A FIRST COUNTABLE, INITIALLY ω_1-COMPACT BUT NON-COMPACT SPACE

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Abstract. We force a first countable, normal, locally compact, initially ω_1-compact but non-compact space X of size ω_2. The one-point compactification of X is a non-first countable compactum without any (non-trivial) converging ω_1-sequence.

1. Introduction

A topological space is *initially κ-compact* if any open cover of size $\leq \kappa$ has a finite subcover or, equivalently, any subset of size $\leq \kappa$ has a complete accumulation point. Under CH an initially ω_1-compact T_3 space of countable tightness is compact, this was observed by E. van Douwen and, independently, A. Dow [4]. They both raised the natural question whether this is actually provable in ZFC. In [2] D. Fremlin and P. Nyikos proved this implication under PFA and in [5] this was established in numerous other models as well.

However, in [9] M. Rabus gave a negative answer to the van Douwen–Dow question. He generalized the method of J. Baumgartner and S. Shelah, which had been used in [3] to force a thin very tall superatomic Boolean algebra, and constructed by forcing a Boolean algebra B such that the Stone space $St(B)$ minus a suitable point is a counterexample of size ω_2 to the van Douwen–Dow question. In both forcings the use of a so-called Δ-function plays an essential role.

In [6] we directly forced a topology τ_f on ω_2 that yields a locally compact and normal counterexample from any Δ-function f, provided that CH holds in the ground model. Moreover, it was also shown in [6] that, with some extra work and extra set-theoretic assumptions, the

2000 *Mathematics Subject Classification.* 54A25,03E35.

Key words and phrases. initially ω_1-compact, first countable, right separated space, forcing, Δ-function, normal, locally compact.

The first and the third authors were supported by the Hungarian National Foundation for Scientific Research grants no. K 68262 and K 61600.

The second author was partially supported by Polish Ministry of Science and Higher Education research grant N N201 386234 .

counterexample can be made not just countably tight but even Frèchet-Urysohn. In this paper we get a further improvement by forcing a first countable, normal, locally compact, initially ω_1-compact but non-compact space X.

Actually, Alan Dow conjectured that applying the method of [8] (that "turns" a compact space into a first countable one) to the space of Rabus in [9] yields an ω_1-compact but non-compact first countable space. How one can carry out such a construction was outlined by the second author in the preprint [7]. However, [7] only sketches some arguments as the language adopted there, which follows that of [9], does not seem to allow direct combinatorial control over the space which is forced. This explains why the second author hesitated to publish [7].

One missing element of [7] was a language similar to that of [6] which allows working with the points of the forced space in a direct combinatorial way. In this paper we combine the approach of [6] with the ideas of [7] to obtain directly an ω_1-compact but non-compact first countable space. Consequently, our proofs follow much more closely the arguments of [6] than those of [9] or their analogues in [7].

As before, we again use a Δ-function to make our forcing CCC but we need both CH and a Δ-function with some extra properties to obtain first countability.

It is immediate from the countable compactness of X that its one-point compactification X^* is not first countable. In fact, one can show that the character of the point at infinity $*$ in X^* is ω_2. As X is initially ω_1-compact, this means that every (transfinite) sequence converging from X to $*$ must be of type cofinal with ω_2. Since X is first countable, this trivially implies that there is no non-trivial converging sequence of type ω_1 in X^*. In other words: the convergence spectrum of the compactum X^* omits ω_1. As far as we know, this is the first and only (consistent) example of this sort.

2. A general construction

First we introduce a general method to construct locally compact, zero-dimensional spaces. This generalizes the method for the construction of locally compact right-separated (i.e. scattered) spaces that was described in [6].

Definition 2.1. Let ϑ be an ordinal, X be a 0-dimensional space, and fix a clopen subbase (i.e. a subbase consisting of clopen sets) $\mathcal{S}$ of X such that $X \in \mathcal{S}$ and

$$S \in \mathcal{S} \setminus \{X\} \text{ implies } (X \setminus S) \in \mathcal{S}. \tag{1}$$

Let $\mathrm{K} : \vartheta \times \mathcal{S} \longrightarrow \mathcal{P}(\vartheta)$ be a function satisfying

$$\mathrm{K}(\delta, S) \subset \mathrm{K}(\delta, X) \subset \delta, \tag{2}$$

for any $\delta \in \vartheta$ and $S \in \mathcal{S}$, and set

$$U(\delta, S) = (\{\delta\} \times S) \cup (\mathrm{K}(\delta, S) \times X). \tag{3}$$

We shall denote by τ_{K} the topology on $\vartheta \times X$ generated by the family

$$\mathcal{U}_{\mathrm{K}} = \{U(\delta, S)\,,\, (\vartheta \times X) \setminus U(\delta, S) : \delta < \vartheta, S \in \mathcal{S}\} \tag{4}$$

as a subbase. Write $X_{\mathrm{K}} = \langle \vartheta \times X, \tau_{\mathrm{K}} \rangle$.

If a is a set of ordinals and s is an arbitrary set we write

$$\big[a\big]^2 \otimes s = \{\langle \zeta, \xi, \sigma\rangle : \zeta, \xi \in a, \zeta < \xi, \sigma \in s\}. \tag{5}$$

Theorem 2.2. *(1) Assume that ϑ, X, $\mathcal{S}$ and K are as in definition 2.1 above. Then the space $X_K = \langle \vartheta \times X, \tau_K \rangle$ is 0-dimensional and Hausdorff and the subspace $\{\alpha\} \times X$ is homeomorphic to X for each $\alpha < \vartheta$.*
(2) Assume, in addition, that X is compact and
(K1) if $S \cap S' = \emptyset$ then $\mathrm{K}(\delta, S) \cap \mathrm{K}(\delta, S') = \emptyset$,
(K2) if $X = \cup \mathcal{S}'$ for some $\mathcal{S}' \in \big[\mathcal{S}\big]^{<\omega}$ then

$$\mathrm{K}(\delta, X) = \cup\{\mathrm{K}(\delta, S) : S \in \mathcal{S}'\},$$

(K3) there is a function i with $\mathrm{dom}(i) = \big[\vartheta\big]^2 \otimes \mathcal{S}$ such that for each $\langle \delta, \delta', S\rangle \in \big[\vartheta\big]^2 \otimes \mathcal{S}$ we have
(i1) $i(\delta, \delta', S) \in \big[\delta\big]^{<\omega}$ and
(i2) $\mathrm{K}(\delta, X) * \mathrm{K}(\delta', S) \subset \cup\{\mathrm{K}(\nu, X) : \nu \in i(\delta, \delta', S)\}$,

where

$$\mathrm{K}(\delta, X) * \mathrm{K}(\delta', S) = \begin{cases} \mathrm{K}(\delta, X) \cap \mathrm{K}(\delta', S) & \text{if } \delta \notin \mathrm{K}(\delta', S), \\ \mathrm{K}(\delta, X) \setminus \mathrm{K}(\delta', S) & \text{if } \delta \in \mathrm{K}(\delta', S). \end{cases} \tag{6}$$

Then all members of $\mathcal{U}_{\mathrm{K}}$ are compact, hence X_{K} is locally compact.

Proof. (1). X_{K} is 0-dimensional because it is generated by a clopen subbase. To see that X_{K} is Hausdorff, assume that $\langle \delta, x\rangle \neq \langle \delta', x'\rangle \in \vartheta \times X$, $\delta \leq \delta'$. If $\delta < \delta'$ then $U(\delta, X) \subset (\delta + 1) \times X$ separates these points. If $\delta = \delta'$ then there is $S \in \mathcal{S}$ with $x \in S$ and $x' \notin S$, but then $U(\delta, S)$ separates $\langle \delta, x\rangle$ and $\langle \delta, x'\rangle$. The trivial proof that $\{\alpha\} \times X$ is homeomorphic to X is left to the reader.

(2). We write $U(\delta) = U(\delta, X)$ for $\delta < \vartheta$ and $U[F] = \cup\{U(\alpha) : \alpha \in F\}$ for $F \subset \vartheta$. We shall prove, by induction on δ, that $U(\delta)$ is compact; this clearly implies that every $U(\delta, S)$ is also compact. We note that

(K1) and (K2) together imply $U(\delta, X\backslash S) = U(\delta)\backslash U(\delta, S)$ whenever $S \in \mathcal{S}\backslash\{X\}$.

Assume now that $U(\alpha)$ is compact for each $\alpha < \delta$. To see that then $U(\delta)$ is also compact, by Alexander's subbase lemma, it suffices to show that any cover of $U(\delta)$ by members of $\mathcal{U}_{\mathrm{K}}$ has a finite subcover.

So let

$$U(\delta) \subset \bigcup\{U_i : i \in I\} \cup \bigcup\{U_j : j \in J\},$$

where $U_i = U(\delta_i, S_i)$ for $i \in I$ and $U_j = (\vartheta \times X) \setminus U(\delta_j, S_j)$ for $j \in J$.

Case 1: *$\delta_j < \delta$ for some $j \in J$.*

Then we have

$$U(\delta) \setminus U_j = U(\delta) \setminus ((\vartheta \times X) \setminus U(\delta_j, S_j)) \subset U(\delta_j, S_j) \subset U(\delta_j),$$

hence $U(\delta) \setminus U_j$ is compact because $U(\delta_j)$ is by the inductive assumption.

Case 2: *$(\{\delta\} \times X) \cap U_j \neq \emptyset$ for some $j \in J$ with $\delta_j > \delta$.*

Then $(\{\delta\} \times X) \subset U_j$ and $\delta \notin \mathrm{K}(\delta_j, S_j)$, so by (K3)

$$\mathrm{K}(\delta, X) \cap \mathrm{K}(\delta_j, S_j) = \mathrm{K}(\delta, X) * \mathrm{K}(\delta_j, S_j) \subset \mathrm{K}[i(\delta, \delta_j, S_j)].$$

Consequently, we have

$$U(\delta) \setminus U_j = U(\delta) \cap U(\delta_j, S_j) \subset U[i(\delta, \delta_j, S_j)]$$

and $U[i(\delta, \delta_j, S_j)]$ is compact by the inductive assumption.

Case 3: *$(\{\delta\} \times X) \cap U_i \neq \emptyset$ for some $i \in I$ with $\delta_i \neq \delta$.*

In this case $\delta < \delta_i$ and $\delta \in \mathrm{K}(\delta_i, S_i)$, hence by (K3)

$$\mathrm{K}(\delta, X) \setminus \mathrm{K}(\delta_i, S_i) = \mathrm{K}(\delta, X) * \mathrm{K}(\delta_i, S_i) \subset \mathrm{K}[i(\delta, \delta_i, S_i)].$$

Thus

$$U(\delta) \setminus U_i = U(\delta) \setminus U(\delta_i, S_i) \subset U[i(\delta, \delta_i, S_i)]$$

and $U[i(\delta, \delta_i, S_i)]$ is compact by the inductive assumption.

Now, in all the three cases it is clear that $\{U_k : k \in I \cup J\}$ contains a finite subcover of $U(\delta)$.

Case 4: *If $(\{\delta\} \times X) \cap U_k \neq \emptyset$ then $\delta_k = \delta$ for each $k \in I \cup J$.*

Since X is compact there are finite sets $I' \in \left[I\right]^{<\omega}$ and $J' \in \left[J\right]^{<\omega}$ such that $\delta_k = \delta$ for each $k \in I' \cup J'$, moreover

$$X = \cup\{S_i : i \in I'\} \cup \cup\{X \setminus S_j : j \in J'\},$$

and then, by (K2),

$$\mathrm{K}(\delta, X) = \cup\{\mathrm{K}(\delta, S_i) : i \in I'\} \cup \cup\{\mathrm{K}(\delta, X \setminus S_j) : j \in J'\}.$$

But these equalities clearly imply

$$U(\delta) \subset \cup\{U_i : i \in I'\} \cup \cup\{U_j : j \in J'\}.$$

□

To describe a natural base of the space X_{K}, we fix some more notation. For $\delta < \vartheta$, $\mathcal{S}' \in \left[\mathcal{S}\right]^{<\omega}$ and $F \in \left[\delta\right]^{<\omega}$ we shall write

$$B(\delta, \mathcal{S}', F) = \cap\{U(\delta, S) : S \in \mathcal{S}'\} \setminus U[F].$$

For a point $x \in X$ we set $\mathcal{S}(x) = \{S \in \mathcal{S} : x \in S\}$, moreover we put

$$\mathcal{B}(\delta, x) = \{B(\delta, \mathcal{S}',\ F) : \mathcal{S}' \in \left[\mathcal{S}(x)\right]^{<\omega}, F \in \left[\delta\right]^{<\omega}\}. \tag{7}$$

Lemma 2.3. *Assume that ϑ, X, $\mathcal{S}$ and* K *are as in part (2) of the previous theorem 2.2. Then for each $\delta < \vartheta$ and $x \in X$ the family $\mathcal{B}(\delta, x)$ forms a neighbourhood base of the point $\langle \delta, x\rangle$ in X_{K}.*

Proof. Since $\mathcal{B}(\delta, x)$ consists of compact neighbourhoods of the point $\langle \delta, x\rangle$ and is closed under finite intersections, it suffices to show that $\cap\mathcal{B}(\delta, x) = \{\langle \delta, x\rangle\}$. To see this, consider any $\langle \delta', x'\rangle \in \vartheta \times X$ distinct from $\langle \delta, x\rangle$.

If $\delta' > \delta$ then $\langle \delta' x'\rangle \notin U(\delta) = B(\delta, X, \emptyset) \in \mathcal{B}(\delta, x)$. If $\delta' < \delta$ then $\langle \delta', x'\rangle \notin U(\delta)\setminus U(\delta') = B(\delta, X, \{\delta'\}) \in \mathcal{B}(\delta, x)$. Finally, if $\delta' = \delta$ then pick $S \in \mathcal{S}$ with $x \in S$ and $x' \notin S$. Then

$$\langle \delta', x'\rangle \notin U(\delta, S) = B(\delta, S, \emptyset) \in \mathcal{B}(\delta, x).$$

□

As we already mentioned above, our construction of the locally compact spaces X_{K} generalizes the construction of locally compact right-separated spaces given in [6]. In fact, the latter is the special case when X is a singleton space (and $\mathcal{S}$ is the only possible subbase $\{X\}$). We may actually say that in the space X_{K} the compact open sets $U(\delta)$ right separate the copies $\{\delta\} \times X$ of X rather than the points.

Actually, a locally compact, right separated, and initially ω_1-compact but non-compact space cannot be first countable. (Indeed, this is because the scattered height of such a space must exceed ω_1.) So the transition to a more complicated procedure is necessary if we want to make our example first countable but keep it locally compact.

We now present a much more interesting example of our general construction, where X will be the Cantor set $\mathbb{C}$ and $\mathcal{S}$ will be a natural subbase of $\mathbb{C}$. For technical reasons, we put $\mathbb{C} = 2^{\mathbb{N}}$ instead of 2^{ω}, where $\mathbb{N} = \omega\setminus\{0\}$.

The clopen subbase $\mathcal{S}$ of $\mathbb{C}$ is the one that determines the product topology and is defined as follows. If $n > 0$ and $\varepsilon < 2$ then let $[n, \varepsilon] = \{f \in \mathbb{C} : f(n) = \varepsilon\}$. We then put

$$\mathcal{S} = \{[n, \varepsilon] : n > 0\,,\ \varepsilon < 2\} \cup \{\mathbb{C}\}.$$

Then $\mathcal{S}$ satisfies 2.1.(1), moreover if $\mathcal{S}' \subset \mathcal{S} \setminus \{\mathbb{C}\}$ covers $\mathbb{C} = 2^{\mathbb{N}}$ then there is $n \in \mathbb{N}$ such that both $[n, 0], [n, 1] \in \mathcal{S}'$.

In order to apply our general scheme, we still need to fix an ordinal ϑ, a function $\mathrm{K} : \vartheta \times \mathcal{S} \longrightarrow \mathcal{P}(\vartheta)$ satisfying 2.1.(2), and another function i with $\operatorname{dom}(i) = \left[\vartheta\right]^2 \otimes \mathcal{S}$ such that all the requirements of theorem 2.1 are satisfied. In our present particular case this may be achieved in a slightly different form that turns out to be simpler and more convenient for the purposes of our forthcoming forcing argument.

If h is a function and $a \subset \operatorname{dom}(h)$ we write $h[a] = \cup\{h(\xi) : \xi \in a\}$ (this piece of notation has been used before). If x and y are two non-empty sets of ordinals with $\sup x < \sup y$ then we let

$$x * y = \begin{cases} x \cap y & \text{if } \sup x \notin y, \\ x \setminus y & \text{if } \sup x \in y. \end{cases}$$

Note that this operation $*$ is not symmetric, on the contrary, if $x * y$ is defined then $y * x$ is not.

Definition 2.4. A pair of functions $H : \vartheta \times \omega \longrightarrow \mathcal{P}(\vartheta)$ and $i : \left[\vartheta\right]^2 \otimes \omega \longrightarrow \left[\vartheta\right]^{<\omega}$ are said to be ϑ-*suitable* if the following three conditions hold for all $\alpha, \beta \in \vartheta$ and $n \in \omega$:

(H1) $\alpha \in H(\alpha, n) \subset H(\alpha, 0) \subset \alpha + 1$,
(H2) $i(\alpha, \beta, n) \in \left[\alpha\right]^{<\omega}$,
(H3) if $\alpha < \beta$ then $H(\alpha, 0) * H(\beta, n) \subset H[i(\alpha, \beta, n)]$.

Concerning (H3) note that we have

$$\max H(\alpha, 0) = \alpha < \max H(\beta, n) = \beta,$$

hence $H(\alpha, 0) * H(\beta, n)$ is defined.

Given a ϑ-suitable pair (H, i) as above, let us define the functions

$$\mathrm{K} : \vartheta \times \mathcal{S} \longrightarrow \mathcal{P}(\vartheta) \ \text{ and } \ i' : \left[\vartheta\right]^2 \otimes \mathcal{S} \longrightarrow \left[\vartheta\right]^{<\omega}$$

as follows:

$$\mathrm{K}(\alpha, \mathbb{C}) = H(\alpha, 0) \cap \alpha, \tag{8}$$

$$\mathrm{K}(\alpha, [n, 1]) = H(\alpha, n) \cap \alpha, \tag{9}$$

$$\mathrm{K}(\alpha, [n, 0]) = H(\alpha, 0) \setminus H(\alpha, n), \tag{10}$$

$$i'(\alpha, \beta, \mathbb{C}) = i(\alpha, \beta, 0), \tag{11}$$

$$i'(\alpha, \beta, [n, \varepsilon]) = i(\alpha, \beta, 0) \cup i(\alpha, \beta, n). \tag{12}$$

It is straightforward to check then that K and i' satisfy all the requirements of theorem 2.2. Because of this, with some abuse of notation, we shall denote the topology τ_{K} also by τ_H and the space $\langle \vartheta \times \mathbb{C}, \tau_{\mathrm{K}} \rangle$ by X_H.

For our subbasic compact open sets we have

$$U(\alpha) = U(\alpha, \mathbb{C}) = H(\alpha, 0) \times \mathbb{C}, \tag{13}$$

and to simplify notation we write

$$U(\alpha, [n, \varepsilon]) = U(\alpha, n, \varepsilon). \tag{14}$$

Using this terminology, we may now formulate lemma 2.3 for this example in the following manner.

Lemma 2.5. *If (H, i) is an ϑ-suitable pair then for every $\langle \alpha, x \rangle \in \vartheta \times \mathbb{C}$ the compact open sets*

$$B(\alpha, x, n, F) = \bigcap \{U(\alpha, j, x(j)) \,:\, 1 \le j \le n, \,\} \setminus U[F]$$

with $n \in \mathbb{N}$ and $F \in [\alpha]^{<\omega}$ form a neighbourhood base of the point $\langle \alpha, x \rangle$ in the space X_H.

What we are set out to do now is to force an ω_2-suitable pair (H, i) such that the space X_H is as required. As mentioned, for this we need a special kind of Δ-function and this will be discussed in the next section.

3. Δ-functions

Definition 3.1. Let $f : \left[\omega_2\right]^2 \longrightarrow \left[\omega_2\right]^{\le\omega}$ be a function with $f(\{\alpha, \beta\}) \subset \alpha \cap \beta$ for $\{\alpha, \beta\} \in \left[\omega_2\right]^2$. Actually, in what follows, we shall simply write $f(\alpha, \beta)$ instead of $f(\{\alpha, \beta\})$.

We say that two finite subsets x and y of ω_2 are *very good for f* provided that for $\tau, \tau_1, \tau_2 \in x \cap y$, $\alpha \in x \setminus y$, $\beta \in y \setminus x$ and $\gamma \in (x \setminus y) \cup (y \setminus x)$ we always have

$\Delta 1)$ $\tau < \alpha, \beta \Longrightarrow \tau \in f(\alpha, \beta)$,
$\Delta 2)$ $\tau < \alpha \Longrightarrow f(\tau, \beta) \subset f(\alpha, \beta)$,
$\Delta 3)$ $\tau < \beta \Longrightarrow f(\tau, \alpha) \subset f(\beta, \alpha)$,
$\Delta 4)$ $\gamma, \tau_1 < \tau_2 \Longrightarrow f(\gamma, \tau_1) \subset f(\gamma, \tau_2)$.

$\Delta 5)$ $\tau_1 < \gamma < \tau_2 \Longrightarrow \tau_1 \in f(\gamma, \tau_2)$.
The sets x and y are said to be *good for* f iff $\Delta 1)$–$\Delta 3)$ hold.

We say that $f : [\omega_2]^2 \longrightarrow [\omega_2]^{\le\omega}$ with $f(\alpha,\beta) \subset \alpha\cap\beta$ is a *strong* Δ*-function*, or a Δ*-function*, respectively, if every uncountable family of finite subsets of ω_2 contains two sets x and y which are very good for f, or good for f, respectively.

We will prove in Lemma 3.3 that it is consistent with CH that there is a strong Δ-function.

In the proof of the countable compactness of our space we shall need the following simple consequence of [6, Lemma 1.2] that yields an additional property of Δ-functions provided that CH holds.

Lemma 3.2. *Assume that CH holds, f is a Δ-function, and $B \in [\omega_2]^\omega$. Then for any finite collection $\{T_i : i < m\} \subset [\omega_2]^{\omega_2}$ we may select a strictly increasing sequence $\langle \gamma_i : i < m\rangle$ with $\gamma_i \in T_i$ such that $B \subset f(\gamma_i, \gamma_j)$ whenever $i < j < m$.*

Proof. Fix a family $\{c_\alpha : \alpha < \omega_2\} \subset [\omega_2]^m$ such that $c_\alpha < c_\beta$ for $\alpha < \beta$, moreover $c_\alpha = \{\gamma_i^\alpha : i < m\}$ and ${\gamma_i}^\alpha \in T_i$ for all $\alpha < \omega_2$ and $i < m$. By [6, Lemma 1.2] there are m ordinals $\alpha_0 < \alpha_1 < \dots < \alpha_{m-1} < \omega_2$ such that

$$B \subset \bigcap\{f(\xi,\eta) : \xi \in c_{\alpha_i}, \eta \in c_{\alpha_j}, i < j < m\}.$$

Clearly, then $\gamma_i = {\gamma_i}^{\alpha_i}$ for $i < m$ are as required. □

Now, we have come to the main result of this section.

Lemma 3.3. *It is consistent with CH that there is a strong Δ-function.*

Proof of Lemma 3.3. There are several natural ways of constructing such a strong Δ-function f. One can do it by forcing, following and modifying a bit the construction given in [3]. One can use Velleman's simplified morasses (see [11]) and put

$$f(\alpha,\beta) = X \cap \alpha \cap \beta$$

where X is an element of minimal rank of the morass that contains both α and β.

In this paper we chose to follow Todorčevič's approach that uses his canonical coloring ρ: $[\omega_2]^2 \to \omega_1$ obtained from a $\square_{\omega_1}$-sequence (see [10, 7.3.2 and 7.4.8]). From this coloring ρ he defines f by

$$f(\alpha,\beta) = \{\xi < \alpha : \rho(\xi,\beta) \le \rho(\alpha,\beta)\}$$

and proves that this f is a Δ-function in our terminology of 3.1 (see [10, 7.4.9 and 7.4.10]). (We should warn the reader, however, that he calls this a D-function instead of a Δ-function in [10].)

He also establishes the following canonical inequalities for ρ (see [10, 7.3.7 and 7.3.8]):

$$(i) \qquad |\{\xi < \alpha : \ \rho(\xi, \alpha) \leq \nu\}| < \omega_1$$

$$(ii) \qquad \rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$$

$$(iii) \qquad \rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$$

for $\alpha < \beta < \gamma < \omega_2$ and $\nu < \omega_1$. We will now use these inequalities to prove that this f is even a strong Δ-function.

Let $\mathcal{A}$ be an uncountable family of finite subsets of ω_2. Note that it is enough to find an uncountable $\mathcal{A}' \subseteq \mathcal{A}$ such that Δ4) and Δ5) of 3.1 hold for every two elements of $\mathcal{A}'$, since then we may apply to $\mathcal{A}'$ the fact that f is a Δ-function to obtain two elements of $\mathcal{A}$ that are very good for f.

We may assume w.l.o.g. that $\mathcal{A}$ forms a Δ-system with root $\Delta \subseteq \omega_2$. Note that then the set

$$D = \{\xi \in \omega_2 : \exists \tau_1, \tau_2, \tau_3 \in \Delta, \quad \xi < \tau_1, \quad \rho(\xi, \tau_1) \leq \rho(\tau_2, \tau_3)\}$$

is countable by (i). Define $\mathcal{A}' \subseteq \mathcal{A}$ to be the set of all elements $a \in \mathcal{A}$ which satisfy $(a - \Delta) \cap D = \emptyset$. The countability of D implies that $\mathcal{A}'$ is uncountable, moreover we have

$$(1) \qquad \rho(\gamma, \tau_1) > \rho(\tau_2, \tau_3)$$

for all $\tau_1, \tau_2, \tau_3 \in \Delta$ and $\gamma \in a - \Delta$ with $a \in \mathcal{A}'$ and $\gamma < \tau_1$.

Now we prove that both Δ4) and Δ5) of 3.1 hold for every two sets $x, y \in \mathcal{A}'$ which will complete the proof of the lemma. Let $\tau_1, \tau_2 \in \Delta = x \cap y$ and $\gamma \in (x \setminus y) \cup (y \setminus x)$.

Note that if $\tau_1, \gamma < \tau_2$, then

$$(2) \qquad \rho(\gamma, \tau_1) \leq \rho(\gamma, \tau_2).$$

This follows from (iii) and (1).

Now we prove Δ4). Consider two cases. First when $\tau_1 < \gamma < \tau_2$. Assume $\xi \in f(\tau_1, \gamma)$, that is $\xi < \tau_1$ and

$$(3) \qquad \rho(\xi, \gamma) \leq \rho(\tau_1, \gamma),$$

By (ii) we have $\rho(\xi, \tau_2) \leq \max(\rho(\xi, \gamma), \rho(\gamma, \tau_2))$ which by (3) is less or equal to $\max(\rho(\tau_1, \gamma), \rho(\gamma, \tau_2)) = \rho(\gamma, \tau_2)$ by (2). But this means that $\xi \in f(\gamma, \tau_2)$ and so gives the inclusion of Δ4).

The second case is when $\gamma < \tau_1 < \tau_2$. Assume $\xi \in f(\gamma, \tau_1)$, that is $\xi < \gamma$ and

$$(4) \qquad \rho(\xi, \tau_1) \leq \rho(\gamma, \tau_1)\,.$$

By (ii) we have that $\rho(\xi, \tau_2) \leq \max(\rho(\xi, \tau_1), \rho(\tau_1, \tau_2))$ which by (4) is less or equal to $\max(\rho(\gamma, \tau_1), \rho(\tau_1, \tau_2))$. But we have

$$\max(\rho(\gamma, \tau_1), \rho(\tau_1, \tau_2)) \leq \rho(\gamma, \tau_2)$$

by (1) and (2), hence $\rho(\xi, \tau_2) \leq \rho(\gamma, \tau_2)$ and so $\xi \in f(\gamma, \tau_2)$ that again gives the inclusion of $\Delta 4$).

Finally, we prove $\Delta 5$). Assume $\tau_1 < \gamma < \tau_2$, then by (1) we have $\rho(\tau_1, \tau_2) \leq \rho(\gamma, \tau_2)$ and so the definition of f gives that $\tau_1 \in f(\gamma, \tau_2)$, as required in $\Delta 5$).

□

4. The forcing notion

Now we describe a natural notion of forcing with finite approximations that produces an ω_2-suitable pair (H, i). The forcing depends on a parameter f that will be chosen to be a strong Δ-function, like the one constructed in 3.3.

Definition 4.1. For each function $f : [\omega_2]^2 \longrightarrow [\omega_2]^{\leq\omega}$ satisfying $f(\alpha, \beta) \subset \alpha \cap \beta$ for any $\{\alpha, \beta\} \in [\omega_2]^2$ we define the poset $(P_f, \leq)$ as follows. The elements of P_f are all quadruples $p = \langle a, h, n, i\rangle$ satisfying the following five conditions (P1) – (P5):

(P1) $a \in [\omega_2]^{<\omega}$, $n \in \omega$, $h : a \times n \to \mathcal{P}(a)$, $i : [a]^2 \otimes n \to \mathcal{P}(a)$,
(P2) $\max h(\xi, j) = \xi$ for all $\langle \xi, j\rangle \in a \times n$,
(P3) $h(\xi, j) \subset h(\xi, 0)$ for all $\langle \xi, j\rangle \in a \times n$,
(P4) $i(\xi, \eta, j) \subseteq f(\xi, \eta)$ whenever $\langle \xi, \eta, j\rangle \in [a]^2 \otimes n$,
(P5) if $\langle \xi, \eta, j\rangle \in [a]^2 \otimes n$ then $h(\xi, 0) * h(\eta, j) \subset h[i(\xi, \eta, j)]$,

where, with some abuse of our earlier notation, we write

$$h[b] = \cup\{h(\alpha, 0) : \alpha \in b\} \tag{15}$$

for $b \subset a$. We say that $p \leq q$ if and only if $a_p \supseteq a_q$, $n_p \geq n_q$, $h_p(\xi, j) \cap a_q = h_q(\xi, j)$ for all $\langle \xi, j\rangle \in a_q \times n_q$, moreover $i_p \supset i_q$.

Assume that the sets

$$D_{\alpha,n} = \{p \in P_f : \alpha \in a_p \text{ and } n < n_p\}$$

are dense in P_f for all pairs $\langle \alpha, n\rangle \in \omega_2 \times \omega$. Then if $\mathcal{G}$ is a P_f-generic filter over V we may define , in $V[\mathcal{G}]$, the function H with $\operatorname{dom} H = \omega_2 \times \omega$ and the function i with $\operatorname{dom}(i) = [\omega_2]^2 \otimes \omega$ as follows:

$$H(\alpha, n) = \cup\{h_p(\alpha, n) : p \in \mathcal{G}, \langle \alpha, n\rangle \in \operatorname{dom}(h_p)\}, \tag{16}$$

$$i = \cup\{i_p : p \in \mathcal{G}\}. \tag{17}$$

Theorem 4.2. *Assume that CH holds and f is a strong Δ-function. Then P_f is CCC and (H, i) is an ω_2-suitable pair in $V[\mathcal{G}]$. Moreover, the locally compact, 0-dimensional, and Hausdorff space $X_H = \langle \omega_2 \times \mathbb{C}, \tau_H \rangle$ defined as in 2.4 satisfies, in $V[\mathcal{G}]$, the following properties:*

(i) $U(\delta) = H(\delta, 0) \times \mathbb{C}$ *is compact open for each* $\delta \in \omega_2$,
(ii) X_H *is first countable,*
(iii) $\forall A \in [\omega_2 \times \mathbb{C}]^{\omega_1}\ \exists \alpha \in \omega_2\ |A \cap U(\alpha)| = \omega_1$,
(iv) $\forall Y \in [\omega_2 \times \mathbb{C}]^{\omega}$ *either the closure* $\overline{Y}$ *is compact or there is* $\alpha < \omega_2$ *such that* $(\omega_2 \setminus \alpha) \times \mathbb{C} \subset \overline{Y}$.

Consequently, X_H is a locally compact, 0-dimensional, normal, first countable, initially ω_1-compact but non-compact space in $V[\mathcal{G}]$.

The rest of this paper is devoted to the proof of Theorem 4.2.

5. The forcing is CCC

The CCC property of P_f is crucial for us because it implies that ω_2 is preserved in the generic extension $V[\mathcal{G}]$. Indeed, properties (H1)–(H3) of definition 2.4 (for $\vartheta = \omega_2^V$) are easily deduced from of conditions (P1)–(P5) in 4.1 using straight-forward density arguments. So if ω_2 is preserved then we immediately conclude that (H, i) is an ω_2-suitable pair in $V[\mathcal{G}]$.

Definition 5.1. Two conditions $p_0 = \langle a_0, h_0, n, i_0 \rangle$ and $p_1 = \langle a_1, h_1, n, i_1 \rangle$ from P_f are said to be *good twins* provided that

(1) p_0 and p_1 are *isomorphic*, i.e. $|a_0| = |a_1|$ and the natural order-preserving bijection e between a_0 and a_1 is an isomorphism between p_0 and p_1:
  (i) $h_1(e(\xi), j) = e[h_0(\xi, j)]$ for $\xi \in a_0$ and $j < n$,
  (ii) $i_1(e(\xi), e(\eta), j)) = e[i_0(\xi, \eta, j)]$ for $\langle \xi, \eta, j \rangle \in \left[a_0\right]^2 \otimes n$,
  (iii) $e(\xi) = \xi$ whenever $\xi \in a_0 \cap a_1$ and $j < n$;
(2) $i_1(\xi, \eta, j) = i_0(\xi, \eta, j)$ for each $\{\xi, \eta\} \in \left[a_0 \cap a_1\right]^2$;
(3) a_0 and a_1 are good for f.

The good twins p_0 and p_1 are called *very good twins* if a_0 and a_1 are very good for f.

Definition 5.2. If $p = \langle a, h, n, i \rangle$ and $p' = \langle a', h', n, i' \rangle$ are good twins we define the *amalgamation $p^* = \langle a^*, h^*, n, i^* \rangle$ of p and p'* as follows:

Let $a^* = a \cup a'$. For $\eta \in h[a \cap a'] \cup h'[a \cap a']$ define

$$\delta_\eta = \min\{\delta \in a \cap a' : \eta \in h(\delta, 0) \cup h'(\delta, 0)\}.$$

Now, for any $\xi \in a^*$ and $m < n$ let

(18)

$$h^*(\xi,m) = \begin{cases} h(\xi,m) \cup h'(\xi,m) & \text{if } \xi \in a \cap a', \\ h(\xi,m) \cup \{\eta \in a' \setminus a : \delta_\eta \text{ is defined and } \delta_\eta \in h(\xi,m)\} & \text{if } \xi \in a \setminus a', \\ h'(\xi,m) \cup \{\eta \in a \setminus a' : \delta_\eta \text{ is defined and } \delta_\eta \in h'(\xi,m)\} & \text{if } \xi \in a' \setminus a. \end{cases}$$

Finally for $\langle \xi, \eta, m\rangle \in \left[a^*\right]^2 \otimes n$ let

$$i^*(\xi,\eta,m) = \begin{cases} i(\xi,\eta,m) & \text{if } \xi,\eta \in a, \\ i'(\xi,\eta,m) & \text{if } \xi,\eta \in a', \\ f(\xi,\eta) \cap a^* & \text{otherwise.} \end{cases} \tag{19}$$

(Observe that i^* is well-defined because p and p' are good twins). We will write $p^* = p + p'$ for the amalgamation of p and p'.

Lemma 5.3. *If p and p' are good twins then their amalgamation, $p^* = p + p'$, is a common extension of p and p' in P_f.*

Proof. First we prove a claim.

Claim 5.3.1. *Let $\alpha \in a$, $\eta \in a \cap a'$, and $m < \omega$. Assume that δ_α is defined and either $m = 0$ or $\delta_\alpha < \eta$. Then*

$$\alpha \in h(\eta,m) \text{ iff } \delta_\alpha \in h(\eta,m). \tag{20}$$

(Clearly, we also have a symmetric version of this statement for $\alpha \in a'$.)

Proof of claim 5.3.1. Assume first that $\alpha \in h(\eta,m) \subset h(\eta,0)$. Then clearly $\delta_\alpha \in h(\eta,m)$ if $\delta_\alpha = \eta$. So assume $\delta_\alpha < \eta$. Since $i(\delta_\alpha,\eta,m) \subset a \cap a'$ and $\max i(\delta_\alpha,\eta,m) < \delta_\alpha$ we have $\alpha \notin h[i(\delta_\alpha,\eta,m)]$ by the choice of δ_α. Thus from $p \in P_f$ we have

$$\alpha \notin h(\delta_\alpha,0) * h(\eta,m), \tag{21}$$

hence $h(\delta_\alpha,0) * h(\eta,m) \neq h(\delta_\alpha,0) \cap h(\eta,m)$. But then $h(\delta_\alpha,0) * h(\eta,m) = h(\delta_\alpha,0) \setminus h(\eta,m)$, so $\delta_\alpha \in h(\eta,m)$.

If, on the other hand, $\delta_\alpha \in h(\eta,m)$ then either $\delta_\alpha = \eta$ and so $\alpha \in h(\eta,0) = h(\eta,m)$ because $m = 0$, or $\delta_\alpha < \eta$ and we have

$$\alpha \notin h[i(\delta_\alpha,\eta,m)] \supset h(\delta_\alpha,0) * h(\eta,m) = h(\delta_\alpha,0) \setminus h(\eta,m).$$

Thus $\alpha \in h(\eta,m)$ in both cases. □

Next we check $p^* \in P_f$. Conditions 4.1.(P1)–(P4) for p^* are clear by the construction, so we should verify 4.1.(P5). Let $\langle \xi,\eta,m\rangle \in \left[a^*\right]^2 \otimes n$ and $\alpha \in h^*(\xi,0) * h^*(\eta,m)$, we need to show that $\alpha \in h^*[i^*(\xi,\eta,m]$. We will distinguish several cases.

Case 1. $\xi,\eta \in a$ *(or symmetrically, $\xi,\eta \in a'$).*

Since $h^*(\xi,0) \cap a = h(\xi,0)$ and $h^*(\eta,m) \cap a = h(\eta,m)$ we have $[h^*(\xi,0) * h^*(\eta,m)] \cap a = h(\xi,0) * h(\eta,m)$ by the definition of the operation $*$. Thus we have $\alpha \in h[i(\xi,\eta,m)] \subset h^*[i^*(\xi,\eta,m)]$ in case $\alpha \in a$.

Assume now that $\alpha \in a' \setminus a$. Then $\alpha \in h^*(\xi,0)$ implies that δ_α is defined and $\delta_\alpha \in h(\xi,0)$. Indeed, if $\xi \in a \setminus a'$ this is immediate from (18). For $\xi \in a \cap a'$, however, this follows from (the second version of) Claim 5.3.1 and the fact that $\delta_\alpha \in h'(\xi,0)$ implies $\delta_\alpha \in h(\xi,0)$.

We also have $\alpha \in h^*(\eta,m)$ iff $\delta_\alpha \in h(\eta,m)$, by (18) if $\eta \in a \setminus a'$ and by Claim 5.3.1 if $\eta \in a \cap a'$ (as $\delta_\alpha \le \xi < \eta$). But then $\alpha \in h^*(\xi,0) * h^*(\eta,m)$ implies $\delta_\alpha \in h(\xi,0) * h(\eta,m)$, hence there is $\nu \in i(\xi,\eta,m)$ such that $\delta_\alpha \in h(\nu,0)$. This again implies $\alpha \in h^*(\nu,0)$ either by (18) or by Claim 5.3.1, consequently, $\alpha \in h^*[i(\xi,\eta,m)] = h^*[i^*(\xi,\eta,m)]$.

Case 2. *$\xi \in a \setminus a'$, $\eta \in a' \setminus a$, and $\alpha \in a$ (or the same with a and a' switched).*

If $\xi \in h^*(\eta,m)$ then δ_ξ is defined and $\delta_\xi < \eta$, moreover

$$\alpha \in h^*(\xi,0) * h^*(\eta,m) = h^*(\xi,0) \setminus h^*(\eta,m) \tag{22}$$

implies $\alpha \notin h^*(\eta,m)$. If $\xi \notin h^*(\eta,m)$ then

$$\alpha \in h^*(\xi,0) * h^*(\eta,m) = h^*(\xi,0) \cap h^*(\eta,m), \tag{23}$$

implies $\alpha \in h^*(\eta,m)$, hence δ_α is defined and $\delta_\alpha < \eta$. Thus

$$\delta^* = \min\big\{\delta \in a \cap a' : \{\alpha,\xi\} \cap h(\delta,0) \neq \emptyset\big\} \tag{24}$$

is defined and $\delta^* < \eta$. If $\delta^* < \xi$ then we must have $\delta^* = \delta_\alpha$ and so, as p and p' are good twins, $\delta_\alpha \in f(\xi,\eta) \cap a^* = i^*(\xi,\eta,m)$. Consequently, $\alpha \in h(\delta_\alpha,0) \subset h^*[i^*(\xi,\eta,m)]$ holds.

Now, assume that $\xi < \delta^*$. We know that $\delta^* = \delta_\alpha$ or $\delta^* = \delta_\xi$, but not both because $|\{\alpha,\xi\} \cap h^*(\eta,m)| = 1$. But then we also have

$$|\{\alpha,\xi\} \cap h(\delta^*,0)| = 1. \tag{25}$$

Indeed, $|\{\alpha,\xi\} \cap h(\delta^*,0)| > 0$ is obvious and $\{\alpha,\xi\} \subset h(\delta^*,0)$ would imply that δ_α and δ_ξ are both defined and distinct, contradicting the definition of the bigger of the two. Now, (25) and $\alpha \in a \cap h^*(\xi,0) = h(\xi,0)$ together imply $\alpha \in h(\xi,0) * h(\delta^*,0) \subset h[i(\xi,\delta^*,0)]$. But

$$i(\xi,\delta^*,0) \subset f(\xi,\delta^*) \subset f(\xi,\eta)$$

because a and a' are good for f. Consequently, $i(\xi,\delta^*,0) \subset i^*(\xi,\eta,m)$, implying that $\alpha \in h^*[i^*(\xi,\eta,m)]$.

Case 3. *$\xi \in a \setminus a'$, $\eta \in a' \setminus a$, and $\alpha \in a'$ (or the same with a and a' switched).*

In this case $\alpha \in h^*(\xi, 0)$ implies that δ_α is defined and $\delta_\alpha < \xi$, hence $\delta_\alpha \in f(\xi, \eta)$ because a and a' are good for f. Since $i^*(\xi, \eta, m) = f(\xi, \eta) \cap a^*$ we conclude that $\alpha \in h'(\delta_\alpha, 0) \subset h^*[i^*(\xi, \eta, m)]$.

Since we have covered all the possible cases, it follows that p^* satisfies 4.1.(P5), that is, $p^* \in P_f$. That $p^* \leq p, p'$ is then immediate from the construction, hence the proof of our lemma is completed. □

Proof of theorem 4.2: P_f is CCC. In every uncountable collection of conditions from P_f there are two which are good twins for f and, by Lemma 5.3, they are compatible. □

As was pointed out at the beginning of this section, we may now conclude that (H, i) is an ω_2-suitable pair in $V[\mathcal{G}]$. This establishes the first part of Theorem 4.2 up to and including (i).

6. First countability

Proof of theorem 4.2: X_H is first countable. Since X_H is locally compact and Hausdorff it suffices to show that every point of X_H has countable pseudo-character or, in other words, every singleton is a G_δ.

To see this, fix $\langle \alpha, x \rangle \in \omega_2 \times \mathbb{C}$. We claim that there is a countable set $\Gamma \subset \alpha$ such that

$$\bigcap_{n \in \mathbb{N}} U(\alpha, n, x(n)) \subset U[\Gamma] \cup \{\langle \alpha, x \rangle\}. \tag{26}$$

Since every $U(\gamma)$ is clopen, this implies that

$$\{\langle \alpha, x \rangle\} = \bigcap_{n \in \mathbb{N}} U(\alpha, n, x(n)) \cap \bigcap \{ X_H \setminus U(\gamma) : \gamma \in \Gamma\}$$

is indeed a G_δ.

Our following lemma clearly implies (26). To formulate it, we first fix some notation. In $V[\mathcal{G}]$, for $\alpha \in \omega_2$, $1 \leq m < \omega$ and $\Gamma \subset \omega_2$ we write

$$H^1(\alpha, m) = H(\alpha, m) \setminus \{\alpha\}, \tag{27}$$

$$H^0(\alpha, m) = H(\alpha, 0) \setminus H(\alpha, m), \tag{28}$$

$$H[\Gamma] = \cup\{H(\gamma, 0) : \gamma \in \Gamma\}. \tag{29}$$

Recall that with this notation we have

$$U(\alpha, n, \varepsilon) = (H^\varepsilon(\alpha, n) \times \mathbb{C}) \cup (\{\alpha\} \times [n, \varepsilon]) .$$

Lemma 6.1. *In $V[\mathcal{G}]$, for each $\langle \alpha, x \rangle \in \omega_2 \times \mathbb{C}$ there is a countable set $\Gamma \subset \alpha$ such that*

$$\bigcap_{n \in \mathbb{N}} H^{x(n)}(\alpha, n) \subset H[\Gamma] . \tag{30}$$

Proof. Suppose, arguing indirectly, that the lemma is false. Then, in $V[\mathcal{G}]$, for each countable set $A \subset \alpha$ there is $\gamma_A \in \alpha$ such that

$$\gamma_A \in \bigcap_{n\in\mathbb{N}} H^{x(n)}(\alpha, n) \setminus H[A]. \tag{31}$$

From now on, we work in the ground model V. For every $\zeta < \omega_1$ let $A_\zeta \subseteq \alpha$ be a countable subset such that $\zeta' \le \zeta < \omega_1$ implies $A_{\zeta'} \subseteq A_\zeta$ and $\bigcup_{\zeta<\omega_1} A_\zeta = \alpha$.

Let $p_\zeta = \langle a_\zeta, h_\zeta, n_\zeta, i_\zeta\rangle \in P_f$ be a condition such that $\alpha \in a_\zeta$ and for some $\gamma_\zeta \in \alpha \cap a_\zeta$ we have

$$p_\zeta \Vdash \gamma_\zeta \in \bigcap_{n\in\mathbb{N}} H^{x(n)}(\alpha, n) \setminus H[A_\zeta]. \tag{32}$$

Using standard Δ-system and counting arguments and the properties of the strong Δ-function f, we may find $\zeta_1 < \zeta_2 < \omega_1$ such that

$$\alpha \cap a_{\zeta_1} \subset A_{\zeta_2}, \tag{33}$$

moreover p_{ζ_1}, p_{ζ_2} are very good twins for f.

Let $p = p_{\zeta_1} + p_{\zeta_2}$ with $p = \langle a, h, n, i\rangle$ be their amalgamation as in 5.2. We now further extend p to a condition of the form $r = \langle a, h_r, n+1, i_r\rangle$ with the following stipulations:

(r1) $h_r \supset h$,
(r2) $h_r(\xi, n) = \{\xi\}$ for $\xi \in a \setminus \{\alpha\}$,
(r3) $h_r(\alpha, n) = \{\alpha\} \cup \big(h(\alpha, 0) \cap h[\alpha \cap a_{\zeta_1}]\big)$,
(r4) $i_r \supset i$,
(r5) $i_r(\eta, \xi, n) = \emptyset$ for $\eta < \xi \in a \setminus \{\alpha\}$,
(r6) $i_r(\eta, \alpha, n) = a \cap f(\eta, \alpha)$ for $\eta < \alpha$.

It is not clear at all that r is a condition, but if it is we have reached a contradiction. Indeed, if $r \in P_f$ then $r \le p_{\zeta_2}$, so $r \Vdash \gamma_{\zeta_2} \notin H[A_{\zeta_2}]$, hence $\gamma_{\zeta_2} \notin h[\alpha \cap a_{\zeta_1}]$ by (33). But then by (r3) we have

$$\gamma_{\zeta_2} \notin h_r(\alpha, n). \tag{34}$$

On the other hand, since $\gamma_{\zeta_1} \in \alpha \cap a_{\zeta_1} \subset h[\alpha \cap a_{\zeta_1}]$ we have

$$\gamma_{\zeta_1} \in h_r(\alpha, n) \tag{35}$$

by (r3). But this is a contradiction because, by (32), the first of these relations implies $r \Vdash x(n) = 0$ while the second implies $r \Vdash x(n) = 1$.

So it remains to show that $r \in P_f$. Items (P1) - (P4) of Definition 4.1 are clear. Also, (P5) holds if $j < n$ because $p \in P_f$. Thus we only have to check (P5) for triples of the form $\langle \eta, \xi, n\rangle$.

If $\eta < \xi \ne \alpha$ we have $\eta \notin h(\xi, n) = \{\xi\}$, and so $h_r(\eta, 0) * h_r(\xi, n) = h_r(\eta, 0) \cap h_r(\xi, n) \subseteq \eta \cap \{\xi\} = \emptyset$, hence (P5) of Definition 4.1 holds trivially. So assume now that $\eta < \alpha$. In view of the definition of r, our task is to show the following two assertions:

(I) if $\eta \in h_r(\alpha, n)$ then $h(\eta, 0) \setminus h_r(\alpha, n) \subset h[a \cap f(\eta, \alpha)]$,

(II) if $\eta \notin h_r(\alpha, n)$ then $h(\eta, 0) \cap h_r(\alpha, n) \subset h[a \cap f(\eta, \alpha)]$.

The fact that $p = p_{\zeta_1} + p_{\zeta_2}$ and properties $\Delta 4)$ and $\Delta 5)$ of our strong Δ-function f will play an essential role in the proofs of (I) and (II).

Proof of (I). First note that by the definition of r we have

$$h(\eta, 0) \setminus h_r(\alpha, n) = h(\eta, 0) \setminus (h(\alpha, 0) \cap h[\alpha \cap a_{\zeta_1}]) = (h(\eta, 0) \setminus h(\alpha, 0)) \cup (h(\eta, 0) \setminus h[\alpha \cap a_{\zeta_1}]). \tag{36}$$

Since $h(\eta, 0) \setminus h(\alpha, 0) \subset h[i(\eta, \alpha, 0)] \subset h[a \cap f(\eta, \alpha)]$ is obvious, it is enough to show that

(I') if $\eta \in h_r(\alpha, n)$, then $h(\eta, 0) \setminus h[\alpha \cap a_{\zeta_1}] \subset h[a \cap f(\eta, \alpha)]$.

If $\eta \in a_{\zeta_1}$ then $h(\eta, 0) \setminus h[\alpha \cap a_{\zeta_1}] = \emptyset$ and we are done. So assume now that $\eta \notin a_{\zeta_1}$, that is $\eta \in a_{\zeta_2} \setminus a_{\zeta_1}$. Now $\eta \in h[\alpha \cap a_{\zeta_1}]$ means that there is a $\xi \in \alpha \cap a_{\zeta_1}$ with $\eta \in h(\xi, 0)$. By the definition 5.2 (18) of the amalgamation then there is $\delta \in a_{\zeta_1} \cap a_{\zeta_2}$ such that $\eta < \delta \le \xi$ and $\eta \in h_{\zeta_2}(\delta, 0)$. Since $p_{\zeta_2} \in P_f$ this implies

$$h_{\zeta_2}(\eta, 0) \setminus h_{\zeta_2}(\delta, 0) \subseteq h_{\zeta_2}[i_{\zeta_2}(\eta, \delta, 0)]. \tag{37}$$

A similar argument, referring back to definition 5.2 (18), yields us that $h(\eta, 0) \setminus h_{\zeta_2}(\eta, 0) \subset h[\alpha \cap a_{\zeta_1}]$, and as $h_{\zeta_2}(\delta, 0) \subset h(\delta, 0) \subset h[\alpha \cap a_{\zeta_1}]$ we may conclude that

$$h(\eta, 0) \setminus h[\alpha \cap a_{\zeta_1}] \subset h_{\zeta_2}[i_{\zeta_2}(\eta, \delta, 0)] \subset h[i_{\zeta_2}(\eta, \delta, 0)]. \tag{38}$$

Since $\eta \in a_{\zeta_2} \setminus a_{\zeta_1}$ and $\delta, \alpha \in a_{\zeta_1} \cap a_{\zeta_2}$, we have $f(\eta, \delta) \subset f(\eta, \alpha)$ by $\Delta 4)$. Consequently,

$$i_{\zeta_2}(\eta, \delta, 0) \subset a_{\zeta_2} \cap f(\eta, \delta) \subset a \cap f(\eta, \alpha), \tag{39}$$

completing the proof of (I') and hence of (I).

Proof of (II). If $\eta \notin h_r(\alpha, n)$ then either $\eta \notin h(\alpha, 0)$ or $\eta \notin h[\alpha \cap a_{\zeta_1}]$. If $\eta \notin h(\alpha, 0)$ then $p \in P_f$ implies

$$h(\eta, 0) \cap h_r(\alpha, n) \subset h(\eta, 0) \cap h(\alpha, 0) = h(\eta, 0) * h(\alpha, 0) \subset h[i(\eta, \alpha, 0)] \subset h[a \cap f(\eta, \alpha]). \tag{40}$$

So assume that $\eta \notin h[\alpha \cap a_{\zeta_1}]$, clearly then $\eta \notin a_{\zeta_1}$ as well. Consider any $\beta \in h(\eta, 0) \cap h_r(\alpha, n)$, we have to show that $\beta \in h[a \cap f(\eta, \alpha)]$.

Case 1. $\beta \in a_{\zeta_1}$. By using definition 5.2 (18) again, then $\beta \in h(\eta, 0)$ implies that there is a $\delta \in \eta \cap a_{\zeta_1} \cap a_{\zeta_2}$ with $\beta \in h_{\zeta_2}(\delta, 0)$. But then $\delta \in f(\eta, \alpha)$ by property $\Delta 5)$ of strong Δ-functions, hence we are done.

Case 2. $\beta \notin a_{\zeta_1}$. In this case $\beta \in h[\alpha \cap a_{\zeta_1}]$ implies that there is a $\delta \in \alpha \cap a_{\zeta_1} \cap a_{\zeta_2}$ such that $\beta \in h_{\zeta_2}(\delta, 0)$, hence $\beta \in h_{\zeta_2}(\eta, 0) \cap h_{\zeta_2}(\delta, 0)$. Moreover, $\eta \notin h[\alpha \cap a_{\zeta_1}]$ implies $\eta \notin h_{\zeta_2}(\delta, 0)$. Thus if $\eta < \delta$ then $p_{\zeta_2} \in P_f$ and $h_{\zeta_2}(\eta, 0) \cap h_{\zeta_2}(\delta, 0) = h_{\zeta_2}(\eta, 0) * h_{\zeta_2}(\delta, 0)$ imply that $\beta \in h_{\zeta_2}(\gamma, 0)$ for some $\gamma \in i(\eta, \delta, 0) \subset f(\eta, \delta)$. But we have $f(\eta, \delta) \subset f(\eta, \alpha)$ by $\Delta 4$), so $\gamma \in a \cap f(\eta, \alpha)$ and we are done.

Finally, if $\delta < \eta$ then $\delta \in f(\eta, \alpha)$ because f satisfies $\Delta 5$), moreover we have $\beta \in h_{\zeta_2}(\delta, 0) \subset h(\delta, 0)$ and the proof of (II) is completed. □

This then completes the proof of Lemma 6.1 and thus of the first countability of the space X_H. □

7. ω_1-COMPACTNESS

In this section we establish part (iii) of theorem 4.2. This implies that every uncountable subset of X_H has uncountable intersection with a compact set, hence every set of size ω_1 has a complete accumulation point.

Lemma 7.1. *If $p = \langle a, h, n, i \rangle \in P_f$ and $\beta \in \omega_2$ with $\beta > \max a$ then there is a condition $q \le p$ such that $a \subset h_q(\beta, 0)$.*

Proof. We define the condition $q = \langle a \cup \{\beta\}, h_q, n, i_q \rangle$ with the following stipulations: $h_q \supset h$, $i_q \supset i$, $h_q(\beta, j) = a \cup \{\beta\}$ for $j < n$, $i_q(\alpha, \beta, j) = \emptyset$ for $\alpha \in a$ and $j < n$. It is straight-forward to check that $q \in P_f$ is as required. □

Lemma 7.2. *In $V[\mathcal{G}]$, for each set $A \in [\omega_2 \times \mathbb{C}]^{\omega_1}$ there is $\beta \in \omega_2$ such that $|A \cap U(\beta)| = \omega_1$.*

Proof. Let $\dot{A}$ be a P_f-name for A and assume that $p \in \mathcal{G}$ with

$$p \Vdash \dot{A} = \{\dot{z}_\xi : \xi < \omega_1\} \in \left[\omega_2 \times \mathbb{C}\right]^{\omega_1}.$$

We may assume that p also forces that $\{\dot{z}_\xi : \xi < \omega_1\}$ is a one-one enumeration of $\dot{A}$. For each $\xi < \omega_1$ we may pick $p_\xi \le p$ and $\alpha_\xi \in \omega_2$ with $\alpha_\xi \in a_{p_\xi}$ such that $p_\xi \Vdash \dot{z}_\xi = \langle \alpha_\xi, \dot{x}_\xi \rangle$. Let $\sup\{\alpha_\xi : \xi < \omega_1\} < \beta < \omega_2$. By lemma 7.1 for each $\xi < \omega_1$ there is a condition $q_\xi \le p_\xi$ such that $\alpha_\xi \in h_{q_\xi}(\beta, 0)$, hence $q_\xi \Vdash \dot{z}_\xi \in U(\beta)$. But P_f satisfies CCC, so there is $q \in \mathcal{G}$ such that $q \Vdash |\{\xi \in \omega_1 : q_\xi \in \mathcal{G}\}| = \omega_1$. Clearly, then $q \Vdash |\dot{A} \cap U(\beta)| = \omega_1$. □

8. COUNTABLE COMPACTNESS

In this section we show that part (iv) of theorem 4.2 holds: in $V[\mathcal{G}]$, the closure of any infinite subset of X_H is either compact or contains a "tail" of X_H, that is $(\omega_2 \setminus \alpha) \times \mathbb{C}$ for some $\alpha < \omega_2$. Of course, this implies that X_H is countably compact and thus, together with the results of the previous section, establishes the initial ω_1-compactness of X_H. Moreover, it also implies that X_H is normal, for of any two disjoint closed sets in X_H (at least) one has to be compact.

We start by proving an extension result for conditions in P_f. We shall use the following notation that is analogous to the one that was introduced before lemma 6.1.

$$h^1(\alpha, m) = h(\alpha, m), \tag{41}$$

$$h^0(\alpha, m) = h(\alpha, 0) \setminus h(\alpha, m). \tag{42}$$

Lemma 8.1. *Assume that* $p = \langle a, h, n, i \rangle \in P_f$, $\alpha \in a$, *and* $\varepsilon : n \longrightarrow 2$ *is a function with* $\varepsilon(0) = 1$. *Then for every* $\eta \in \alpha \setminus a$ *there is a condition of the form* $q = \langle a \cup \{\eta\}, h_q, n, i_q \rangle \in P_f$ *such that* $q \leq p$ *and*

$$\eta \in \bigcap_{m<n} h_q^{\varepsilon(m)}(\alpha, m) \setminus h_q[a \cap \alpha]. \tag{43}$$

Proof. We define h_q and i_q with the following stipulations:

$h_q(\eta, m) = \{\eta\}$ for $m < n$,
$h_q(\alpha, m) = h(\alpha, m) \cup \{\eta\}$ if $m < n$ and $\varepsilon(m) = 1$,
$h_q(\alpha, m) = h(\alpha, m)$ if $m < n$ and $\varepsilon(m) = 0$,
$h_q(\nu, m) = h(\nu, m) \cup \{\eta\}$ if $\nu \in a \setminus \{\alpha\}$, $m < n$, and $\alpha \in h(\nu, m)$,
$h_q(\nu, m) = h(\nu, m)$ if $\nu \in a \setminus \{\alpha\}$, $m < n$, and $\alpha \notin h(\nu, m)$,
$i_q \supset i$, $i_q(\eta, \nu, m) = \emptyset$ if $\nu \in a \setminus \eta$, and $i_q(\nu, \eta, m) = \emptyset$ if $\nu \in a \cap \eta$.

To show $q \in P_f$ we need to check only (P5). But this follows from the fact that if $\eta \in h_q(\nu, 0) * h_q(\mu, m)$ then, as can be checked by examining a number of cases, we have $\nu, \mu \in a$ and $\alpha \in h(\nu, 0) * h(\mu, m)$ as well. By $p \in P_f$ then there is a $\xi \in i(\nu, \mu, m)$ with $\alpha \in h(\xi, 0)$ which implies $\eta \in h_q(\xi, 0)$ because $\varepsilon(0) = 1$, so we are done. Thus $q \in P_f$, $q \leq p$, and q clearly satisfies all our requirements. □

Lemmas 7.1 and 8.1 can be used to show that

$$D_{\alpha,n} = \{p \in P_f : \alpha \in a_p \text{ and } n < n_p\}$$

is dense in P_f for all pairs $\langle \alpha, n \rangle \in \omega_2 \times \omega$, showing that $\mathrm{dom}(H) = \omega_2 \times \omega$ and $\mathrm{dom}(i) = \left[\omega_2\right]^2 \otimes \omega$.

Our next lemma is a partial result on the way to what we promised to show in this section.

Lemma 8.2. *Assume that, in* $V[\mathcal{G}]$*, we have* $D \in V \cap \big[\omega_2\big]^\omega$ *and* $Y = \{\langle \delta, x_\delta \rangle : \delta \in D\} \subset \omega_2 \times \mathbb{C}$*. Then*

$$(\omega_2 \setminus \sup(D)) \times \mathbb{C} \subset \overline{Y}.$$

Proof. By lemma 2.5 it suffices to prove that

$$V[\mathcal{G}] \models \Big(\bigcap_{1 \le m < n} U(\alpha, m, \varepsilon(m)) \setminus U[b] \Big) \cap Y \neq \emptyset \tag{44}$$

whenever $\alpha \in \omega_2 \setminus \sup D$, $n \in \mathbb{N}$, $\varepsilon : n \longrightarrow 2$ with $\varepsilon(0) = 1$, and $b \in \big[\alpha\big]^{<\omega}$. So fix these and pick a condition $p = \langle a, h, k, i \rangle \in P_f$ such that $\alpha \in a$, $b \subset a$, and $n < k$. (We know that the set E of these conditions is dense in P_f.) Let us then choose $\delta \in D \setminus a$. By lemma 8.1 there is a condition $q \le p$ such that

$$\delta \in \bigcap_{1 \le m < n} h_q^{\varepsilon(m)}(\alpha, m) \setminus h_q[b]. \tag{45}$$

Then

$$q \Vdash \langle \delta, x_\delta \rangle \in \bigcap_{1 \le m < n} U(\alpha, m, \varepsilon(m)) \setminus U[b], \tag{46}$$

hence

$$q \Vdash \Big(\bigcap_{1 \le m < n} U(\alpha, m, \varepsilon(m)) \setminus U[b] \Big) \cap Y \neq \emptyset\,. \tag{47}$$

Since $p \in E$ was arbitrary, the set of q's satisfying the last forcing relation is also dense in P_f, so we are done. □

We need a couple more, rather technical, results before we can turn to the proof of part (iv) of theorem 4.2. First we give a definition.

Definition 8.3. (1) Assume that $p = \langle a, h, n, i \rangle \in P_f$ and $a < b \in [\omega_2]^{<\omega}$ are such that $a \subset f(\gamma, \gamma')$ for any $\{\gamma, \gamma'\} \in [b]^2$. Then we define the *b-extension of* p to be the condition q of the form $q = \langle a \cup b, h_q, n, i_q \rangle$ with $h \subset h_q$, $i \subset i_q$, and the following stipulations:

(R1) $h_q(\gamma, \ell) = a \cup \{\gamma\}$ for $\gamma \in b$ and $\ell < n$,
(R2) $i_q(\gamma', \gamma, \ell) = a$ for $\gamma', \gamma \in b$ with $\gamma' < \gamma$ and $\ell < n$,
(R3) $i_q(\xi, \gamma, \ell) = \emptyset$ for $\xi \in a$, $\gamma \in b$, and $\ell < n$.

(2) If $q \in P_f$ and $b \subset a_q$ then $s \le q$ is said to be a *b-fair extension of* q iff $h_s(\gamma, j) = h_s(\gamma, 0)$ holds for any $\gamma \in b$ and $n_q \le j < n_s$.

Our following result shows that the b-extension severely restricts any further extensions.

Lemma 8.4. *Assume that* $p = \langle a, h, n, i\rangle \in P_f$, $a < b$, *and* q *is the* b*-extension of* p. *If* $s \le q$ *is any extension of* q *then*

$$h_s[a] = h_s(\gamma', 0) \cap h_s(\gamma, \ell) \tag{48}$$

whenever $\langle \gamma', \gamma, \ell\rangle \in \big[b\big]^2 \otimes n$. *If, in addition,* s *is a* b*-fair extension of* q *then (48) holds for all* $\langle \gamma', \gamma, \ell\rangle \in \big[b\big]^2 \otimes n_s$.

Proof. We have $\gamma' \notin h_s(\gamma)$ by (R1) and $s \le q$, hence if $\ell < n$ then (P5) and (R2) imply

$$h_s(\gamma', 0) \cap h_s(\gamma, \ell) = h_s(\gamma', 0) * h_s(\gamma, \ell) \subset h_s[i_s(\gamma', \gamma, \ell)] = h_s[a]. \tag{49}$$

Similarly, for all $\xi \in a$, $\gamma'' \in b$, and $\ell'' < n$ we have

$$h_s(\xi, 0) \setminus h_s(\gamma'', \ell'') = h_s(\xi, 0) * h_s(\gamma'', \ell'') \subset h_s[i_s(\xi, \gamma'', \ell'')] = h_s[\emptyset] = \emptyset, \tag{50}$$

which implies $h_s[a] \subset h_s(\gamma'', \ell'')$. But then $h_s[a] \subset h_s(\gamma', 0) \cap h_s(\gamma, \ell)$ which together with (49) yields (48).

Now, if s is a b-fair extension of q and $\langle \gamma', \gamma, \ell\rangle \in \big[b\big]^2 \otimes n_s$ with $n \le \ell < n_s$ then we have (48) because $h_s(\gamma, 0) = h_s(\gamma, \ell)$ and $h_s[a] = h_s(\gamma', 0) \cap h_s(\gamma, 0)$. □

In our next result we are going to make use of the following simple observation.

Fact 8.5. *If* $p = \langle a, h, n, i\rangle \in P_f$ *and* $X \subset a$ *is an initial segment of* a *then* $p \restriction X = \Big\langle X,\ h \restriction X \times n,\ n,\ i \restriction \big[X\big]^2 \otimes n \Big\rangle \in P_f$ *as well.*

Lemma 8.6. *Let* $p, q, s \in P_f$ *be conditions and* $Q \subset S < E < F$ *be sets of ordinals such that*

$$a_p = Q \cup E\,,\ a_q = Q \cup E \cup F\,,\ a_s = S \cup E \cup F\,,$$

q *is the* F*-extension of* p, *and* s *is an* F*-fair extension of* q. *Assume, moreover, that* $|E| = k$ *with* $E = \{\gamma_i : i < k\}$ *the increasing enumeration of* E *and* $|F| = 2k$, $F = \{\gamma_{i,0}, \gamma_{i,1} : i < k\}$ *with* $\gamma_{i,0} < \gamma_{i,1}$ *satisfying*

$$\forall i < k\, \forall \xi \in S\, [\, f(\xi, \gamma_i) = f(\xi, \gamma_{i,0}) = f(\xi, \gamma_{i,1})\,]. \tag{51}$$

Let us now define $r = \langle a_r, h_r, n_r, i_r\rangle$ *as follows:*

(A) $a_r = S \cup E$, $n_r = n_s$,

(B) for $\xi \in a_r$ *and* $j < n_r$ *let*

$$h_r(\xi, j) = \begin{cases} h_s(\xi, j) \cup (S \setminus h_s[a_p]) & \text{if } \xi = \gamma_i \text{ and } \gamma_0 \in h_s(\gamma_i, j), \\ h_s(\xi, j) & \text{otherwise,} \end{cases}$$

(C) for $\langle \xi, \eta, j \rangle \in [a_r]^2 \otimes n_r$

$$i_r(\xi, \eta, j) = \begin{cases} i_s(\xi, \eta, j) & \text{if } \xi, \eta \in a_p \text{ or } \xi, \eta \in S, \\ f(\xi, \eta) \cap a_s & \text{otherwise.} \end{cases}$$

Then $r \in P_f$, $r \leq p$, $r \leq s \restriction S \in P_f$, *and* $S \setminus h_s[a_p] \subset h_r(\gamma_0, 0)$.

Proof. It is clear from our assumptions and the construction of r that the only thing we need to establish is $r \in P_f$. To see that, it suffices to check that r satisfies (P5) because the other requirements are obvious. So let $\langle \xi, \eta, j \rangle \in [a_r]^2 \otimes n_r$. We have to show

$$h_r(\xi, 0) * h_r(\eta, j) \subset h_r[i_r(\xi, \eta, j)]. \tag{52}$$

If $\eta \in S$ then $h_r(\xi, 0) * h_r(\eta, j) \subset h_r[i_r(\xi, \eta, j)]$ holds because $r \restriction S = s \restriction S \in P_f$. So, from here on, we assume that $\eta = \gamma_i$ for some $i < k$.

Let us first point out that, as q is the F-extension of p and s is an F-fair extension of q, by lemma 8.4 we have

$$h_s[a_p] = h_s(\gamma_{i,0}, 0) \cap h_s(\gamma_{i,1}, j) \tag{53}$$

for any $i < k$ and $j < n_r$. Also, to shorten notation, we shall write

$$C = S \setminus h_s[a_p].$$

Case 1. $\xi \in S$.

Subcase 1.1. $\xi \notin h_r(\gamma_i, j)$.

Then $\xi \notin h_s(\gamma_i, j)$ as well, so we have both

$$h_r(\xi, 0) * h_r(\gamma_i, j) = h_r(\xi, 0) \cap h_r(\gamma_i, j) \tag{54}$$

and

$$h_s(\xi, 0) * h_s(\gamma_i, j) = h_s(\xi, 0) \cap h_s(\gamma_i, j) \subset h_s[i_s(\xi, \gamma_i, j)] \subset h_r[i_r(\xi, \gamma_i, j)]. \tag{55}$$

If $\gamma_0 \notin h_s(\gamma_i, j)$ then $h_r(\gamma_i, j) = h_s(\gamma_i, j)$ and also $h_r(\xi, 0) = h_s(\xi, 0)$, hence (54) and (55) imply (52).

Assume now that $\gamma_0 \in h_s(\gamma_i, j)$, hence $h_r(\gamma_i, j) = h_s(\gamma_i, j) \cup C$.

Claim. $h_s(\xi, 0) \cap C \subset h_r[i_r(\xi, \gamma_i, j)]$.

Since now $\xi \notin C$, by (53) we have $\xi \in h_s(\gamma_{i,0}, 0) \cap h_s(\gamma_{i,1}, 0)$. Thus, using twice that s satisfies (P5), we have

$$
\begin{aligned}
(56)\quad h_s(\xi,0) \cap C = h_s(\xi,0) \setminus (h_s(\gamma_{i,0},0) \cap h_s(\gamma_{i,1},0)) = \\
(h_s(\xi,0) \setminus h_s(\gamma_{i,0},0)) \cup (h_s(\xi,0) \setminus h_s(\gamma_{i,1},0)) = \\
(h_s(\xi,0) * h_s(\gamma_{i,0},0)) \cup (h_s(\xi,0) * h_s(\gamma_{i,1},0)) \subset \\
h_s[i_s(\xi,\gamma_{i,0},0)] \cup h_s[i_s(\xi,\gamma_{i,1},0)].
\end{aligned}
$$

If $\xi \in Q \subset a_p$ then $h_r(\xi,0) \cap C = \emptyset$, so the Claim holds trivially. So we can assume that $\xi \notin Q$. Then, by clause (C) of 8.6, for each $\varepsilon \in \{0,1\}$ we have

$$(57)\qquad i_r(\xi,\gamma_i,j) = f(\xi,\gamma_i) \cap a_s = f(\xi,\gamma_{i,\varepsilon}) \cap a_s \supset i_s(\xi,\gamma_{i,\varepsilon},0).$$

Clearly, (56) and (57) together yield the Claim.

But then we have

$$
\begin{aligned}
(58)\quad h_r(\xi,0) * h_r(\gamma_i,j) = h_r(\xi,0) \cap (h_s(\gamma_i,j) \cup C) = \\
(h_s(\xi,0) \cap h_s(\gamma_i,j)) \cup (h_s(\xi,0) \cap C) \subset h_r[i_r(\xi,\eta,j)]
\end{aligned}
$$

by (54), (55), and the Claim.

Subcase 1.2. $\xi \in h_r(\gamma_i,j)$.

If $\xi \in h_s(\gamma_i,j)$ then

$$
\begin{aligned}
h_r(\xi,0) * h_r(\gamma_i,j) = h_r(\xi,0) \setminus h_r(\gamma_i,j) \subset h_s(\xi,0) \setminus h_s(\gamma_i,j) \\
= h_s(\xi,0) * h_s(\gamma_i,j) \subset h_s[i_s(\xi,\gamma_i,j)] \subset h_r[i_r(\xi,\gamma_i,j)]
\end{aligned}
$$

and we are done.

So we can assume that $\xi \notin h_s(\gamma_i,j)$. Then $\xi \in C$, $h_r(\gamma_i,j) = h_s(\gamma_i,j) \cup C$, and $\gamma_0 \in h_s(\gamma_i,j)$. By (53) we can fix $\varepsilon < 2$ such that $\xi \notin h_s(\gamma_{i,\varepsilon},0)$, consequently we have

$$
\begin{aligned}
(59)\quad h_r(\xi,0) * h_r(\gamma_i,j) = h_s(\xi,0) \setminus (h_s(\gamma_i,j) \cup C) \subset h_s(\xi,0) \setminus C = \\
h_s(\xi,0) \setminus (S \setminus (h_s(\gamma_{i,0}, 0 \cap h_s(\gamma_{i,1},0))) = \\
h_s(\xi,0) \cap (h_s(\gamma_{i,0}, 0 \cap h_s(\gamma_{i,1},0))) \subset \\
h_s(\xi,0) \cap h_s(\gamma_{i,\varepsilon},0) = h_s(\xi,0) * h_s(\gamma_{i,\varepsilon},0) = h_s[i_s(\xi,\gamma_{i,\varepsilon},0)].
\end{aligned}
$$

But then again by clause (C) of 8.6

$$(60)\qquad i_r(\xi,\gamma_i,j) = f(\xi,\gamma_i) \cap a_s = f(\xi,\gamma_{i,\varepsilon}) \cap a_s \supset i_s(\xi,\gamma_{i,\varepsilon},0).$$

(59) and (60) clearly imply (52).

Case 2. $\xi = \gamma_\ell$ *for some* $\ell < i$.

Then $i_s(\gamma_\ell, \gamma_i, j) = i_r(\gamma_\ell, \gamma_i, j)$, hence we have

$$h_s(\gamma_\ell, 0) * h_s(\gamma_i, j) \subset h_r[i_r(\gamma_\ell, \gamma_i, j)]. \tag{61}$$

Examining the definition of h_r in clause (B) of 8.6 and using that $C \cap h_s(\gamma_\ell, 0) = \emptyset$ we get
(62)

$$h_r(\gamma_\ell, 0) * h_r(\gamma_i, j) = \begin{cases} h_s(\gamma_\ell, 0) * h_s(\gamma_i, j) & \text{if } \gamma_0 \notin h_s(\gamma_\ell, 0) * h_s(\gamma_i, j), \\ (h_s(\gamma_\ell, 0) * h_s(\gamma_i, j)) \cup C & \text{if } \gamma_0 \in h_s(\gamma_\ell, 0) * h_s(\gamma_i, j). \end{cases}$$

This and (61) show that we are done if $\gamma_0 \notin h_s(\gamma_\ell, 0) * h_s(\gamma_i, j)$.

So assume that $\gamma_0 \in h_s(\gamma_\ell, 0) * h_s(\gamma_i, j)$. Then there is $\zeta \in i_s(\gamma_\ell, \gamma_i, j)$ with $\gamma_0 \in h_s(\zeta)$. But then $\gamma_0 \leq \zeta < \gamma_\ell$ implies that $\zeta \in E$, hence $\zeta = \gamma_m$ for some $m < \ell$. Because of this and by the choice of h_r we have

$$C \subset h_r(\gamma_m) \subset h_r[i_r(\gamma_\ell, \gamma_i, j)]. \tag{63}$$

But (61), (62), and (63) together imply (52), completing the proof of $r \in P_f$. □

Proof of theorem 4.2: Property (*iv*). Our aim is to prove that the following statement holds in $V[\mathcal{G}]$:

(*iv*) *If the closure* $\overline{Y}$ *of a set* $Y \in [X_H]^\omega$ *is not compact then there is* $\alpha < \omega_2$ *such that* $(\omega_2 \setminus \alpha) \times \mathbb{C} \subset \overline{Y}$.

We shall make use of the following easy lemma.

Lemma 8.7. *A set* $Z \subset X_H$ *has compact closure if and only if*

$$\Gamma = \{\gamma : \exists\, x\, \langle \gamma, x \rangle \in Z\} \subset H[F]$$

for some finite set $F \subset \omega_2$.

Proof of the lemma. If $\overline{Z}$ is compact then there is a finite set $F \subset \omega_2$ such that $\overline{Z} \subset U[F]$. Clearly, then $\Gamma \subset H[F]$.

Conversely, if $\Gamma \subset H[F]$ for a finite $F \subset \omega_2$ then $Z \subset U[F]$, hence $\overline{Z} \subset U[F]$ as well. But as $U[F]$ is compact, so is $\overline{Z}$. □

Given two sets $X, E \subset \omega_2$ with $X < E$ we shall write

$$cl_f(X, E) = (\text{the } f\text{-closure of } X \cup E) \cap \sup(X). \tag{64}$$

Fact 8.8. *If* $\xi \in cl_f(X, E)$ *and* $\eta \in cl_f(X, E) \cup E$ *then* $f(\xi, \eta) \subset cl_f(X, E)$.

Let us now fix a regular cardinal ϑ that is large enough so that $\mathcal{H}_\vartheta$, the structure of sets whose transitive closure has cardinality $< \vartheta$, contains everything relevant.

Lemma 8.9. *Assume that*

(65) $$V[\mathcal{G}] \models \Gamma \in [\omega_2]^\omega \text{ is not covered by finitely many } H(\xi,0)$$

and $\dot\Gamma$ is a P_f-name for Γ. If M is a σ-closed elementary submodel of $\mathcal{H}_\theta$ (in V) such that $f, \dot\Gamma \in M$, $|M| = \omega_1$, and $\delta = M \cap \omega_2 \in \omega_2$ then

(66) $$V[\mathcal{G}] \models \Gamma \cap H(\delta,0) \setminus H[D] \neq \emptyset \text{ for each finite } D \subset \delta.$$

Proof of the lemma 8.9. Fix $D \in [\delta]^{<\omega}$ and a condition $p \in P_f$ with $D \cup \{\delta\} \subset a_p$ such that

(67) $$p \Vdash \text{“}\dot\Gamma \in [\omega_2]^\omega \text{ is not covered by finitely many } H(\xi,0)\text{”}.$$

We shall be done if we can find a condition $r \le p$ and an ordinal $\alpha \in a_r$ such that

(68) $$r \Vdash \text{“}\alpha \in \dot\Gamma\text{” and } \alpha \in h_r(\delta,0) \setminus h_r[D].$$

Let $Q = a_p \cap \delta$, $E = a_p \setminus \delta$, and $\{\gamma_i : i < k\}$ be the increasing enumeration of E. In particular, then we have $\gamma_0 = \delta$.

To achieve our aim, we first choose a *countable* elementary submodel N of $\mathcal{H}_\theta$ such that $M, \dot\Gamma, p \in N$ and put

$$A = \delta \cap N \text{ and } B = cl_f(A \cup Q, E)\,.$$

Note that we have $A, B \in M$ because M is σ-closed. For each $i < k$ the function $f(\,.\,,\gamma_i) \restriction B$ is in M, hence so is the set

$$T_i = \{\gamma \in \omega_2 : \forall \beta \in B\ f(\beta,\gamma) = f(\beta,\gamma_i)\}\,,$$

and $\gamma_i \in T_i \setminus M$ implies $|T_i| = \omega_2$.

By Lemma 3.2 there is a set of $2k$ ordinals

$$F = \{\gamma_{i,\varepsilon} : i < k, \varepsilon < 2\}$$

with $\gamma_{i,\varepsilon} \in T_i$ and $\gamma_{i,0} < \gamma_{i,1}$ for each $i < k$ such that

(69) $$B \cup E \subset \bigcap\{f(\gamma_{i,\varepsilon}, \gamma_{i',\varepsilon'}) : \{\langle i,\varepsilon\rangle, \langle i',\varepsilon'\rangle\} \in [k \times 2]^2\}.$$

Since $a_p \subset B \cup E < F$, (69) implies that we can form the F-extension $q = \langle a_p \cup F, h_q, n_p, i_q\rangle \in P_f$ of p, see definition 8.3.

As $p \Vdash$ “$H[Q \cup E] \not\supset \dot\Gamma$”, there is a condition $t \le q$ and an ordinal α such that

(70) $$t \Vdash \text{“}\alpha \in \dot\Gamma \setminus H[Q \cup E]''.$$

Clearly we can assume that $\alpha \in a_t$, and then

(71) $$t \Vdash \text{“}\alpha \in \dot\Gamma\text{” and } \alpha \in a_t \setminus h_t[Q \cup E].$$

Since $\dot\Gamma \in N \cap M$ and P_f is CCC, we have $\alpha \in M \cap N \cap \omega_2 = N \cap \delta$. As P_f is CCC and $\alpha, \dot\Gamma \in M \cap N$ we may choose a maximal antichain

$W \subset \{w \le p : w \Vdash \alpha \in \dot\Gamma\}$ with $W \in N \cap M$ and hence $W \subset N \cap M$. By taking a further extension we can assume that $t \le w$ for some $w \in W$.

We claim that, putting $S = B \cap a_t$, we have

$$i_t(\xi,\eta,j) \subset S \cup E \text{ for each } \langle \xi,\eta,j\rangle \in \big[S \cup E \cup F\big]^2 \otimes n_p\,. \tag{72}$$

Indeed, if $\xi \in S \subset B$ then fact 8.8 and $\gamma_{i,\varepsilon} \in T_i$ imply $f(\xi,\eta) \subset B$ and so $i_t(\xi,\eta,j) \subset S$, and if $\xi,\eta \in E \cup F$ then

$$i_t(\xi,\eta,j) = i_q(\xi,\eta,j) \subset a_p = Q \cup E \subset S \cup E$$

because q is the F-extension of p.

Let us now make the following definitions:

(s1) $a_s = S \cup E \cup F$,
(s2) $h_s(\xi,j) = h_t(\xi,j) \cap S = h_t(\xi,j) \cap a_s$ for $\xi \in S$ and $j < n_t$,
(s3) $i_s \restriction \big[S\big]^2 \otimes n_t = i_t \restriction \big[S\big]^2 \otimes n_t$,
(s4) for $\eta \in E \cup F$ and $j < n_t$ let

$$h_s(\eta,j) = \begin{cases} h_t(\eta,j) \cap a_s & \text{if } j < n_p,\\ h_t(\eta,0) \cap a_s & \text{if } n_p \le j < n_t,\end{cases} \tag{73}$$

(s5) for $\eta \in E \cup F$, $\xi \in a_s \cap \eta$ and $j < n_t$ let

$$i_s(\xi,\eta,j) = \begin{cases} i_t(\xi,\eta,j) & \text{if } j < n_p,\\ i_t(\xi,\eta,0) & \text{if } n_p \le j < n_t.\end{cases} \tag{74}$$

Then (72) and $t \in P_f$ imply that $s = \langle a_s, h_s, n_t, i_s\rangle \in P_f$, moreover s is an F-fair (even $E \cup F$-fair) extension of q.

Note that $t \le w$ and $a_w \subset A \subset B$ implies $a_w \subset S$, hence by the definition of the condition s we have $s \le w$ and even $s \restriction S \le w$.

Things were set up in such a way that we can apply lemma 8.6 to the three conditions $s \le q \le p$ and the sets $Q \subset S < E < F$ to get a condition $r \in P_f$ such that

- $r \le p$, $r \le s \restriction S \le w$,
- $\alpha \in S \setminus h_s[a_p] \subset h_s(\gamma_0)$.

Since $\delta = \gamma_0$ and $D \subset a_p$, we have $\alpha \in h_r(\delta) \setminus h_r[D]$. Moreover, $r \le s \restriction S \le w$ implies $r \Vdash$ "$\alpha \in \dot\Gamma$". So r satisfies (68), which completes the proof of our lemma. □

Assume now, to finish the proof of (iv), that

$$V[\mathcal{G}] \models\ Y \in \big[\omega_2 \times \mathbb{C}\big]^\omega \text{ and } \overline{Y} \text{ is not compact.} \tag{75}$$

Then, by lemma 8.7, $\Gamma = \{\gamma : \exists x \in \mathbb{C}\ \langle \gamma, x\rangle \in Y\} \in [\omega_2]^\omega$ can not be covered by finitely many $H(\xi,0)$. Let $\dot\Gamma$ be a P_f-name for Γ.

Claim: *If M is a σ-closed elementary submodel of $\mathcal{H}_\theta$ with $f, \dot{\Gamma} \in M$, $|M| = \omega_1$, $\delta = M \cap \omega_2 \in \omega_2$ then $(\{\delta\} \times \mathbb{C}) \cap \overline{Y} \neq \emptyset$.*

Assume, on the contrary, that $(\{\delta\} \times \mathbb{C}) \cap \overline{Y} = \emptyset$. Then, as $U(\delta) \cap \overline{Y}$ is compact, $U(\delta) \cap Y \subset U(\delta) \cap \overline{Y} \subset U[D]$ for some finite set $D \subset \delta$ consequently we have $\Gamma \cap H(\delta, 0) \subset H[D]$. this, however, contradicts lemma 8.9 by which

$$\Gamma \cap H[\delta, D] \neq \emptyset \text{ for each finite } D \subset \delta. \tag{76}$$

This contradiction proves our claim.

Since CH holds in V, the set S of ordinals $\delta \in \omega_2$ that arise in the form $\delta = M \cap \omega_2$ for an elementary submodel $M \prec \mathcal{H}_\theta$ as in the above claim is unbounded (even stationary) in ω_2. Let A be the set of the first ω elements of S. Then $A \in V \cap [\omega_2]^\omega$ and our claim implies that, in $V[\mathcal{G}]$, for each $\delta \in A$ there is $x_\delta \in \mathbb{C}$ with $\langle \delta, x_\delta \rangle \in \overline{Y}$. But then, by lemma 8.2, for $\alpha = \sup A$ we have

$$(\omega_2 \setminus \alpha) \times \mathbb{C} \subset \overline{\{\langle \delta, x_\delta \rangle : \delta \in A\}} \subset \overline{Y}. \tag{77}$$

This completes the proof of theorem 4.2. □

Acknowledgements. Part of this work was done while the second author was visiting the Alfréd Rényi Institute of Mathematics of the Hungarian Academy of Sciences in November 2008. He would like to thank the other authors for their hospitality and providing the partial financial support.

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