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October 28, 2009

Abstract

We give a construction under CH of a non-metrizable compact Hausdorff space K such that any uncountable semi-biorthogonal sequence in $C(K)$ must be of a very specific kind. The space K has many nice properties, such as being hereditarily separable, hereditarily Lindelöf and a 2-to-1 continuous preimage of a metric space, and all Radon measures on K are separable. However K is not a Rosenthal compactum.

We introduce the notion of a *bidiscrete system* in a compact space K. These are subsets of K^2 which determine biorthogonal systems of a special kind in $C(K)$ that we call *nice*. We note that every infinite compact Hausdorff space K has a bidiscrete system and hence a nice biorthogonal system of size $d(K)$, the density of K. [1]

0   Introduction

All topological spaces mentioned here are Hausdorff. As is traditional in general Banach space theory, Banach spaces we mention are considered as real Banach spaces even though nothing essential changes in the context of complex spaces. Throughout let K stand for an infinite compact topological space.

Let X be a Banach space and C a closed convex subset of X. A point $x_0 \in C$ is a *point of support for C* if there is a functional $\varphi \in X^*$ such that $\varphi(x_0) \leq \varphi(x)$ for all $x \in C$, and $\varphi(x_0) < \varphi(x')$ for some $x' \in C$.

Rolewicz [13] proved in 1978 that every separable closed convex subset Y of a Banach space contains a point which is not a point of support for Y, and asked if every

non-separable Banach space must contain a closed convex set containing only points of support. In fact, this topic was already considered by Klee [8] in 1955 and the above theorem follows from 2.6 in that paper, by the same proof and taking x_is to form a dense set in C [2]. However it was Rolewicz's paper which started a whole series of articles on this topic, and his question has not yet been settled completely. It is known that the answer to Rolewicz's question is independent of ZFC, and it is still not known if the negative answer follows from CH. In §2 we construct a CH example of a nonseparable Banach space of the form $C(K)$ which violates a strenghtening of the requirements in the Rolewicz's question.

The proof in §2 uses certain systems of pairs of points of K, whose structure seems to us to be of independent interest. They appear implicitly in many proofs about biorthogonal systems in spaces of the form $C(K)$, see [6], but their existence is in fact entirely a property of the compact space K. We call such systems *bidiscrete systems*. They are studied in §3. Specifically, we prove in Theorem 3.5 that if K is an infinite compact Hausdorff space then K has a bidiscrete system of size $d(K)$, the density of K. This theorem has not been stated in this form before, but we note that an argument by Todorčević in [15] can be easily extended to give this result.

We now give some historical background. Mathematical background will be presented in Section 1.

Borwein and Vanderwerff [3] proved in 1996 that, in a Banach space X, the existence of a closed convex set all of whose points are support points is equivalent to the existence of an uncountable semi-biorthogonal sequence for X, where semi-biorthogonal sequences are defined as follows:

Definition 0.1. Let X be a Banach space. A sequence $\langle (f_\alpha, \varphi_\alpha) : \alpha < \alpha^* \rangle$ in $X \times X^*$ is said to be a *semi-biorthogonal sequence* if for all $\alpha, \beta < \alpha^*$ we have:

- $\varphi_\alpha(f_\alpha) = 1$,

- $\varphi_\alpha(f_\beta) = 0$ if $\beta < \alpha$,

- $\varphi_\alpha(f_\beta) \geq 0$ if $\beta > \alpha$.

We remind the reader of the better known notion of *a biorthogonal system* $\{(f_\alpha, \varphi_\alpha) : \alpha < \alpha^*\}$ in $X \times X^*$ which is defined to satisfy the first item of the Definition 0.1 with the last two items are strengthened to

- $\varphi_\alpha(f_\beta) = 0$ if $\beta \neq \alpha$.

Notice that the requirements of a semi-biorthogonal sequence make it clear that we really need a well ordering of the sequence in the definition, but that the definition of a biorthogonal system does not require an underlying well-ordering. There is nothing special about the values 0 and 1 in the above definitions, of course, and we could replace

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[2]We thank Libor Vesely for pointing out this, not widely recognised, connection.

them by any pair (a, b) of distinct values in $\mathbb{R}$ and even let $b = b_\alpha$ vary with α. Equally, we could require the range of all f_α to be in $[0, 1]$ or some other fixed nonempty closed interval.

Obviously, any well-ordering of a biorthogonal system gives a semi-biorthogonal sequence. On the other hand, there is an example by Kunen under CH of a nonmetrizable compact scattered space K for which $X = C(K)$ does not have an uncountable biorthogonal system, as proved by Borwein and Vanderwerff in [3]. Since K is scattered, it is known that X must have an uncountable semi-biorthogonal sequence (see [6] for a presentation of a similar example under ♣ and a further discussion). Let us say that a Banach space is a *Rolewicz space* if it is nonseparable but does not have an uncountable semi-biorthogonal sequence.

In his 2006 paper [15], Todorčević proved that under Martin's Maximum (MM) every non-separable Banach space has an uncountable biorthogonal system, so certainly it has an uncountable semi-biorthogonal sequence. Hence, under MM there are no Rolewicz spaces. On the other hand, Todorčević informed us that he realized in 2004 that a forcing construction in [1] does give a consistent example of a Rolewicz space. Independently, also in 2004 (published in 2009), Koszmider gave a similar forcing construction in [9]. It is still not known if there has to be a Rolewicz space under CH.

Our motivation was to construct a Rolewicz space of the form $X = C(K)$ under CH. Unfortunately, we are not able to do so, but we obtain in Theorem 2.1 a space for which we can at least show that it satisfies most of the known necessary conditions for a Rolewicz space and that it has no uncountable semi-bidiscrete sequences of the kind that are present in the known failed candidates for such a space, for example in $C(S)$ where S is the split interval.

Specifically, it is known that if K has a non-separable Radon measure or if it is scattered then $C(K)$ cannot be Rolewicz ([5], [6]) and our space does not have either of these properties. Further, it is known that a compact space K for which $C(K)$ is a Rolewicz space must be both HS and HL ([10], [3]) while not being metrizable, and our space has these properties, as well. It follows from the celebrated structural results on Rosenthal compacta by Todorčević in [14] that a Rosenthal compactum cannot be a Rolewicz space, and our space is not Rosenthal compact. Finally, our space is not metric but it is a 2-to-1 continuous preimage of a metric space. This is a property possessed by the forcing example in [9] and it is interesting because of a theorem from [14] which states that every non-metric Rosenthal compact space which does not contain an uncountable discrete subspace is a 2-to-1 continuous preimage of a metric space. Hence the example in [9] is a space which is not Rosenthal compact and yet it satisfies these properties, and so is our space.

1 Background

Definition 1.1. Let $X = C(K)$ be the Banach space of continuous functions on a compact space K. We say that a sequence $\langle (f_\alpha, \phi_\alpha) : \alpha < \alpha^* \rangle$ in $X \times X^*$ is a *nice*

semi-biorthogonal sequence if it is a semi-biorthogonal sequence and there are points $\langle x_\alpha^l : l = 0, 1, \alpha < \alpha^* \rangle$ in K such that $\phi_\alpha = \delta_{x_\alpha^1} - \delta_{x_\alpha^0}$, where δ denotes the Dirac measure. We similarly define nice biorthogonal systems.

As Definition 1.1 mentions points of K and $C(K)$ does not uniquely determine K [3], the definition is actually topological rather than analytic. We shall observe below that the existence of a nice semi-biorthogonal sequence of a given length or of a nice biorthogonal system of a given size in $C(K)$ is equivalent to the existence of objects which can be defined in terms which do not involve the dual $C(K)^*$.

Definition 1.2. (1) A system $\{(x_\alpha^0, x_\alpha^1) : \alpha < \kappa\}$ of pairs of points in K (i.e. a subfamily of K^2) is called *a bidiscrete system in K* if there exist functions $\{f_\alpha : \alpha < \kappa\} \subseteq C(K)$ satisfying that for every $\alpha, \beta < \kappa$:

- $f_\alpha(x_\alpha^l) = l$ for $l \in \{0, 1\}$,

- if $\alpha \neq \beta$ then $f_\alpha(x_\beta^0) = f_\alpha(x_\beta^1)$.

(2) We similarly define semi-bidiscrete sequences in K as sequences $\langle (x_\alpha^0, x_\alpha^1) : \alpha < \alpha^* \rangle$ of points in K^2 that satisfy the first requirement of (1) but instead of the second the following two requirements:

- if $\alpha > \beta$ then $f_\alpha(x_\beta^0) = f_\alpha(x_\beta^1)$,

- if $\alpha < \beta$ then $f_\alpha(x_\beta^0) = 1 \implies f_\alpha(x_\beta^1) = 1$.

Observation 1.3. For a compact space K, $\{(x_\alpha^0, x_\alpha^1) : \alpha < \alpha^*\} \subseteq K^2$ is a bidiscrete system iff there are $\{f_\alpha : \alpha < \alpha^*\} \subseteq C(K)$ such that $\{(f_\alpha, \delta_{x_1^\alpha} - \delta_{x_0^\alpha}) : \alpha < \alpha^*\}$ is a nice biorthogonal system for the Banach space $X = C(K)$. The analogous statement holds for nice semi-bidiscrete sequences.

Proof. We only prove the statement for nice biorthogonal systems, the proof for the nice semi-biorthogonal sequences is the same. If we are given a system exemplifying (1), then $\delta_{x_\alpha^1}(f_\beta) - \delta_{x_\alpha^0}(f_\beta) = f_\beta(x_\alpha^1) - f_\beta(x_\alpha^0)$ has the values as required. On the other hand, if we are given a nice biorthogonal system of pairs $\{(f_\alpha, \delta_{x_\alpha^1} - \delta_{x_\alpha^0}) : \alpha < \alpha^*\}$ for X, define for $\alpha < \alpha^*$ the function $g_\alpha \in C(K)$ by $g_\alpha(x) = f_\alpha(x) - f_\alpha(x_\alpha^0)$. Then $\{(x_\alpha^0, x_\alpha^1) : \alpha < \alpha^*\}$ satisfies (1), as witnessed by $\{g_\alpha : \alpha < \alpha^*\}$. ★$_{1.3}$

In the case of a 0-dimensional space K we are often able to make a further simplification by requiring that the functions f_α exemplifying the bidiscreteness of (x_α^0, x_α^1) take only the values 0 and 1. This is clearly equivalent to asking for the existence of a family $\{H_\alpha : \alpha < \alpha^*\}$ of clopen sets in K such that each H_α separates x_α^0 and x_α^1 but

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[3]see e.g. Miljutin's theorem [11], [12] which states that for K, L both uncountable compact metrizable, the spaces $C(K)$ and $C(L)$ are isomorphic.

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not x_β^0 and x_β^1 for $\beta \neq \alpha$. We call such bidiscrete systems *very nice*. We can analogously define a *very nice* semi-bidiscrete sequence, where the requirements on the clopen sets become $x_\alpha^l \in H_\alpha \iff l = 1$, $\beta < \alpha \implies [x_\beta^0 \in H_\alpha \iff x_\beta^1 \in H_\alpha]$ and $[\beta > \alpha \wedge x_\beta^0 \in H_\alpha] \implies x_\beta^1 \in H_\alpha$.

We shall use the expression *very nice (semi-)biorthogonal system (sequence)* in $C(K)$ to refer to a nice (semi-)biorthogonal system (sequence) obtained as in the proof of Claim 1.3 from a very nice (semi-)bidiscrete system (sequence) in K.

Example 1.4. (1) Let K be the split interval (or double arrow) space, namely the ordered space $K = [0, 1] \times \{0, 1\}$, ordered lexicographically. Then

$$\{((x, 0), (x, 1)) : x \in [0, 1]\}$$

forms a very nice bidiscrete system in K. This is exemplified by the two-valued continuous functions $\{f_x : x \in [0, 1]\}$ defined by $f_x(r) = 0$ if $r \leq (x, 0)$ and $f_x(r) = 1$ otherwise.

(2) Suppose that κ is an infinite cardinal and $K = 2^\kappa$. For $l \in \{0, 1\}$ and $\alpha < \kappa$ we define $x_\alpha^l \in K$ by letting $x_\alpha^l(\beta) = 1$ if $\beta < \alpha$, $x_\alpha^l(\beta) = 0$ if $\beta > \alpha$, and $x_\alpha^l(\alpha) = l$. The clopen sets $H_\alpha = \{f \in K : f(\alpha) = 1\}$ show that the pairs $\{(x_\alpha^0, x_\alpha^1) : \alpha < \kappa\}$ form a very nice bidiscrete system in the Cantor cube $K = 2^\kappa$.

In [15], Theorem 10, it is proved under MA_{ω_1} that every Banach space of the kind $X = C(K)$ for a nonmetrizable compact K admits an uncountable nice biorthogonal system. Moreover, at the end of the proof it is stated that for a 0-dimensional K this biorthogonal system can even be assumed to be very nice (in our terminology).

As nice semi-biorthogonal sequences may be defined using only K and $X = C(K)$ and do not involve the dual X^*, in constructions where an enumerative tool such as CH is used it is easier to control nice systems than the general ones. In our CH construction below of a closed subspace K of 2^{ω_1} we would at least like to destroy all uncountable nice semi-biorthogonal sequences by controlling semi-bidiscrete sequences in K. We are only able to do this for semi-bidiscrete sequences which are not already determined by the first ω-coordinates, in the sense of the following Definition 1.5 : In our space K any uncountable nice semi-biorthogonal sequence must be ω-determined.

Definition 1.5. A family $\{(x_\alpha^0, x_\alpha^1) : \alpha < \alpha^*\} \subseteq 2^{\omega_1} \times 2^{\omega_1}$ is said to be *ω-determined* if

$$(\forall s \in 2^\omega) \{\alpha : x_\alpha^0 \restriction \omega = x_\alpha^1 \restriction \omega = s\} \text{ is countable.}$$

For $K \subseteq 2^{\omega_1}$ we define an *ω-determined semi-biorthogonal sequence in $C(K)$* to be any nice semi-biorthogonal sequence $\langle (f_\alpha, \delta_{x_\alpha^1} - \delta_{x_\alpha^0}) : \alpha < \alpha^* \rangle$ for which the associated semi-bidiscrete sequence $\langle (x_\alpha^0, x_\alpha^1) : \alpha < \alpha^* \rangle$ forms an ω-determined family.

2   The CH construction

Theorem 2.1. Under CH, there is a compact space $K \subseteq 2^{\omega_1}$ with the following properties:

- K is not metrizable, but is a 2-to-1 continuous preimage of a metric space,

- K is HS and HL,

- every Radon measure on K is separable,

- K has no isolated points,

- K is not Rosenthal compact,

- any uncountable nice semi-biorthogonal sequence in $C(K)$ is ω-determined.

Proof. We divide the proof into two parts. In the first we give various requirements on the construction, and show that if these requirements are satisfied the space meeting the claim of the theorem can be constructed. In the second part we show that these requirements can be met.

2.0.1   The requirements

Our space will be a closed subspace of 2^{ω_1}. Every such space can be viewed as the limit of an inverse system of spaces, as we now explain.

Definition 2.2. For $\alpha \leq \beta \leq \omega_1$, define $\pi_\alpha^\beta : 2^\beta \to 2^\alpha$ by $\pi_\alpha^\beta(f) = f \upharpoonright \alpha$.

Suppose that K is a closed subspace of 2^{ω_1}, then for $\alpha \leq \omega_1$ we let $K_\alpha = \pi_\alpha^{\omega_1}(K)$. So, if $\alpha \leq \beta$ then K_α is the π_α^β-projection of K_β. For $\alpha < \omega_1$ let

$$A_\alpha = \pi_\alpha^{\alpha+1}(\{x \in K_{\alpha+1} : x(\alpha) = 0\}), B_\alpha = \pi_\alpha^{\alpha+1}(\{x \in K_{\alpha+1} : x(\alpha) = 1\}).$$

The following statements are then true:

R1. K_α is a closed subset of 2^α, and $\pi_\alpha^\beta(K_\beta) = K_\alpha$ whenever $\alpha \leq \beta \leq \omega_1$.

R2. For $\alpha < \omega_1$, A_α and B_α are closed in K_α, $A_\alpha \cup B_\alpha = K_\alpha$, and $K_{\alpha+1} = A_\alpha \times \{0\} \cup B_\alpha \times \{1\}$.

Now K can be viewed as the limit of the inverse system $\mathcal{K} = \{K_\alpha : \alpha < \omega_1, \pi_\alpha^\beta \upharpoonright K_\beta : \alpha \leq \beta < \omega_1\}$. Therefore to construct the space K it is sufficient to specify the system $\mathcal{K}$, and as long as the requirements **R1** and **R2** are satisfied, the resulting space K will be a compact subspace of 2^{ω_1}. This will be our approach to constructing K, that is we define K_α by induction on α to satisfy various requirements that we list as **Rx**.

The property HS+HL will be guaranteed by a use of irreducible maps, as in [4]. Recall that for spaces X, Y, a map $f : X \to Y$ is called *irreducible* on $A \subseteq X$ iff for any proper closed subspace F of A we have that $f(F)$ is a proper subset of $f(A)$. We shall have a special requirement to let us deal with HS+HL, but we can already quote Lemma 4.2 from [4], which will be used in the proof. It applies to any space K of the above form.

Lemma 2.3. Assume that K and K_α satisfy **R1** and **R2** above. Then K is HL+HS iff for all closed $H \subseteq K$, there is an $\alpha < \omega_1$ for which $\pi_\alpha^{\omega_1}$ is irreducible on $(\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(H))$.

In addition to the requirements given above we add the following basic requirement **R3** which assures that K has no isolated points.

R3. For $n < \omega$, $K_n = A_n = B_n = 2^n$. For $\alpha \geq \omega$, A_α and B_α have no isolated points.

Note that the requirement **R3** implies that for each $\alpha \geq \omega$, K_α has no isolated points; so it is easy to see that the requirements guarantee that K is a compact subspace of 2^{ω_1} and that it has no isolated points. Further, $K_\omega = 2^\omega$ by **R1** and **R3**. The space K is called *simplistic* if for all α large enough $A_\alpha \cap B_\alpha$ is a singleton. For us 'large enough' will mean 'infinite', i.e. during the construction we shall obey the following:

R4. For all $\alpha \in [\omega, \omega_1)$ we have $A_\alpha \cap B_\alpha = \{s_\alpha\}$ for some $s_\alpha \in K_\alpha$.

By **R4** we can make the following observation which will be useful later:

Observation 2.4. Suppose that $x \in K_\alpha, y \in K_\beta$ for some $\omega \leq \alpha \leq \beta$ and $x \not\subseteq y$, $y \not\subseteq x$ with $\Delta(x, y) \geq \omega$. Then $x \restriction \Delta(x, y) = y \restriction \Delta(x, y) = s_{\Delta(x,y)}$.

As usual, we used here the notation $\Delta(x, y) = \min\{\alpha : x(\alpha) \neq y(\alpha)\}$.

Requirement **R4** implies that K is not 2nd countable, hence not metrizable. The following is folklore in the subject, but one can also see [2] for a detailed explanation and stronger theorems:

Fact 2.5. Every Radon measure on a simplistic space is separable.

Now we come back to the property HS+HL. To assure this we shall construct an auxiliary Radon measure μ on K. This measure will be used, similarly as in the proof from Section §4 in [4], to assure that for every closed subset H of K we have $H = (\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(H))$ for some countable coordinate α. In fact, what we need for our construction is not the measure μ itself but a sequence $\langle \mu_\alpha : \alpha < \omega_1 \rangle$ where each μ_α is a Borel measure on K_α and these measures satisfy that for each $\alpha \leq \beta < \omega_1$ and Borel set $B \subseteq K_\beta$, we have $\mu_\beta(B) = \mu_\alpha(\pi_\alpha^\beta(B))$. As a side remark the sequence $\langle \mu_\alpha : \alpha < \omega_1 \rangle$ will uniquely determine a Radon measure $\mu = \mu_{\omega_1}$ on K. To uniquely determine each Borel (=Baire) measure μ_α it is sufficient to decide its values on the clopen subsets of K_α. We formulate a requirement to encapsulate this discussion:

R5. For $\alpha \leq \omega_1$, μ_α is a finitely additive probability measure on the clopen subsets of K_α, and $\mu_\alpha = \mu_\beta(\pi_\alpha^\beta)^{-1}$ whenever $\omega \leq \alpha \leq \beta \leq \omega_1$. For $\alpha \leq \omega$, μ_α is the usual product measure on the clopen subsets of $K_\alpha = 2^\alpha$.

Let $\widehat{\mu}_\alpha$ be the Borel measure on K_α generated by μ_α. It is easy to verify that **R1**-**R5** imply that for $\alpha \leq \omega$, $\widehat{\mu}_\alpha$ is the usual product measure on $K_\alpha = 2^\alpha$, and that for any α, $\widehat{\mu}_\alpha$ gives each non-empty clopen set positive measure and measure 0 to each point in K_α. We shall abuse notation and use μ_α for both $\widehat{\mu}_\alpha$ and its restriction to the clopen sets. Note that by the usual Cantor tree argument these properties assure that in every set of positive measure there is an uncountable set of measure 0; this observation will be useful later on.

The following requirements will help us both to obtain HS+HL and to assure that K is not Rosenthal compact. To formulate these requirements we use CH to enumerate the set of pairs $\{(\gamma, J) : \gamma < \omega_1 , \, J \subseteq 2^\gamma \text{ is Borel}\}$ as $\{(\delta_\alpha, J_\alpha) : \omega \leq \alpha < \omega_1\}$ so that $\delta_\alpha \leq \alpha$ for all α and each pair appears unboundedly often.

Suppose that $\omega \leq \alpha < \omega_1$ and K_α and μ_α are defined. We define the following subsets of K_α:

$C_\alpha = (\pi_{\delta_\alpha}^\alpha)^{-1}(J_\alpha)$, if $J_\alpha \subseteq K_{\delta_\alpha}$; $C_\alpha = \emptyset$ otherwise.
$L_\alpha = C_\alpha$ if C_α is closed; $L_\alpha = K_\alpha$ otherwise.
$Q_\alpha = L_\alpha \setminus \bigcup\{O : O \text{ is open and } \mu_\alpha(L_\alpha \cap O) = 0\}$
$N_\alpha = (L_\alpha \setminus Q_\alpha) \cup C_\alpha$, if $\mu_\alpha(C_\alpha) = 0$; $N_\alpha = (L_\alpha \setminus Q_\alpha)$ otherwise.

Let us note that L_α is a closed subset of K_α and that $Q_\alpha \subseteq L_\alpha$ is also closed and satisfies $\mu_\alpha(Q_\alpha) = \mu_\alpha(L_\alpha)$, and hence $\mu_\alpha(N_\alpha) = 0$. Also observe that Q_α has no isolated points, as points have μ_α measure 0.

We now recall from [4] what is meant by A and B being *complementary regular closed subsets* of a space X: this means that A and B are both regular closed with $A \cup B = X$, while $A \cap B$ is nowhere dense in X. Finally, we state the following requirements:

R6. For any $\beta \geq \alpha \geq \omega$, $s_\beta \notin (\pi_\alpha^\beta)^{-1}(N_\alpha)$;

R7. For any $\beta \geq \alpha \geq \omega$, $A_\beta \cap (\pi_\alpha^\beta)^{-1}(Q_\alpha)$ and $B_\beta \cap (\pi_\alpha^\beta)^{-1}(Q_\alpha)$ are complementary regular closed subsets of $(\pi_\alpha^\beta)^{-1}(Q_\alpha)$.

The following claim and lemma explain our use of irreducible maps, and the use of measure as a tool to achieve the HS+HL properties of the space. The proof is basically the same as in [4] but we give it here since it explains the main point and also to show how our situation actually simplifies the proof from [4]. For any α, we use the notation $[s]$ for a finite partial function s from α to 2 to denote the basic clopen set $\{f \in 2^\alpha : s \subseteq f\}$, or its relativization to a subspace of 2^α, as it is clear from the context. [4]

Claim 2.6. Assume the requirements **R1**-**R5** and **R7**. Then for each $\beta \in [\alpha, \omega_1]$ the projection π_α^β is irreducible on $(\pi_\alpha^\beta)^{-1}(Q_\alpha)$.

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[4]The notation also does not specify α but again following the tradition, we shall rely on α being clear from the context.

Proof of the Claim. We use induction on $\beta \geq \alpha$.

The step $\beta = \alpha$ is clear. Assume that we know that the projection π_α^β is irreducible on $(\pi_\alpha^\beta)^{-1}(Q_\alpha)$ and let us prove that $\pi_\alpha^{\beta+1}$ is irreducible on $(\pi_\alpha^{\beta+1})^{-1}(Q_\alpha)$. Suppose that F is a proper closed subset of $(\pi_\alpha^{\beta+1})^{-1}(Q_\alpha)$ satisfying $\pi_\alpha^{\beta+1}(F) = Q_\alpha$. Then by the inductive assumption $\pi_\beta^{\beta+1}(F) = (\pi_\alpha^\beta)^{-1}(Q_\alpha)$. Let $x \in (\pi_\alpha^{\beta+1})^{-1}(Q_\alpha) \setminus F$, so we must have that $x \upharpoonright \beta = s_\beta$. Assume $x(\beta) = 0$, the case $x(\beta) = 1$ is symmetric. Because F is closed, we can find a basic clopen set $[t]$ in $K_{\beta+1}$ containing x such that $[t] \cap F = \emptyset$. Let $s = t \upharpoonright \beta$.

Therefore $s_\beta \in [s]$ holds in K_β, and by **R7** we can find $y \in \text{int}(A_\beta \cap (\pi_\alpha^\beta)^{-1}(Q_\alpha)) \cap [s]$. Using the inductive assumption we conclude $y \in \text{int}(A_\beta \cap (\pi_\beta^{\beta+1})(F)) \cap [s]$, so there is a basic clopen set $[v] \subseteq [s]$ in K_β such that $y \in [v]$ and $[v] \subseteq A_\beta \cap (\pi_\beta^{\beta+1})(F)$. But then $[v]$ viewed as a clopen set in $K_{\beta+1}$ satisfies $[v] \subseteq [t]$ and yet $[v] \cap F \neq \emptyset$.

The limit case of the induction is easy by the definition of inverse limits.

★$_{2.6}$

Lemma 2.7. Assume the requirements **R1-R7** and let H be a closed subset of K. Then there is an $\alpha < \omega_1$ such that $\pi_\alpha^{\omega_1}$ is irreducible on $(\pi_\alpha^{\omega_1})^{-1}(\pi_\alpha^{\omega_1}(H))$.

Proof of the Lemma. For each $\gamma < \omega_1$, let $H_\gamma = \pi_\gamma^{\omega_1}(H)$. Then the $\mu_\gamma(H_\gamma)$ form a non-increasing sequence of real numbers, so we may fix a $\gamma < \omega_1$ such that for all $\alpha \geq \gamma$, $\mu_\alpha(H_\alpha) = \mu_\gamma(H_\gamma)$. Next fix an $\alpha \geq \gamma$ such that $\delta_\alpha = \gamma$ and $J_\alpha = H_\gamma$. Then $L_\alpha = C_\alpha = (\pi_\gamma^\alpha)^{-1}(H_\gamma)$. Hence H_α is a closed subset of L_α with the same measure as L_α, so $Q_\alpha \subseteq H_\alpha \subseteq L_\alpha$, by the definition of Q_α. Recall that by Claim 2.6 we have that $\pi_\alpha^{\omega_1}$ is irreducible on $(\pi_\alpha^{\omega_1})^{-1}(Q_\alpha)$.

Now we claim that $\pi_\alpha^{\omega_1}$ is 1-1 on $(\pi_\alpha^{\omega_1})^{-1}(H_\alpha \setminus Q_\alpha)$. Otherwise, there would be $x \neq y \in (\pi_\alpha^{\omega_1})^{-1}(H_\alpha \setminus Q_\alpha)$ with $x \upharpoonright \alpha = y \upharpoonright \alpha$. Therefore for some $\beta \geq \alpha$ we have $x \upharpoonright \beta = y \upharpoonright \beta = s_\beta$, as otherwise $x = (\pi_\alpha^{\omega_1})^{-1}(\{x \upharpoonright \alpha\})$. In particular $s_\beta \in (\pi_\alpha^\beta)^{-1}(H_\alpha) \subseteq (\pi_\alpha^\beta)^{-1}(L_\alpha)$. On the other hand, if $s_\beta \in (\pi_\alpha^\beta)^{-1}(Q_\alpha)$ then $\{x, y\} \in (\pi_\alpha^{\omega_1})^{-1}(Q_\alpha)$- a contradiction- so $s_\beta \notin (\pi_\alpha^\beta)^{-1}(Q_\alpha)$. This means $s_\beta \in (\pi_\alpha^\beta)^{-1}(N_\alpha)$, in contradiction with **R6**.

Thus, $\pi_\alpha^{\omega_1}$ must be irreducible on $(\pi_\alpha^{\omega_1})^{-1}(H_\alpha)$ as well, and the Lemma is proved.

★$_{2.7}$

Now we comment on how to assure that K is not Rosenthal compact. A remarkable theorem of Todorčević from [14] states that every non-metric Rosenthal compactum contains either an uncountable discrete subspace or a homeomorphic copy of the split interval. As our K, being HS+HL, cannot have an uncountable discrete subspace, it will suffice to show that it does not contain a homeomorphic copy of the split interval.

Claim 2.8. Suppose that the requirements **R1-R7** are met. Then

(1) all μ-measure 0 sets in K are second countable and

(2) K does not contain a homeomorphic copy of the split interval.

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Proof of the Claim. (1) Suppose that M is a μ-measure 0 Borel set in K and let $N = \pi_\omega^{\omega_1}(M)$, hence N is of measure 0 in 2^ω. Let $\alpha \in [\omega, \omega_1)$ be such that $\delta_\alpha = \omega$ and $J_\alpha = N$. Then $C_\alpha = (\pi_\omega^\alpha)^{-1}(N)$ and hence $\mu_\alpha(C_\alpha) = 0$ and so $C_\alpha \subseteq N_\alpha$. Requirement **R6** implies that for $\beta \geq \alpha$, $(\pi_\beta^{\omega_1})^{-1}(s_\beta) \cap M = \emptyset$, so the topology on M is generated by the basic clopen sets of the form $[s]$ for $\mathrm{dom}(s) \subseteq \alpha$. So M is 2nd countable.

(2) Suppose that $H \subseteq K$ is homeomorphic to the split interval. Therefore H is compact and therefore closed in K. In particular $\mu(H)$ is defined.

If $\mu(H) = 0$ then by (1), H is 2nd countable, a contradiction. If $\mu(H) > 0$ then there is an uncountable set $N \subseteq H$ with $\mu(N) = 0$. Then N is uncountable and 2nd countable, contradicting the fact that all 2nd countable subspaces of the split interval are countable. $\bigstar_{2.8}$

Now we comment on how we assure that any uncountable nice semi-biorthogonal system in $C(K)$ is ω-determined, i.e. any uncountable semi-bidiscrete sequence in K forms an ω-determined family of pairs of points. For this we make one further requirement:

R8. If $\alpha, \beta \in [\omega, \omega_1)$ with $\alpha < \beta$ then $s_\beta \upharpoonright \alpha \neq s_\alpha$.

Claim 2.9. Requirements **R1**-**R8** guarantee that any uncountable semi-bidiscrete sequence in K is ω-determined.

Proof of the Claim. Suppose that $\langle (x_\alpha^0, x_\alpha^1) : \alpha < \omega_1 \rangle$ forms an uncountable semi-bidiscrete sequence in K that is not ω-determined. By the definition of a semi-bidiscrete sequence, the (x_α^0, x_α^1)'s are distinct pairs of distinct points. Therefore there must be $s \in 2^\omega$ such that $A = \{\alpha : x_\alpha^0 \upharpoonright \omega = x_\alpha^1 \upharpoonright \omega = s\}$ is uncountable. We have at least one $l < 2$ such that $\{x_\alpha^l : \alpha \in A\}$ is uncountable, so assume, without loss of generality, that this is true for $l = 0$.

Let α, β, γ be three distinct members of A. Then by Observation 2.4 we have

$$x_\alpha^0 \upharpoonright \Delta(x_\alpha^0, x_\beta^0) = x_\beta^0 \upharpoonright \Delta(x_\alpha^0, x_\beta^0) = s_{\Delta(x_\alpha^0, x_\beta^0)}$$

and similarly

$$x_\alpha^0 \upharpoonright \Delta(x_\alpha^0, x_\gamma^0) = x_\gamma^0 \upharpoonright \Delta(x_\alpha^0, x_\gamma^0) = s_{\Delta(x_\alpha^0, x_\gamma^0)}.$$

By **R8** we conclude that $\Delta(x_\alpha^0, x_\beta^0)$ is the same for all $\beta \in A \setminus \{\alpha\}$ and we denote this common value by Δ_α. Thus for $\beta \in A \setminus \{\alpha\}$ we have $x_\beta^0 \upharpoonright \Delta_\alpha = s_{\Delta_\alpha}$, but applying the same reasoning to β we obtain $x_\alpha^0 \upharpoonright \Delta_\beta = s_{\Delta_\beta}$ and hence by **R8** again we have $\Delta_\alpha = \Delta_\beta$. Let δ^* denote the common value of Δ_α for $\alpha \in A$.

Again, taking distinct $\alpha, \beta, \gamma \in A$ we have $x_\alpha^0 \upharpoonright \delta^* = x_\beta^0 \upharpoonright \delta^* = x_\gamma^0 \upharpoonright \delta^*$ and that $x_\alpha^0(\delta^*), x_\beta^0(\delta^*)$ and $x_\gamma^0(\delta^*)$ are pairwise distinct. This is, however, impossible as the latter have values in $\{0, 1\}$. $\bigstar_{2.9}$

Finally we show that the space K is a 2-to-1 continuous preimage of a compact metric space. We simply define $\varphi : K \to 2^\omega$ as $\varphi(x) = x \upharpoonright \omega$. This is clearly continuous. To show that it is 2-to-1 we first prove the following:

Claim 2.10. In the space K above, for any $\alpha \neq \beta$ we have $s_\alpha \restriction \omega \neq s_\beta \restriction \omega$.

Proof of the Claim. Otherwise suppose that $\alpha < \beta$ and yet $s_\alpha \restriction \omega = s_\beta \restriction \omega$. By **R8** we have $s_\alpha \not\subseteq s_\beta$, so $\omega \leq \delta = \Delta(s_\alpha, s_\beta) < \beta$. By Observation 2.4 applied to any $x \supseteq s_\alpha$ and $y \supseteq s_\beta$ from K, we have $s_\alpha \restriction \delta = x \restriction \delta = y \restriction \delta = s_\beta \restriction \delta = s_\delta$. But this would imply $s_\delta \subseteq s_\beta$, contradicting **R8**. $\bigstar_{2.10}$

Now suppose that φ is not 2-to-1, that is there are three elements $x, y, z \in K$ such that $x \restriction \omega = y \restriction \omega = z \restriction \omega$. Let $\alpha = \delta(x, y)$ and $\beta = \delta(x, z)$, so $\alpha, \beta \geq \omega$. By Observation 2.4 we have $x \restriction \alpha = y \restriction \alpha = s_\alpha$, $x \restriction \beta = z \restriction \beta = s_\beta$, so by requirement **R8** we conclude $\alpha = \beta$. Note that then $y(\alpha) = z(\alpha)$ and so $\delta = \Delta(y, z) > \alpha$ and $y \restriction \delta = s_\delta \supseteq s_\beta$, in contradiction with **R8**. Therefore φ is really 2-to-1.

2.0.2 Meeting the requirements

Now we show how to meet all these requirements. It suffices to show what to do at any successor stage $\alpha + 1$ for $\alpha \in [\omega, \omega_1)$, assuming all the requirements have been met at previous stages.

First we choose s_α. By **R5** for any $\gamma < \alpha$ we have $\mu_\gamma(\{s_\gamma\}) = 0$ and $\mu_\alpha((\pi_\gamma^\alpha)^{-1}(s_\gamma)) = 0$. Hence the set of points $s \in K_\alpha$ for which $s \restriction \gamma = s_\gamma$ for some $\gamma < \alpha$ has measure 0, so we simply choose s_α outside of $\bigcup_{\gamma < \alpha} (\pi_\gamma^\alpha)^{-1}(s_\gamma)$, as well as outside of $\bigcup_{\gamma < \alpha} (\pi_\gamma^\alpha)^{-1}(N_\gamma)$ (to meet **R6**), which is possible as the μ_α measure of the latter set is also 0.

Now we shall use an idea from [4]. We fix a strictly decreasing sequence $\langle V_n : n \in \omega \rangle$ of clopen sets in K_α such that $V_0 = K_\alpha$ and $\bigcap_{n < \omega} V_n = \{s_\alpha\}$. We shall choose a function $f : \omega \to \omega$ such that letting

$$A_\alpha = \bigcup_{n < \omega} (V_{f(2n)} \setminus V_{f(2n+1)}) \cup \{s_\alpha\}$$

and

$$B_\alpha = \bigcup_{n < \omega} (V_{f(2n+1)} \setminus V_{f(2n)}) \cup \{s_\alpha\}$$

will meet all the requirements. Once we have chosen A_α and B_α, we let

$$K_{\alpha+1} = A_\alpha \times \{0\} \cup B_\alpha \times \{1\}.$$

For a basic clopen set $[s] = \{g \in K_{\alpha+1} : g \supseteq s\}$, where s is a finite partial function from $\alpha + 1$ to 2 and $\alpha \in \text{dom}(s)$, we let $\mu_{\alpha+1}([s]) = 1/2 \cdot \mu_\alpha([s \restriction \alpha])$. We prove below that this extends uniquely to a Baire measure on $K_{\alpha+1}$.

The following is basically the same (in fact simpler) argument which appears in [4]. We state and prove it here for the convenience of the reader.

Claim 2.11. The above choices of A_α, B_α, and $\mu_{\alpha+1}$, with the choice of any function f which is increasing fast enough, will satisfy all the requirements **R1**-**R8**.

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Proof of the Claim. Requirements **R1**-**R4** are clearly met with any choice of f. To see that **R5** is met, let us first prove that $\mu_{\alpha+1}$ as defined above indeed extends uniquely to a Baire measure on $K_{\alpha+1}$. We have already defined $\mu_{\alpha+1}([s])$ for s satisfying $\alpha \in \text{dom}(s)$. If $\alpha \notin \text{dom}(s)$ then we let $\mu_{\alpha+1}([s]) = \mu_\alpha(\pi_\alpha^{\alpha+1}[s])$. It is easily seen that this is a finitely additive measure on the basic clopen sets, which then extends uniquely to a Baire measure on $K_{\alpha+1}$. It is also clear that this extension satisfies **R5**.

Requirements **R6** and **R8** are met by the choice of s_α, so it remains to see that we can meet **R7**. For each $\gamma \in [\omega, \alpha]$, if $s_\alpha \in (\pi_\gamma^\alpha)^{-1}(Q_\gamma)$, fix an ω-sequence $\bar{t}_\gamma$ of distinct points in $(\pi_\gamma^\alpha)^{-1}(Q_\gamma)$ converging to s_α. Suppose that $\bar{t}_\gamma$ is defined and that both $A_\alpha \setminus B_\alpha$ and $B_\alpha \setminus A_\alpha$ contain infinitely many points from $\bar{t}_\gamma$. Then we claim that $A_\alpha \cap (\pi_\gamma^\alpha)^{-1}(Q_\gamma)$ and $B_\alpha \cap (\pi_\gamma^\alpha)^{-1}(Q_\gamma)$ are complementary regular closed subsets of $(\pi_\gamma^\alpha)^{-1}(Q_\gamma)$. Note that we have already observed that Q_γ does not have isolated points, so neither does $(\pi_\gamma^\alpha)^{-1}(Q_\gamma)$. Hence, since $\{s_\alpha\} \supseteq A_\alpha \cap (\pi_\gamma^\alpha)^{-1}(Q_\gamma) \cap B_\alpha \cap (\pi_\gamma^\alpha)^{-1}(Q_\gamma)$, we may conclude that this intersection is nowhere dense in both $A_\alpha \cap (\pi_\gamma^\alpha)^{-1}(Q_\gamma)$ and $B_\alpha \cap (\pi_\gamma^\alpha)^{-1}(Q_\gamma)$. Finally, $A_\alpha \cap (\pi_\gamma^\alpha)^{-1}(Q_\gamma)$ and $B_\alpha \cap (\pi_\gamma^\alpha)^{-1}(Q_\gamma)$ are regular closed because we have assured that s_α is in the closure of both.

Therefore we need to choose f so that for every relevant γ, both $A_\alpha \setminus B_\alpha$ and $B_\alpha \setminus A_\alpha$ contain infinitely many points of $\bar{t}_\gamma$. Enumerate all the relevant sequences $\bar{t}_\gamma$ as $\{\bar{z}^k\}_{k<\omega}$. Our aim will be achieved by choosing f in such a way that, for every n, both sets $V_{f(2n)} \setminus V_{f(2n+1)}$ and $V_{f(2n+1)} \setminus V_{f(2n+2)}$ contain a point of each $\bar{z}^k$ for $k \leq n$. ★$_{2.11}$

This finishes the proof of the theorem. ★$_{2.1}$

3 Bidiscrete systems

The main result of this section is Theorem 3.5 below. In the course of proving Theorem 10 in §7 of [15], Todorčević actually proved that if K is not hereditarily separable then it has an uncountable bidiscrete system. Thus his proof yields Theorem 3.5 for $d(K) = \aleph_1$ and the same argument can be easily extended to a full proof of 3.5.

Let us first state some general observations about bidiscrete systems.

Observation 3.1. Suppose that K is a compact Hausdorff space and $H \subseteq K$ is closed, while $\{(x_\alpha^0, x_\alpha^1) : \alpha < \kappa\}$ is a bidiscrete system in H, as exemplified by functions f_α $(\alpha < \kappa)$. Then there are functions g_α $(\alpha < \kappa)$ in $C(K)$ such that $f_\alpha \subseteq g_\alpha$ and g_α $(\alpha < \kappa)$ exemplify that $\{(x_\alpha^0, x_\alpha^1) : \alpha < \kappa\}$ is a bidiscrete system in K.

Proof. Since H is closed we can, by Tietze's Extension Theorem, extend each f_α continuously to a function g_α on K. The conclusion follows from the definition of a bidiscrete system. ★$_{3.1}$

Claim 3.2. Suppose that K is a compact space and $F_i \subseteq G_i \subseteq K$ for $i \in I$ are such that the G_i's are disjoint open, the F_i's are closed and in each F_i we have a bidiscrete system S_i. Then $\bigcup_{i \in I} S_i$ is a bidiscrete system in K.

Proof. For $i \in I$ let the bidiscreteness of S_i be witnessed by $\{g^i_\alpha : \alpha < \kappa_i\} \subseteq C(F_i)$. We can, as in Observation 3.1, extend each g^i_α to $h^i_\alpha \in C(K)$ which exemplify that S_i is a bidiscrete system in K.

Now we would like to put all these bidscrete systems together, for which we need to find appropriate witnessing functions. For any $i \in I$ we can apply Urysohn's Lemma to find functions $f_i \in C(K)$ such that f_i is 1 on F_i and 0 on the complement of G_i. Let us then put, for any α and i, $f^i_\alpha = g^i_\alpha \cdot f_i$. Now, it is easy to verify that the functions $\{f^i_\alpha : \alpha < \kappa_i, i \in I\}$ witness that $\bigcup_{i \in I} S_i$ is a bidiscrete system in K. ★$_{3.2}$

Clearly, Observation 3.1 is the special case of Claim 3.2 when I is a singleton and $G_i = K$.

Claim 3.3. If the compact space K has a discrete subspace of size $\kappa \geq \omega$ then it has a bidiscrete system of size κ, as well.

Proof. Suppose that $D = \{x_\alpha : \alpha < \kappa\}$ (enumerated in a one-to-one manner) is discrete in K with open sets U_α witnessing this, i.e. $D \cap U_\alpha = \{x_\alpha\}$ for all $\alpha < \kappa$. For any $\alpha < \kappa$ we may fix a function $f_\alpha \in C(K)$ such that $f_\alpha(x_{2\alpha+1}) = 1$ and $f_\alpha(x) = 0$ for all $x \notin U_{2\alpha+1}$. Obviously, then $\{f_\alpha : \alpha < \kappa\}$ exemplifies that $\{(x_{2\alpha}, x_{2\alpha+1}) : \alpha < \kappa\}$ is a bidiscrete system in K.

The converse of Claim 3.3 is false, however the following is true.

Claim 3.4. Suppose that $B = \{(x^0_\alpha, x^1_\alpha) : \alpha < \kappa\}$ is a bidiscrete system in K. Then B is a discrete subspace of K^2.

Proof. Assume that the functions $\{f_\alpha : \alpha < \kappa\} \subseteq C(K)$ exemplify the bidiscreteness of B. Then $O_\alpha = f^{-1}_\alpha((-\infty, 1/2)) \times f^{-1}_\alpha((1/2, \infty))$ is an open set in K^2 containing (x^0_α, x^1_α). Also, if $\beta \neq \alpha$ then $(x^0_\beta, x^1_\beta) \notin O_\alpha$, hence B is a discrete subspace of K^2. ★$_{3.4}$

Now we turn to formulating and proving the main result of this section.

Theorem 3.5. If K is an infinite compact Hausdorff space then K has a bidiscrete system of size $d(K)$. If K is moreover 0-dimensional then there is a very nice bidiscrete system in K of size $d(K)$.

Proof. The proofs of the two parts of the theorem are the same, except that in the case of a 0-dimensional space every time that we take functions witnessing bidiscreteness, we need to observe that these functions can be assumed to take values only in $\{0, 1\}$. We leave it to the reader to check that this is indeed the case.

The case $d(K) = \aleph_0$ is very easy, as it is well known that every infinite Hausdorff space has an infinite discrete subspace and so we can apply Claim 3.3. So, from now on we assume that $d(K) > \aleph_0$.

Recall that a Hausdorff space (Y, σ) is said to be *minimal Hausdorff* provided that there does not exist another Hausdorff topology ρ on Y such that $\rho \subsetneq \sigma$, i.e. ρ is strictly coarser than σ. The following fact is well known and easy to prove, and it will provide a key part of our argument:

Fact 3.6. Any compact Hausdorff space is minimal Hausdorff.

Lemma 3.7. Suppose that X is a compact Hausdorff space with $d(X) \geq \kappa > \aleph_0$ in which every non-empty open (equivalently: regular closed) subspace has weight $\geq \kappa$. Then X has a bidiscrete system of size κ.

Proof of the Lemma. We shall choose $x_\alpha^0, x_\alpha^1, f_\alpha$ by induction on $\alpha < \kappa$ so that the pairs (x_α^0, x_α^1) form a bidiscrete system, as exemplified by the functions f_α. Suppose that $x_\beta^0, x_\beta^1, f_\beta$ have been chosen for $\beta < \alpha < \kappa$.

Let C_α be the closure of the set $\{x_\beta^0, x_\beta^1 : \beta < \alpha\}$. Therefore $d(C_\alpha) < \kappa$ and, in particular, $C_\alpha \neq X$. Let $F_\alpha \subseteq X \setminus C_\alpha$ be non-empty regular closed, hence $w(F_\alpha) \geq \kappa$.

Let τ_α be the topology on F_α generated by the family

$$\mathcal{F}_\alpha = \{f_\beta^{-1}(-\infty, q) \cap F_\alpha, \, f_\beta^{-1}(q, \infty) \cap F_\alpha : \beta < \alpha, \, q \in \mathbf{Q}\},$$

where $\mathbf{Q}$ denotes the set of rational numbers. Then $|\mathcal{F}_\alpha| < \kappa$ (as $\kappa > \aleph_0$), hence the weight of τ_α is less than κ, consequently τ_α is strictly coarser than the subspace topology on F_α. Fact 3.6 implies that τ_α is not a Hausdorff topology on F_α, hence we can find two distinct points $x_\alpha^0, x_\alpha^1 \in F_\alpha$ which are not T_2-separated by any two disjoint sets in τ_α and, in particular, in $\mathcal{F}_\alpha$. This clearly implies that $f_\beta(x_\alpha^0) = f_\beta(x_\alpha^1)$ for all $\beta < \alpha$.

Now we use the complete regularity of X to find $f_\alpha \in C(X)$ such that f_α is identically 0 on the closed set $C_\alpha \cup \{x_\alpha^0\}$ and $f_\alpha(x_\alpha^1) = 1$. It is straight-forward to check that $\{f_\alpha : \alpha < \kappa\}$ indeed witnesses the bidiscreteness of $\{(x_\alpha^0, x_\alpha^1) : \alpha < \kappa\}$. ★$_{3.7}$

Let us now continue the proof of the theorem. We let κ stand for $d(K)$ and let

$$\mathcal{P} = \{\emptyset \neq O \subseteq K : O \text{ open such that } [\emptyset \neq U \text{ open} \subseteq O \implies d(U) = d(O)]\}.$$

We claim that $\mathcal{P}$ is a π-base for K, i.e. that every non-empty open set includes an element of $\mathcal{P}$. Indeed, suppose this is not case, as witnessed by a non-empty open set U_0. Then $U_0 \notin \mathcal{P}$, so there is a non-empty open set $\emptyset \neq U_1 \subseteq U_0$ with $d(U_1) < d(U_0)$ (the case $d(U_1) < d(U_0)$ cannot occur). Then U_1 itself is not a member of $\mathcal{P}$ and therefore we can find a non-empty open set $\emptyset \neq U_2 \subseteq U_1$ with $d(U_2) < d(U_1)$, etc. In this way we would obtain an infinite decreasing sequence of cardinals, a contradiction.

Let now $\mathcal{O}$ be a maximal disjoint family of members of $\mathcal{P}$. Since $\mathcal{P}$ is a π-base for K the union of $\mathcal{O}$ is clearly dense in K. This implies that if we fix any dense subset D_O of O for all $O \in \mathcal{O}$ then $\bigcup\{D_O : O \in \mathcal{O}\}$ is dense in K, as well. This, in turn, implies that $\sum\{d(O) : O \in \mathcal{O}\} \geq d(K) = \kappa$.

If $|\mathcal{O}| = \kappa$ then we can select a discrete subspace of K of size κ by choosing a point in each $O \in \mathcal{O}$, so the conclusion of our theorem follows by Corollary 3.3.

So now we may assume that $|\mathcal{O}| < \kappa$. In this case, since $\kappa > \aleph_0$, letting $\mathcal{O}' = \{O \in \mathcal{O} : d(O) > \aleph_0\}$, we still have $\sum\{d(O) : O \in \mathcal{O}'\} \geq \kappa$. Next, for each $O \in \mathcal{O}'$ we choose a non-empty open set G_O such that its closure $\overline{G}_O \subseteq O$. Then we have, by the definition of $\mathcal{P}$, that $d(\overline{G}_O) = d(G_O) = d(O)$. By the same token, every non-empty

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open subspace of the compact space $\overline{G}_O$ has density $d(O)$, and hence weight $\geq d(O)$. Therefore we may apply Lemma 3.7 to produce a bidiscrete system S_O of size $d(O)$ in $\overline{G}_O$. But then Claim 3.2 enables us to put these systems together to obtain the bidcrete system $S = \bigcup\{S_O : O \in \mathcal{O}'\}$ in K of size $\sum\{d(O) : O \in \mathcal{O}'\} \geq \kappa$. $\bigstar_{3.5}$

It is immediate from Theorem 3.5 and Observation 3.1 that if C is a closed subspace of the compactum K with $d(C) = \kappa$ then K has a bidiscrete system of size κ. We recall that the hereditary density $\mathrm{hd}(X)$ of a space X is defined as the supremum of the densities of all subspaces of X.

Fact 3.8. For any compact Hausdorff space K, $\mathrm{hd}(K) = \sup\{d(C) : C \text{ closed} \subseteq K\}$.

From this fact and what we said above we immediately obtain the following corollary of Theorem 3.5.

Corollary 3.9. If K is a compact Hausdorff space with $\mathrm{hd}(K) \geq \lambda^+$ for some $\lambda \geq \omega$, then K has a bidiscrete system of size λ^+.

We finish by listing some open questions.

Question 3.10. (1) Does every compact space K admit a bidiscrete system of size $\mathrm{hd}(K)$?
(2) Define
$$\mathrm{bd}(K) = \sup\{|S| \ : \ S \text{ is a bidiscrete system in } K\}.$$
Is there always a bidiscrete system in K of size $\mathrm{bd}(K)$?
(3) Suppose that K is a 0-dimensional compact space which has a bidiscrete system of size κ. Does then K also have a very nice bidiscrete system of size κ (i.e. such that the witnessing functions take values only in $\{0,1\}$)? Is it true that any bidiscrete system in a 0-dimensional compact space is very nice?
(4) (This is Problem 4 from [7]): Is there a ZFC example of a compact space K that has no discrete subspace of size $d(K)$?
(5) If the square K^2 of a compact space K contains a discrete subspace of size κ, does then K admit a bidiscrete system of size κ (or does at least $C(K)$ have a biorthogonal system of size κ)? This question is of especial interest for $\kappa = \omega_1$.

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