ON d-SEPARABILITY OF POWERS AND $C_p(X)$

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Abstract. A space is called d-separable if it has a dense subset representable as the union of countably many discrete subsets. We answer several problems raised by V. V. Tkachuk by showing that

(1) $X^{d(X)}$ is d-separable for every T_1 space X;
(2) if X is compact Hausdorff then X^ω is d-separable;
(3) there is a 0-dimensional T_2 space X such that X^{ω_2} is d-separable but X^{ω_1} (and hence X^ω) is not;
(4) there is a 0-dimensional T_2 space X such that $C_p(X)$ is not d-separable.

The proof of (2) uses the following new result: If X is compact Hausdorff then its square X^2 has a discrete subspace of cardinality $d(X)$.

A space is called d-separable if it has a dense subset representable as the union of countably many discrete subsets. Thus d-separable spaces form a common generalization of separable and metrizable spaces. A. V. Arhangelskii was the first to study d-separable spaces in [1], where he proved for instance that any product of d-separable spaces is again d-separable. In [9], V. V. Tkachuk considered conditions under which a function space of the form $C_p(X)$ is d-separable and also raised a number of problems concerning the d-separability of both finite and infinite powers of certain spaces. He again raised some of these problems in his lecture presented at the 2006 Prague Topology Conference. In this note we give solutions to basically all his problems concerning infinite powers and to one concerning $C_p(X)$.

Let us start by fixing some notation. As usual, see e. g. [3], we denote the density of a space X by $d(X)$. Also following [3] we use $\widehat{s}(X)$ to denote the smallest cardinal λ such that X has *no* discrete subspace of size λ. Thus $\widehat{s}(X) > \kappa$ means that X does have a discrete subspace of cardinality κ. With this we are now in a position to present our first result.

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Theorem 1. *Let κ be an infinite cardinal and X be a T_1 space satisfying $\widehat{s}(X^\kappa) > d(X)$. Then the power X^κ is d-separable.*

Proof. If X itself is discrete then all powers of X are obviously d-separable, hence in what follows we assume that X is not discrete. Consequently, we may pick an accumulation point of X that we fix from now on and denote it by 0. By definition, we may then find a dense subset S of X with $0 \notin S$ and $|S| = d(X) = \delta$. For any non-empty finite set of indices $a \in [\kappa]^{<\omega}$ we have then $|S^a| = \delta$ as well, hence we may fix a one-one indexing $S^a = \{s^a_\xi : \xi < \delta\}$.

Let us next fix an *increasing* sequence $\langle I_n : n < \omega\rangle$ of subsets of κ such that $\bigcup_{n<\omega} I_n = \kappa$ and $|\kappa \setminus I_n| = \kappa$ for each $n < \omega$. It follows from our assumptions then that for every $n < \omega$ there is a *discrete* subspace D_n of the "partial" power $X^{\kappa\setminus I_n}$ such that $|D_n| = \delta$. Thus we may also fix a one-one indexing of D_n of the form

$$D_n = \{y^n_\xi : \xi < \delta\}.$$

The discreteness of D_n means that for each $\xi < \delta$ there is an open set U^n_ξ in $X^{\kappa\setminus I_n}$ such that $U^n_\xi \cap D_n = \{y^n_\xi\}$.

Now fix $n \in \omega$ and pick a non-empty finite subset a of I_n. For each ordinal $\xi < \delta$ we define a point $x^{n,a}_\xi \in X^\kappa$ as follows:

$$x^{n,a}_\xi(\alpha) = \begin{cases} s^a_\xi(\alpha) & \text{if } \alpha \in a, \\ 0 & \text{if } \alpha \in I_n \setminus a, \\ y^n_\xi(\alpha) & \text{if } \alpha \in \kappa \setminus I_n. \end{cases}$$

Having done this, for any $n < \omega$ and $1 \le k < \omega$ we define a subset $E^{n,k} \subset X^\kappa$ by putting

$$E^{n,k} = \{x^{n,a}_\xi : \ a \in [I_n]^k \text{ and } \xi < \delta\}.$$

Now, for n and a as above and for $\xi < \delta$, let $W^{n,a}_\xi$ be the (obviously open) subset of X^κ consisting of those points $x \in X^\kappa$ that satisfy both $x(\alpha) \ne 0$ for all $\alpha \in a$ and $x \restriction (\kappa \setminus I_n) \in U^n_\xi$. Clearly, we have $x^{n,a}_\xi \in W^{n,a}_\xi$ and we claim that

$$W^{n,a}_\xi \cap E^{n,k} = \{x^{n,a}_\xi\}$$

whenever $a \in [I_n]^k$. Indeed, if $b \in [I_n]^k$ and $a \ne b$ then $|a| = |b| = k$ implies that $a \setminus b \ne \emptyset$, hence for any $\alpha \in a \setminus b$ and for any $\eta < \delta$ we have $x^{n,b}_\eta(\alpha) = 0$ showing that $x^{n,b}_\eta \notin W^{n,a}_\xi$. Moreover, for any ordinal $\eta < \delta$ with $\eta \ne \xi$ we have

$$x^{n,a}_\eta \restriction (\kappa \setminus I_n) = y^n_\eta \notin U^n_\xi,$$

hence again $x_\eta^{n,a} \notin W_\xi^{n,a}$. Thus we have shown that each set $E^{n,k}$ is discrete, while their union is trivially dense in X^κ. Consequently, X^κ is indeed d-separable. $\square$

Let us note now that if X is any T_1 space containing at least two points then the power X^κ includes the Cantor cube 2^κ that is known to contain a discrete subspace of size κ. So if we apply this trivial observation to $\kappa = d(X)$, then we obtain immediately from theorem 1 the following corollary which answers problem 4.10 of [9]. This was asking if for every (Tychonov) space X there is a cardinal κ such that X^κ is d-separable.

Corollary 2. *For every T_1 space X the power $X^{d(X)}$ is d-separable.*

Next we show that if X is compact Hausdorff then even X^ω is d-separable, answering the second half of problem 4.2 from [9]. This will follow from the following result that we think is of independent interest.

Theorem 3. *If X is any compact T_2 space then X^2 contains a discrete subspace of size $d(X)$, that is $\widehat{s}(X^2) > d(X)$.*

Proof. Let us assume first that for every non-empty open subspace $G \subset X$ we also have $w(G) \geq d(X) = \delta$. We then define by transfinite induction on $\alpha < \delta$ distinct points $x_\alpha,\ y_\alpha \in X$ together with their *disjoint* open neighbourhoods $U_\alpha,\ V_\alpha$ as follows.

Suppose that $\alpha < \delta$, moreover $x_\beta \in U_\beta$ and $y_\beta \in V_\beta$ have already been defined for all $\beta < \alpha$. Then $\alpha < \delta = d(X)$ implies that there exists a non-empty open set $G_\alpha \subset X$ such that neither x_β nor y_β belongs to $\overline{G_\alpha}$ for $\beta < \alpha$. Let us choose then a non-empty open set H_α such that $\overline{H_\alpha} \subset G_\alpha$ and consider the topology τ_α on $\overline{H_\alpha}$ generated by the traces of the open sets $U_\beta,\ V_\beta$ for all $\beta < \alpha$. Since

$$w(\overline{H_\alpha},\, \tau_\alpha) < \delta \leq w(H_\alpha) \leq w(\overline{H_\alpha}),$$

the topology τ_α is strictly coarser than the compact Hausdorff subspace topology of $\overline{H_\alpha}$ inherited from X, hence τ_α is not Hausdorff. We pick the two points $x_\alpha,\ y_\alpha \in \overline{H_\alpha}$ so that they witness the failure of the Hausdorffness of τ_α. Note that, in particular, this will imply

$$\langle x_\alpha,\, y_\alpha\rangle \notin U_\beta \times V_\beta$$

for all $\beta < \alpha$. We may then choose their disjoint open (in X) neighbourhoods $U_\alpha,\ V_\alpha$ inside G_α. This will clearly imply that we shall also have $\langle x_\alpha,\, y_\alpha\rangle \notin U_\gamma \times V_\gamma$ whenever $\alpha < \gamma < \delta$. Thus, indeed, $\{\langle x_\alpha,\, y_\alpha\rangle : \alpha < \delta\}$ is a discrete subspace of X^2.

Now, assume that X is an arbitrary compact Hausdorff space and call an open set $G \subset X$ *good* if we have $d(H) = d(G)$ for every non-empty open $H \subset G$. Clearly, every non-empty open set has a non-empty good open subset, hence if $\mathcal{G}$ is a maximal disjoint family of good open sets in X then $\bigcup \mathcal{G}$ is dense in X. Consequently we have

$$\sum \{d(G) : G \in \mathcal{G}\} \geq d(X).$$

But for every $G \in \mathcal{G}$ its square G^2 has a discrete subspace D_G with $|D_G| = d(G)$. Indeed, if H is open with $\emptyset \neq \overline{H} \subset G$ then for every non-empty open $U \subset \overline{H}$ we have $w(U) \geq d(U) = d(H) = d(\overline{H})$, so the first part of our proof applies to $\overline{H}$, that is $\overline{H}^2$ (and therefore G^2) has a discrete subspace of size $d(\overline{H}) = d(U)$. It immediately follows that $D = \bigcup \{D_G : G \in \mathcal{G}\}$ is discrete in X^2, moreover

$$|D| = \sum \{d(G) : G \in \mathcal{G}\} \geq d(X),$$

completing our proof.

$\square$

Any compact L-space, more precisely: a non-separable hereditarily Lindelof compact space (e. g. a Suslin line), demonstrates, alas only consistently, that in theorem 3 the square X^2 cannot be replaced by X itself. On the other hand, we should recall here Shapirovskii's celebrated result from [7], see also 3.13 of [3], which states that $d(X) \leq s(X)^+$ holds for any compact T_2 space X. This leads us to the following natural question.

Problem 4. *Is there a ZFC example of a compact T_2 space X that does not contain a discrete subspace of cardinality $d(X)$?*

Since X^2 embeds as a subspace into X^ω, theorems 1 and 3 immediately imply the following.

Corollary 5. *If X is any compact T_2 space then X^ω is d-separable.*

Of course, to get corollary 5 it would suffice to know $\widehat{s}(X^\omega) > d(X)$. Our next result shows, however, that if we know that some finite power of X has a discrete subspace of size $d(X)$ then we may actually obtain a stronger conclusion. To formulate this result we again fix a point $0 \in X$ and introduce the notation

$$\sigma(X^\omega) = \big\{x \in X^\omega \, : \, \{i < \omega : x(i) \neq 0\} \text{ is finite}\big\}.$$

Clearly, $\sigma(X^\omega)$ is dense in X^ω, hence the d-separability of the former implies that of the latter.

Theorem 6. *Let X be a space such that, for some $k < \omega$, the power X^k has a discrete subspace of cardinality $d(X)$. Then $\sigma(X^\omega)$ (and hence X^ω) is d-separable.*

Proof. Let us put again $d(X) = \delta$ and fix a dense set $S \subset X$ with $|S| = \delta$. By assumption, there is a discrete subspace $D \subset X^k$ with a one-one indexing $D = \{d_\xi : \xi < \delta\}$. Also, for each natural number $n \geq 1$ we have $|S^n| = \delta$, so we may fix a one-one indexing $S^n = \{s^n_\xi : \xi < \delta\}$.

Now, for any $1 \leq n < \omega$ and $\xi < \delta$ we define a point $x^n_\xi \in \sigma(X^\omega)$ with the following stipulations:

$$x^n_\xi(i) = \begin{cases} s^n_\xi(i) & \text{if } i < n, \\ d_\xi(i-n) & \text{if } n \leq i < n+k, \\ 0 & \text{if } n+k \leq i < \omega. \end{cases}$$

It is straight-forward to check that each $D_n = \{x^n_\xi : \xi < \delta\} \subset \sigma(X^\omega)$ is discrete, moreover $\bigcup_{n<\omega} D_n$ is dense in $\sigma(X^\omega)$.

$\square$

Actually, before we get too excited, let us point out that the d-separability of X^ω implies that some finite power of X has a discrete subspace of cardinality $d(X)$, in "most" cases, namely if $\mathrm{cf}(d(X)) > \omega$. Indeed, first of all, in this case there is a discrete $D \subset X^\omega$ with $|D| = d(X^\omega) = d(X)$. Secondly, for each point $x \in D$ there is a finite set of co-ordinates $a_x \in [\omega]^{<\omega}$ that supports a neighbourhood U_x of x such that $D \cap U_x = \{x\}$. But by $\mathrm{cf}(|D|) > \omega$ then there is some $a \in [\omega]^{<\omega}$ with $|\{x \in D : a_x = a\}| = |D| = d(X)$, and we are clearly done.

Let us mention though that the d-separability of the power X^ω does not imply that of some finite power of X. In fact, the Čech–Stone remainder ω^* demonstrates this because its ω^{th} power is d-separable by theorem 6 but no finite power of ω^* is d-separable, as it was pointed out in [9, 3.16 (b)].

Next we give a negative solution to one more problem of Tkachuk concerning the d-separability of powers. Problem 4.9 from [9] asks if the d-separability of some infinite power X^κ implies the d-separability of the countable power X^ω. We recall that a strong L-space is a non-separable regular space all finite powers of which are hereditarily Lindelöf.

Theorem 7. *Let X be a strong L-space with $d(X) = \omega_1$. Then X^{ω_1} is d-separable but X^ω is not. Moreover, there is a ZFC example of a 0-dimensional T_2 space Y such that Y^{ω_2} is d-separable but Y^{ω_1} (and hence Y^ω) is not.*

Proof. It is immediate from corollary 2 that X^{ω_1} is d-separable. Also, since all finite powers of X are hereditarily Lindelöf so is X^ω, hence

$$s(X^\omega) = \omega < \omega_1 = d(X^\omega)$$

implies that X^ω cannot be d-separable.

To see the second statement, we use Shelah's celebrated coloring theorem from [8], which says that $Col(\lambda^+,2)$ holds for every uncountable regular cardinal λ, together with theorem [4, 1.11 (i)] saying that $Col(\lambda^+,2)$ implies the existence of a 0-dimensional T_2 space Y that is a strong L_λ space. The latter means that $hL(Y^n) \le \lambda$ for all finite n but $d(Y) > \lambda$. Without loss of generality, we may assume that $d(Y) = \lambda^+$. Thus from from corollary 2 we conclude that the power Y^{λ^+} is d-separable.

On the other hand, a simple counting argument as above yields that

$$s(Y^\lambda) \le hL(Y^\lambda) \le \lambda < \lambda^+ = d(Y) = d(Y^\lambda),$$

hence Y^λ obviously cannot be d-separable. In particular, if $\lambda = \omega_1$ then we obtain our claim. □

Finally, our next result answers the first part of problem 4.1 from [9] that asks for a ZFC example of a (Tychonov) space X such that $C_p(X)$ is not d-separable. (The second part asks the same for compact spaces.)

Theorem 8. *If $Col(\kappa,2)$ holds for some successor cardinal $\kappa = \lambda^+$ then the Cantor cube of weight κ, $D(2)^\kappa$, has a dense subspace X such that $C_p(X)$ is not d-separable. Moreover, if X is a compact strong S_λ space of weight λ^+ then $C_p(X)$ is not d-separable.*

Proof. It was shown in [5, 6.4] (and mentioned in [4, 1.11]) that $Col(\kappa,2)$ implies the existence of a strong κ-HFD$_w$ subspace $Y = \{y_\alpha : \alpha < \kappa\}$ of $D(2)^\kappa$ with the additional property that $y_\alpha(\beta) = 0$ for $\beta < \alpha < \kappa$. It is also well-known (see e. g. [3, 5.4]) that $D(2)^\kappa$ has a dense subspace Z of cardinality λ. Let us now set $X = Y \cup Z$.

As Y is a strong κ-HFD$_w$, we have $s(Y^n) \le hd(Y^n) \le \lambda$ for each finite n and it is easy to see that then we also have $s(X^n) \le hd(X^n) \le \lambda$ whenever $n < \omega$. It was also pointed out in [5, 6.5] that every (relatively) open subset G of Y (and hence of X) satisfies either $|G| \le \lambda$ or $|Y \backslash G| \le \lambda$ (resp. $|X \backslash G| \le \lambda$). This in turn obviously implies that no family $\mathcal{U}$ of open subsets of Y (resp. X) with $|\mathcal{U}| < \kappa$ can separate its points, hence we have

$$iw(X) = iw(Y) = \kappa > \lambda.$$

But then by [9, 3.6] neither $C_p(X)$ nor $C_p(Y)$ is d-separable. As we have noted above, $Col(\omega_2, 2)$ is provable in ZFC, so in particular we may conclude that the Cantor cube of weight ω_2 has a dense subspace X such that $C_p(X)$ is not d-separable.

To see the second statement of our theorem, consider a compact strong S_λ space X. This means that for each natural number n we have $s(X^n) \le hd(X^n) \le \lambda$ but $hL(X) > \lambda$. It is well-known that we may assume without any loss of generality that $w(X) = \lambda^+$ holds as well. But now the compactness of X immediately implies $iw(X) = w(X)$, hence again by [9, 3.6] the function space $C_p(X)$ is not d-separable.

□

It is an intriguing open question if the existence of a cardinal λ for which there is a compact strong S_λ space is provable in ZFC. Note that by theorem 3 there is no compact strong L_λ space for any cardinal λ. On the other hand, the existence of compact strong S (i. e. S_ω) spaces was shown to follow from CH by K. Kunen, see e. g. [2, 2.4] and [6, 7.1].

References

[1] A. V. Arhangelskii, *On d-separable spaces*, in: Proceedings of the Seminar in General Topology, P. S. Alexandrov ed., Mosk. Univ. P. H., 1981, 3–8.

[2] R. de la Vega and K. Kunen, *A Compact Homogeneous S-space*, Top. Appl. 136 (2004), 123 - 127.

[3] I. Juhász, *Cardinal functions – ten years later*, Math. Center Tract no. 123, Amsterdam, 1980

[4] I. Juhász, *Cardinal functions*, In: Recent Progress in General Topology, M. Hušek and J. van Mill, eds., North-Holland, 1992, 417–441.

[5] I. Juhász, *HFD and HFC type spaces*, Top. Appl. 126 (2002), 217–262.

[6] S. Negrepontis, *Banach spaces and topology*, in: Handbook of Set-Theoretic Topology, K. Kunen and J. E. Vaughan, eds., North Holland, Amsterdam, 1984, 1045–1142.

[7] B. Shapirovskii, *Canonical sets and character*, Soviet Math. Dokl. 15 (1974), 1282–1287.

[8] S. Shelah, *Colouring and non-productivity of* $\aleph_2$*-cc,* Ann. Pure Appl. Logic 84 (1997), 153–174.

[9] V. V. Tkachuk, *Function spaces and d-separability*, Questiones Mathematicae **28** (2005), 409–424.

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