{T\}$ as an antichain. If $\mathbb{D}'$ had the (FAE) property there would exist a core $H' \in \mathbb{D}'$ such that H' is incomparable to T and $H' < H$. However, every connected component of $\overline{H'}$ contains an odd cycle, so $H' < H$ implies that $H' \leq T$ since no component of H' can be mapped by a homomorphism into P_3. □

Although we cannot prove that $\mathbb{D}^*$ does not satisfy the (FAE) property, fortunately there is another subset $\mathbb{D}^c$ of $\mathbb{D}$ which is both an upward loose kernel in $\mathbb{D}$ and has the finite antichain extension property in $\mathbb{D}$. To discuss it, first we need an easy observation. A finite directed graph $\vec{C}$ is a *directed cycle* if it is connected and each vertex has indegree and outdegree 1. It is easily seen that each directed cycle is a core.

Observation 5.3. *Let* $\vec{C}$ *be a directed cycle and* T *be a graph such that* $\overline{T}$ *is an arbitrary tree. Then* $T \to \vec{C}$*.*

Proof. Map a vertex v of T to any vertex of the cycle. Next the vertices adjacent to v in $\overline{T}$ can be mapped into vertices of $\vec{C}$ so that directed edges are preserved. Since there is no cycle in $\overline{T}$ we can finish the process easily. $\square$

Let $\mathbb{D}^c$ be the set of all homomorphism classes in $\mathbb{D}$ whose core X has the property that for some $\vec{C}$, $\vec{C} \to X$. Here is a direct consequence of Observation 5.3.

Observation 5.4. *If $G \in \mathbb{D}^c$, $T \in \mathbb{D}$ and $\overline{T}$ is a tree, then $G + T \sim G$.*

Hence we can assume that no component of an element of $\mathbb{D}^c$ can be embedded into a tree. Therefore from now on we assume that each component X of each element of $\mathbb{D}^c$ has the property that $\overline{X}$ contains a cycle.

Theorem 5.5. *The subposet $\mathbb{D}^c$ is an upward loose kernel in $\mathbb{D}$.*

Proof. Let $\mathcal{F} \subseteq \mathbb{D}^c$ be finite, and $X \in \mathbb{D}^c$ but $X \notin \mathcal{F}^{\uparrow}$. We are going to find an $Y \in \mathbb{D}^c$ such that $X \to Y$, $Y \not\to X$, and Y is incomparable to each $F \in \mathcal{F}$.

Let $n := \max\{|X|, |F| : F \in \mathcal{F}\}$. Using the Erdős theorem, obtain a graph Z such that $\chi(\overline{Z}) > n$, girth$(\overline{Z}) > n$, $\overline{Z}$ is connected, and Z contains at least one directed cycle. Then $Z \in \mathbb{D}^c$. Let $Y = X + Z$. Since $Z \in \mathbb{D}^c$ therefore $Y \in \mathbb{D}^c$ as well. Clearly $X \to Y$ while $Y \not\to X$ because $\chi(Y) > |X|$.

Assume that $f : F \to Y$. Then there is a component K of F such that $f : K \to Z$. But $|V(K)| \leq n$ while girth$(Z) > n$, hence the image $f(K)$ is a tree, which contradicts the assumption that no component of an element of $\mathbb{D}^c$ can be mapped into a tree. $\square$

Theorem 5.6. *Let $A_1 \subseteq \mathbb{D}^c$ be a finite antichain. Then there is a non-splitting $\mathbb{D}$-maximal antichain $A \subset \mathbb{D}^c$ with $A \supset A_1$.*

Proof. Since $\mathbb{D}^c$ is an upward loose kernel it can be used in Theorem 3.5 to extend a non-maximal antichain into a non-splitting antichain, maximal in $\mathbb{D}$. $\square$

Theorem 5.7. *$\mathbb{D}^c$ has the finite antichain extension property in $\mathbb{D}$.*

Proof. Let $\mathcal{F} \subseteq \mathbb{D}^c$ be a finite antichain and $X \in \mathbb{D}$. We need to find $Y \in (X^{\updownarrow} \cap \mathbb{D}^c) \setminus \mathcal{F}^{\updownarrow}$. In case $X \notin \mathcal{F}^{\uparrow}$ then Theorem 5.5 provides the required Y.

Assume now that $X \in \mathcal{F}^{\uparrow}$. Then $X \in \mathbb{D}^c$ because there exists $F \in \mathcal{F}$ with $F \to X$ and the image of its directed circle of F is a directed circle in X.

Let $n = \max\{|X|, |F| : F \in \mathcal{F}\}$. Apply Theorem 2.2 with $H = X$ and $m = \ell = n$ to obtain $X' = H'$. Now let $Y = X' + \vec{C}_{kn}$. (The directed circle has kn vertices.) Then $X' \to X$ and $\vec{C}_{kn} \to \vec{C}_k$ therefore $Y \to X$. At the same time $X \not\to Y$ since girth$(Y) > \ell \geq |X|$ therefore the circle $\vec{C}_k$ of X cannot be embedded into Y. The same applies for the directed circles in each $F \in \mathcal{F}$ therefore $F \not\to Y$. Finally we have $X \not\to F$ and so $Y \not\to F$. $\square$

Corollary 5.8. *$\mathbb{D}^c$ does not contain finite maximal antichains.*

Corollary 5.9. *Let $A_1 \subseteq \mathbb{D}^c$ be a finite antichain in $\mathbb{D}$. Then there is a strongly splitting $\mathbb{D}$-maximal antichain $A_1 \subset A \subset \mathbb{D}^c$.*

Proof. This is just the direct application of Theorem 3.4 to the posets $\mathbb{D}$ and $\mathbb{D}^c$. □

Theorem 5.10. *$\mathbb{D}^c$ is cut-free in $\mathbb{D}$.*

Proof. Let $F < G < H$ where $G \in \mathbb{D}^c$ (and therefore $H \in \mathbb{D}^c$ as well). We need a $G' \in \mathbb{D}^c$, which is incomparable to G but $F < G' < H$.

Let $n = \max\{|F|, |G|, |H|\}$. Apply Theorem 2.2 to H with parameters $m = \ell = n$ to obtain the directed graph H'. Since $H \in \mathbb{D}^c$, there is k such that $\vec{C}_k$ is a subgraph of H. Let $G' = F + H' + \vec{C}_{kn}$.

Then $F \to G'$ since F is a subgraph of G'. Furthermore $H' \to H$ due to the fact that $|H| = n \leq m$ and $H \to H$. Due to our assumption on $\mathbb{D}^c$, each component of the graph G contains cycles, and at least one of them, say K, cannot be embedded into F. Therefore if $G \to Y$ then for this component we have $K \to H' + C_{nk}$. However, $\text{girth}(H' + C_{nk}) > |K|$, hence K is embedded into a tree, a contradiction. □

References

[1] R. Ahlswede, P.L. Erdős, N. Graham, A splitting property of maximal antichains, *Combinatorica* **15** (1995) 475–480

[2] D. Duffus, B. Sands, Minimum sized fibres in distributive lattices, *J. Austral. Math. Soc.* **70** (2001) 337-350

[3] D. Duffus, B. Sands, Splitting numbers of grids, *Electronic J. Comb.* **12** (2005) #R17

[4] M. Džamonja, A note on the splitting property in strongly dense posets of size $\aleph_0$, *Radovi Mathematički* **8** (1999) 321–326

[5] Paul Erdős: Graph theory and probability. *Canad. J. Math.* **11** (1959) 34–38

[6] Paul Erdős, A. Hajnal: On chromatic number of graphs and set systems, *Acta Math. Hungar.* **17** (1966), 61–99

[7] Paul Erdős, A. Rényi: On random graphs I. *Publ. Math. Debrecen* **6** (1959) 290–297

[8] P.L. Erdős, Splitting property in infinite posets, *Discrete Math.* **163** (1997) 251–256

[9] Péter L. Erdős, L. Soukup: How to split antichains in infinite posets, to appear in *Combinatorica* (2006) 1–17

[10] J. Foniok, J. Nešetřil, C. Tardif: Generalized dualities and maximal finite antichains in the homomorphism order of relational structures, *KAM-DIMATIA Series* (2006)

[11] P. Hell, J. Nešetřil, *Graphs and Homomorphisms*, Oxford University Press, Oxford (2004)

[12] J. Nešetřil, Sparse Incomparability for relational structures

[13] J. Nešetřil, A. Pultr, On classes of relations and graphs determined by subobjects and factorobjects,*Discrete Math.* **22** (1978) 287–300

[14] J. Nešetřil, V. Rödl: Chromatically optimal rigid graphs, *J. Comb. Theory Ser. (B)* **46** (1989) 133–141

[15] J. Nešetřil, C. Tardif: Duality theorems for finite structures (chacterising gaps and good chacterisations) *J. Comb. Theory Ser. (B)* **80** (2000) 80–97

[16] J. Nešetřil, C. Tardif: On maximal finite antichains in the homomorphism order of directed graphs. *Discuss. Math. Graph Theory* **23** (2003) 325–332

[17] J. Nešetřil, Xuding Zhu: On sparse graphs with given colorings and homomorphisms, *J. Comb. Theory Ser. (B)* **90** (2004), 161–172

[18] A. Pultr, V. Trnková: *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories*, North-Holland, Amsterdam (1980)

[19] E. Welzl: Color families are dense, *Theoret. Comput. Sci.* **17** (1982) 29-41