ELEMENTARY SUBMODELS IN INFINITE COMBINATORICS

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Abstract. We show that usage of elementary submodels is a simple but powerful method to prove theorems, or to simplify proofs in infinite combinatorics. First we introduce all the necessary concepts of logic, then we prove classical theorems using elementary submodels. We also present a new proof of Nash-Williams's theorem on cycle-decomposition of graphs, and finally we obtain some new decomposition theorems by eliminating GCH from some proofs concerning bond-faithful decompositions of graphs.

1. Introduction

The aim of this paper is to explain how to use elementary submodels to prove new theorems or to simplify old proofs in infinite combinatorics. The paper is a combination of an expository article and a research paper: we introduce all the necessary concept and give easy examples to illustrate our method, but the paper also contains new proofs of theorems of Nash-Williams on decomposition of infinite graphs, and some new results concerning bond-faithful decompositions.

Our aim is to popularize a *method* instead of giving just "*black box*" theorems.

In Section 2 we recall and summarize all necessary preliminaries from set theory, combinatorics and logics.

In section 3 we give the first application of elementary submodels, and we explain why it is natural to consider Σ-elementary submodels for some large enough finite family Σ of formulas.

In section 4 we use elementary submodels to prove some classical theorems in combinatorial set theory. All these theorems have the

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following Ramsey-like flavor: *Every large enough structure contains large enough "nice" substructure.*

In section 5 we prove structure theorems of different kind: *Every large structure having certain properties can be partitioned into small "nice" pieces.* A typical example is Nash-Williams theorem on cycle decomposition of graphs without odd cuts. To prove these structure theorems it is not enough to consider just one elementary submodel but we should introduce the concept of the *chain of elementary submodels.*

Finally, in section 6, we give a more elaborated application of chain of elementary submodels to eliminate GCH from a theorem concerning bond-faithful decomposition of graphs.

2. Preliminaries

2.1. **Set theory.** We use the standard notion and notation of set theory, see [5] or [7]. If κ is a cardinal and A is a set, let

$$(1) \qquad \big[A\big]^{<\kappa} = \{a \subset A : |a| < \kappa\}; \qquad \big[A\big]^{\kappa} = \{a \subset A : |a| = \kappa\}.$$

If X and Y are sets let $[X,Y] = \big\{\{x,y\} : x \in X, y \in Y\big\}$.

We denote by V the class of all sets, and by On the class of all ordinals. The *cumulative hierarchy* $\langle V_\alpha : \alpha \in On\rangle$ is defined by transfinite induction on α as follows:

(1) $V_0 = \emptyset$,
(2) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
(3) $V_\beta = \cup\{V_\alpha : \alpha < \beta\}$ if β is a limit ordinal.

Fact 2.1. *$V = \cup\{V_\alpha : \alpha \in On\}$, i.e. for each set x there is an ordinal α such that $x \in V_\alpha$.*

2.2. **Combinatorics.** We use the standard notion and notation of combinatorics, see e.g. [1]. A *graph* G is a pair $\langle V(G), E(G)\rangle$, where $E(G) \subset \big[V(G)\big]^2$. $V(G)$ and $E(G)$ are the set of *vertices and edges*, respectively, of G. If $X \subset V[G]$ then $G[X]$ denotes the induced subgraph on X. If M is a set then let

$$G \restriction M = \langle V(G) \cap M, E(G) \cap M\rangle; \qquad G \setminus M = \langle V(G), E(G) \setminus M\rangle.$$

If G is fixed, and $A \subset V(G)$ then we write $\bar{A}$ for $V(G) \setminus A$. A *cut* of G is a set of edges of the form $E(G) \cap [A, \bar{A}]$ for some $A \subset V(G)$. A *bond* is a non-empty cut which is minimal among the cuts with respect to inclusion.

Fact 2.2. *$\emptyset \neq F \subset E(G)$ is a bond in G iff there are two distinct connected components C_1 and C_2 of $G\setminus F$ such that $F = E(G) \cap [C_1, C_2]$.*

The following statement will be used later several times.

Proposition 2.3. *Assume that H is a subgraph of G, F is a bond in H. If F is not a bond in G then $F \subset \left[D\right]^2$ for some connected component D of G.*

Proof. By fact 2.2 there are two distinct connected components C_1 and C_2 of $H \setminus F$ such that $F = E(H) \cap [C_1, C_2]$. If that C_1 and C_2 are subsets of different connected components of G, $C_1 \subset D_1$ and $C_2 \subset D_2$, then

$$F = [C_1, C_2] \cap E(H) \subset [D_1, D_2] \cap E(G) \subset F \cup ([D_1, D_2] \cap E(G \setminus F)) = F,$$

i.e. $F = [D_1, D_2] \cap E(G)$ and so F is a bond in G by Fact 2.2 above. This is not the case, so C_1 and C_2 are subsets of the same connected component D of $G \setminus F$. Thus $F \subset [C_1, C_2] \subset \left[D\right]^2$. □

Given a graph G for $x \neq y \in V(G)$ denote $\gamma_G(x, y)$ the *edge connectivity* of x and y in G, i.e.

$$\gamma_G(x, y) = \min\{|F| : F \subset E(G) : F \text{ separates } x \text{ and } y \text{ in } G\}.$$

By the weak Erdős-Menger Theorem there are $\gamma_G(x, y)$ many edge disjoint paths between x and y in G.

A *κ-cover* of a graph G is a family $\mathcal{G}$ of subgraphs of G such that every edge of G belongs to exactly κ members of the family $\mathcal{G}$. A *decomposition* is a 1-cover, i.e. a family $\mathcal{H}$ such that $\{E(G') : G' \in \mathcal{G}\}$ is a partition of $E(G)$.

2.3. **Logic.** The language of set theory is the first order language $\mathcal{L}$ containing only one binary relation symbol $\in$. So the formulas of $\mathcal{L}$ are over the alphabet $\{\vee, \neg, (,)\exists, =, \in\} \cup \mathrm{Var}$, where Var is an infinite set of variables. To simplify our formulas we often use abbreviations like $\forall x$, $\rightarrow$, $x \subset y$, $\exists! x$, $\exists x \in y\ \varphi$, etc.

An *$\mathcal{L}$-structure* is a pair $\langle M, E\rangle$, where $E \subset M \times M$. In this paper we will consider only structure in the form $\langle M, \in\restriction M\rangle$ where $\in\restriction M$ is the restriction of the usual membership relation to M, i.e.

$$\in\restriction M = \{\langle x, y\rangle \in M \times M : x \in y\}.$$

We usually write $\langle M, \in\rangle$ or simply M for $\langle M, \in\restriction M\rangle$.

If $\varphi(x_1, \ldots, x_n)$ is a formula, $a_1, \ldots, a_n$ are sets, then let $\varphi(a_1, \ldots, a_n)$ be the formula obtained from $\varphi(x_1, \ldots, x_n)$ by replacing each free occurrence of x_i with a_i. [An occurrence of x_i is *free* it is not within the scope of a quantifier $\exists x_i$.]

If $\varphi(x, x_1, \ldots, x_n)$ is a formula, $a_1, \ldots, a_n$ are sets, then $C = \{a : \varphi(a, a_1, \ldots, a_n)\}$ is a *class*. Especially, every set b is a class: $b = \{a : a \in b\}$. Moreover, all sets form the class V: $V = \{a : a = a\}$. In this

paper we will consider just these classes: the sets and the "universal" class V.

For a formula $\varphi(x_1,\dots,x_n)$, a class M, and for $a_1,\dots,a_n\in M$ we define when

$$M\models\varphi(a_1,\dots,a_n), \tag{2}$$

i.e. when M *satisfies* $\varphi(a_1,\dots,a_n)$, by induction on the complexity of the formulas in the usual way:

(i) $M\models$ "$a_i\in a_j$" iff $a_i\in a_j$,
(ii) $M\models$ "$\varphi\vee\psi$" iff $M\models\varphi$ or $M\models\psi$.
(iii) $M\models$ "$\neg\varphi$" iff $M\models\varphi$ fails.
(iv) $M\models$ "$\exists x\varphi(x,a_1,\dots a_n)$" iff there is an $a\in M$ such that $M\models$ "$\varphi(a,a_1,\dots,a_n)$"

For a formula $\varphi(x_1,\dots,x_n)$ let $\varphi^M(x_1,\dots,x_n)$ be the formula obtained by replacing each quantifier $\exists x$ with $\exists x\in M$ in φ. Clearly for each $a_1,\dots,a_n\in M$,

$$\varphi^M(a_1,\dots,a_n)\text{ iff }M\models\varphi(a_1,\dots,a_n). \tag{3}$$

If $\varphi(x_1,\dots,x_n)$ is a formula, M and N are classes, $M\subset N$, then we say that φ is *absolute between M and N,*

$$M\prec_\varphi N \tag{4}$$

in short, iff for each $a_1,\dots,a_n\in M$

$$M\models\varphi(a_1,\dots,a_n)\text{ iff }N\models\varphi(a_1,\dots,a_n) \tag{5}$$

If Σ is a collection of formulas then write

$$M\prec_\Sigma N \tag{6}$$

iff $M\prec_\varphi N$ for each $\varphi\in\Sigma$.

M is an *elementary submodel* of N,

$$M\prec N \tag{7}$$

iff $M\prec_\varphi N$ for each formula φ.

If φ is absolute between M and V, where V is the class of all sets, then we say that φ is *absolute for M*.

Theorem 2.4 (Löwenheim-Skolem)**.** *For each set N and infinite subset $A\subset N$ there is a set M such that $A\subset M\prec N$ and $|M|=|A|$.*

Since $ZFC\not\vdash Con(ZFC)$ by Gödel Second Incompleteness Theorem, it is not provable in ZFC that there is a set M with $M\models ZFC$. So, since $V\models ZFC$, it is not provable in ZFC that there is a set M with $M\prec V$. Thus, in Löwenheim-Skolem theorem above, the assumption

that N is a set was essential. However, as we will see, the following result can serve as a substitute of Löwenheim-Skolem theorem for classes in certain cases.

Theorem 2.5 (Reflection Principle)**.** *Let Σ be a finite collection of formulas. Then for each cardinal κ there is a cardinal λ such that $V_\lambda \prec_\Sigma V$, and $\left[V_\lambda\right]^{<\kappa} \subset V_\lambda$ (i.e. $\mathrm{cf}(\lambda) \geq \kappa$).*

We need some corollaries of this theorem.

Corollary 2.6. *Let Σ be a finite collection of formulas, κ an infinite cardinal, and x a set.*
(1) There is a set $M \prec_\Sigma V$ with $x \in M$ and $|M| = \kappa$.
(2) If $\kappa > \omega$ is regular then there is a set $M \prec_\Sigma V$ with $x \in M$, $|M| < \kappa$ and $M \cap \kappa \in \kappa$.
(3) If $\kappa^\omega = \kappa$ then then there is a set $M \prec_\Sigma V$ such that $x \in M$, $|M| = \kappa$, $M \cap \kappa^+ \in \kappa^+$, and $\left[M\right]^\omega \subset M$.
(4) If $\kappa > \omega$ is regular then the set

$$S_x = \{M \cap \kappa : x \in M \prec_\Sigma V, M \cap \kappa \in \kappa\}$$

contains a closed unbounded subset of κ.

Proof. We can assume $x \in V_\kappa$. By the Reflection Principle there is a cardinal $\lambda > \kappa$ such that $V_\lambda \prec_\Sigma V$ and $\left[V_\lambda\right]^\kappa \subset V_\lambda$.

(1) Straightforward from Löwenheim-Skolem theorem: since V_λ is a set, $|V_\lambda| \geq \kappa$, and $x \in V_\lambda$ there is $M \prec V_\lambda$ with $x \in M$ and $|M| = \kappa$. Then $M \prec_\Sigma V$.

(2) Construct a sequence $\langle M_n : n < \omega\rangle$ of elementary submodels of V_λ with $|M_n| < \kappa$ as follows. Let M_0 be a countable elementary submodel of V_λ with $x \in M$. If M_n is constructed, let $\alpha_n = \sup(M_n \cap \kappa)$. Since κ is regular we have $\alpha_n < \kappa$. By Löwenheim-Skolem theorem there is an elementary submodel M_{n+1} of V_λ such that of $M_n \cup \alpha_n \subset M_{n+1}$ and $|M_{n+1}| = |M_n \cup \alpha_n| < \kappa$. Finally let $M = \cup\{M_n : n < \omega\}$. Then $M \prec V_\lambda$, and so $M \prec_\Sigma V$, and $M \cap \kappa = \sup \alpha_n \in \kappa$.

(3) Construct an increasing sequence $\langle M_\nu : \nu < \omega_1\rangle$ of elementary submodels of V_λ with $|M_\nu| = \kappa$ as follows. Let M_0 be an elementary submodel of V_λ with $\kappa \cup \{x\} \subset M_0$ and $|M_0| = \kappa$. For limit ν let $M_\nu = \cup\{M_\beta : \beta < \nu\}$. If M_ν is constructed, let $\alpha_\nu = \sup(M_\nu \cap \kappa^+)$. Since $|M_\nu| = \kappa$ we have $\alpha_\nu < \kappa^+$. Let $X_\nu = M_\nu \cup \alpha_\nu \cup \left[M_\nu\right]^\omega$. Then $|X_\nu| \leq \kappa^\omega = \kappa$. By Löwenheim-Skolem theorem there is an elementary submodel $M_{\nu+1}$ of V_λ with $X_\nu \subset M_{\nu+1}$ and $|M_{\nu+1}| = \kappa$. Finally let $M = \cup\{M_\nu : \nu < \omega_1\}$. Since $\kappa \geq \omega_1$, $M \cap \kappa^+ = \sup\{\alpha_\nu : \nu < \omega_1\} \in \kappa^+$.

If $A \in \big[M\big]^{\omega}$ then there is $\nu < \omega_1$ with $A \subset M_\nu$, and so $A \in X_\nu \subset M_{\nu+1} \subset M$.

(4) Construct a continuous increasing chain of elementary submodels $\langle M_\nu : \nu < \kappa\rangle$ of V_λ with $|M_\nu| \leq \nu+\omega$ as follows. Let M_0 be a countable elementary submodel of V_λ with $x \in M$. For limit ν let $M_\nu = \cup\{M_\beta : \beta < \nu\}$. If M_ν is constructed, let $\alpha_\nu = \sup(M_\nu \cap \kappa^+)$. Since $|M_\nu| < \kappa$ and κ is regular we have $\alpha_\nu < \kappa$. Let $X_\nu = M_\nu \cup (\alpha_\nu + 1)$. Since $|X_\nu| \leq \nu + \omega$, by Löwenheim-Skolem theorem there is an elementary submodel M_ν of V_λ with $X_\nu \subset M_\nu$ and $|M_\nu| = |X_\nu|$.

Then $C = \{\alpha_\nu : \nu < \kappa\}$ is a closed unbounded subset of κ and $C \subset S_x$ because $\alpha_\nu \in S_x$ is witnessed by M_ν. □

2.4. **Absoluteness.** A set b is *definable from parameters* $a_1, \dots, a_n$ iff there is a formula $\varphi(x)$ such that

$$\forall x(\varphi(x, a_1, \dots, a_n) \leftrightarrow x = b). \tag{8}$$

We say that b is *definable* iff we do not need any parameters, i.e. $\forall x(\varphi(x) \leftrightarrow x = b)$.

Claim 2.7. *If b is definable from $a_1, \dots a_n \in M$, $M \prec_{\{\exists x\varphi(x,\vec{y}),\varphi(x,\vec{y})\}} V$ then $b \in M$.*

Proof. Since $M \prec_{\exists x\varphi(x,\vec{y}} V$, $\vec{a} \in M$ and so $M \models \exists x\varphi(x, \vec{a})$, there is $b' \in M$ such that $M \models \varphi(b', \vec{a})$. Thus $M \prec_{\varphi(x,\vec{y})} V$ yields $V \models \varphi(b', \vec{a})$, and so $b = b' \in M$. □

Given a class N we say that a formula $\varphi(x_1, \dots, x_n, y)$ *defines the operation* F^N_φ *in* N iff $N \models \forall x_1, \dots, , x_n \exists! y\varphi(x_1, \dots, x_n, y)$, and for each $a_1, \dots, a_n, b \in N$, $F^N_\varphi(a_1, \dots, a_n) = b$ iff $N \models \varphi(a_1, \dots, a_n, b)$. If $V = N$ then we omit the superscript V.

Given a class N we say that the operation F_φ is *absolute for* N provided φ defines an operation in N, and $\varphi(\vec{x}, y)$ is absolute for N.

Claim 2.8. *If the formula φ defines the operation F_φ in V, moreover $M \prec_{\{\forall\vec{x}\exists y\varphi(\vec{x},y),\varphi(\vec{x},y)\}} V$ then φ defines an operation F^M_φ in M, and $F^M_\varphi = F_\varphi \restriction M$.*

Proof. Since $M \prec_{\forall\vec{x}\exists y\varphi(\vec{x},y)} V$, for each $a_1, \dots, a_n \in M$ there is $b \in M$ such that $M \models \varphi(\vec{a}, b)$. Thus $V \models \varphi(\vec{a}, b)$, and so $F_\varphi(\vec{a}) = b \in M$. If $M \models \varphi(\vec{a}, b) \wedge \varphi(\vec{a}, b')$ then $V \models \varphi(\vec{a}, b) \wedge \varphi(\vec{a}, b')$, so $b = b'$. Thus $M \models \forall\vec{x}\exists! y\varphi(\vec{x}, y)$. □

3. First application of elementary submodels.

In this section we present an example

- to illustrate our basic method,
- to indicate the main technical problem of this approach; and also
- to give a solution to that technical problem.

In [9] Nash-Williams proved that *a graph G is decomposable into cycles if and only if it has no odd cut.* In Section 5 we give a new proof of this result. Let us say that a graph G is NW iff it does not have any odd cut. We will prove Nash-Williams Theorem by induction on $|V(G)|$. Since the statement is trivial for countable graphs, it is enough to decompose an uncountable NW-graph G into NW-graphs of smaller cardinality. We will use "small" elementary submodels to cut the graph G into the right pieces. To do so we need two lemmas, the first (and easy) one will serve as the first example of the application of our method.

Lemma 3.1. *If $G = \langle W, E\rangle$ is an NW-graph, $G \in M \prec V$, then $G \restriction M = G[M \cap W]$ is also an NW-graph.*

Proof. Assume on the contrary that $G \restriction M$ has an odd cut $F = \{f_1, \dots, f_{2n+1}\}$. Since any cut is the disjoint union of bonds we can assume that F is a bond. Since F can not be a bond in G, by Proposition 2.3 there is a connected component D of $G \setminus F$ such that $F \subset \big[D\big]^2$. Let $bc \in F$. Then b and c are in D, D is connected, so there is a path $bw_1w_2 \dots w_{m-1}c$ between b and c in G which avoids F.

Claim 3.2. $\big[M\big]^{<\omega} \subset M$.

Proof of the claim. Consider the operations $F_1(x, y) = \{x, y\}$ and $F_2(z) = \cup z$. By Claim 2.8, there are formulas $\sigma_1, \sigma_1', \sigma_2$ and σ_2' such that if $N \prec_{\{\sigma_i, \sigma_i'\}} V$ then N is closed under operation F_i, $i = 1, 2$.

Since $M \prec V$, this yields that M is closed under F_1 and F_2. Since

$$\{a_0, \dots, a_n\} = \cup\{\{a_0, \dots, a_{n-1}\}, \{a_n\}\} \tag{9}$$

we obtain $\big[M\big]^{<\omega} \subset M$ by induction on n. □

Claim 3.3. $\omega \cup \{\omega\} \subset M$.

Proof of the Claim. $\emptyset$ and ω are definable, so by Claim 2.7 there are formulas ρ_1 and ρ_1', and ρ_2 and ρ_2', respectively, such that if $N \prec_{\{\rho_1, \rho_1'\}} V$ then $\emptyset \in N$, and if $N \prec_{\{\rho_2, \rho_2'\}} V$ then $\omega \in N$. Since $M \prec V$, this implies $\emptyset, \omega \in M$.

Consider the operation $F_3(x) = x \cup \{x\}$. By Claim 2.8, there are formulas σ_3 and σ_3' such that if $N \prec_{\{\sigma_3,\sigma_3'\}} V$ then N is closed under operation F_3. Since $M \prec V$, this yields that M is closed under F_3. So $0 \in M$ and $n+1 = F_3(n)$ imply $\omega \subset M$. □

So we have $F \in M$ and $m \in M$. Consider the following formula $\varphi_1(G, m, f, b, c, F)$:

(10) G is a graph, f is a function, $\mathrm{dom}(f) = m$, $\mathrm{ran}(f) \subset V(G)$, $f(0) = b, f(m-1) = c \wedge (\forall i < m-1)\ \{f(i), f(i+1)\} \in E(G) \setminus F$.

Since

$$\exists f\ \varphi_1(G, m, f, b, c, F), \tag{11}$$

the assumption $M \prec_{\exists f \varphi_1(G,m,f,b,c,F)} V$ and $G, m, b, c, F \in M$ imply that the same formula holds in M. So there is $f \in M$ such that

$$\varphi_1(G, m, f, b, c, F). \tag{12}$$

Since $M \prec_{\varphi_1(G,m,f,b,c,F)} V$ we have

$$\varphi_1(G, m, f, b, c, F). \tag{13}$$

To complete the proof we need one more claim.

Claim 3.4. *If $g \in M$ is a function, $x \in \mathrm{dom}(g)$, then $g(x) \in M$.*

Proof of the Claim. Consider the evaluation operation $F_4(g, y) = g(y)$. By Claim 2.8, there are formulas σ_4 and σ_4' such that if $N \prec_{\{\sigma_4,\sigma_4'\}} V$ then N is closed under operation F_4. Since $M \prec V$, this yields that M is closed under the evaluation operation F_4. □

By Claim 3.4 above, $\mathrm{ran}(f) \subset M \cap W$, and so $f(0)f(1)\dots f(m-1)$ is a path between b and c in $G \restriction M$ which avoids F. Contradiction. □

So if M is a "small" elementary submodel of V, then $G \restriction$ is a "small" NW-subgraph of G. Unfortunately, as we explained before the formulation of the Reflection Principle, we can not get any set M with $M \prec V$ by the Second Incompleteness Theorem of Gödel. So we can not apply the lemma above to prove the Nash-William Theorem.

Fortunately, this is just a technical problem because one can observe that in the proof above we have not used the full power of $M \prec V$, we applied the absoluteness only for finitely many formulas between V and M. Namely, we used only the absoluteness for the formulas from the family

$$\Sigma^* = \{\sigma_i, \sigma_i', : i = 1, 2, 3, 4\} \cup \{\rho_j, \rho_j', : j = 1, 2\} \cup \{\exists f \varphi_1, \varphi_1\}. \tag{14}$$

So actually the proof of lemma 3.1 yields the following result:

Lemma 3.5. *If $G = \langle W, E\rangle$ is an NW-graph, $G \in M \prec_\Sigma V$ for some large enough finite set Σ of formulas, then $G \restriction M$ is also an NW-graph.*

In many proofs we will argue in the following way:

(I) using the Reflection Principle we can find a cardinal λ such that V_λ resembles to V in two ways:
  (1) $\left[V_\lambda\right]^\kappa \subset V_\lambda$ for some large enough cardinal κ, and
  (2) $V_\lambda \prec_\Sigma V$ for some large enough finite collection Σ of formulas.

We can not use the model V_λ directly, because it is too large, but

(II) since V_λ is a set, we can use Löwenheim-Skolem Theorem to find a small elementary submodel M of V_λ which contains G.

Then $M \prec_\Sigma V$.

We do not fix Σ in advance. Instead of this we write down the proof, and after that we put all the formulas for which we used the absoluteness into Σ. Actually, apart from the proof of lemma 3.5 above, we will not construct Σ explicitly.

Remark . We will show later that if Σ is large enough then $G \setminus M$ is also an NW-graph, so the pair $\langle G \restriction M, G \setminus M\rangle$ is a decomposition of G into NW-graphs.

3.1. **More on absoluteness.** In Claim 3.6 below we summarize certain observations we made in the proof of lemma 3.1 above.

Claim 3.6. *There is a finite collection Σ_0 of formulas such that if $M \prec_{\Sigma_0} V$ then $\left[M\right]^{<\omega} \subset M$, $\omega \cup \{\omega\} \subset M$, and $f(x) \in M$ for each function $f \in M$ and $x \in \mathrm{dom}(f) \cap M$.*

We need two more easy claims.

Claim 3.7. *There is a finite collection Σ_1 of formulas such that if $M \prec_{\Sigma_1} V$ then for each $A \in M$ if $|A| \subset M$ then $A \subset M$.*

Proof. Let $\Sigma_1 \supset \Sigma_0$ be a finite family of formulas such that

(1) the formulas "*f is a bijection between x and y*" and "$\exists f$ *(f is a bijection between x and y)*" are in Σ_1,
(2) if $M \prec_{\Sigma_1} V$ then M is closed under the "cardinality" operation $A \mapsto |A|$.

Assume that $|A| = \kappa$. Then $\kappa \in M$ by (2). Since $V \models$ "$\exists f$ *f is a bijection between κ and A*" there is $f \in M$ such that $M \models$ "*f is a bijection from κ onto A*". Then f is a bijection from κ to A by (1). So if $a \in A$ then there is $\alpha \in \kappa$ such that $f(\alpha) = a$. However, $f, \alpha \in M$ so $f(\alpha) \in M$ by $\Sigma_1 \supset \Sigma_0$. Thus $A \subset M$. □

Claim 3.8. *If $M \prec_{\Sigma_0\cup\Sigma_1} V$ then for each countable set $A \in M$ we have $A \subset M$.*

Proof. If A is countable then $|A| = \omega \subset M$ by Claim 3.6 because $M \prec_{\Sigma_0} V$. Thus $A \subset M$ by Claim 3.7 because $M \prec_{\Sigma_1} V$. □

4. Classical theorems

In this section we prove some classical theorems using elementary submodels.

A family $\mathcal{A}$ is called Δ-*system with kernel* D iff $A \cap A' = D$ for each $A \neq A' \in \mathcal{A}$. A Δ-*system* is a Δ-system with some kernel.

Theorem 4.1. *Every uncountable family $\mathcal{A}$ of finite sets contains an uncountable Δ-system.*

Proof. We can assume that $\mathcal{A} \subset \left[\omega_1\right]^{<\omega}$.

Let Σ be a large enough finite set of formulas. By Corollary 2.6(1) there is a countable set M such that $\mathcal{A} \in M \prec_\Sigma V$.

Since $\mathcal{A}$ is uncountable, we can pick $A \in \mathcal{A} \setminus M$. Let $D = M \cap A$. Since $\left[M\right]^{<\omega} \subset M$ we have $D \in M$ by Claim 3.6. Let

$$\mathbb{B} = \{\mathcal{B} \subset \mathcal{A} : \mathcal{B} \text{ is a } \Delta\text{-system with kernel } D\}. \tag{15}$$

Since $\mathcal{A}, D \in M$ we have $\mathbb{B} \in M$ as well. Moreover,

$$\exists \mathcal{B}\ (\mathcal{B} \text{ is a } \subset\text{-maximal element of } \mathbb{B}). \tag{16}$$

Since $M \prec_\Sigma V$, and the parameter $\mathbb{B}$ is in M, there is $\mathcal{B} \in M$ such that

$$M \models\ (\mathcal{B} \text{ is a } \subset\text{-maximal element of } \mathbb{B}). \tag{17}$$

Since $M \prec_\Sigma V$, we have

$$\mathcal{B} \text{ is a } \subset\text{-maximal element of } \mathbb{B}. \tag{18}$$

Claim: $\mathcal{B}$ *is uncountable.*
Assume on the contrary that $\mathcal{B}$ is countable. Then, by claim 3.8, $M \prec_\Sigma V$ implies $\mathcal{B} \subset M$. Let $\mathcal{C} = \mathcal{B} \cup \{A\}$. Since $A \notin M$, $\mathcal{C} \supsetneq \mathcal{B}$. If $B \in \mathcal{B}$, then $B \in M$ and so $B \subset M$ and $D \subset A \cap B \subset A \cap M = D$. So $\mathcal{C} \supsetneq \mathcal{B}$ is a Δ-system with kernel D, i.e. $\mathcal{B}$ was not a $\subset$-maximal element of $\mathbb{B}$. This contradiction proves the claim. □

Remark . In each proofs of this section we will argue in the following way. Let $\mathcal{A}$ be a structure of "size" κ. Let $M \prec_\Sigma V$ for some large enough finite family Σ of formulas with $\mathcal{A} \in M$ and $|M| < \kappa$, i.e. M is a "small" elementary submodel which contains, as element, a "large" structure $\mathcal{A}$. Since M has less element than the size of $\mathcal{A}$, there is A

from $\mathcal{A}$ such that $A \notin M$. Then this A has some "trace" D on M. If M is "closed enough" then this trace D is in M. Using this trace we define, in M, a maximal, "nice" substructure $\mathcal{B}$ of $\mathcal{A}$. Then, using the fact that $A \notin M$, we try to prove that $\mathcal{B}$ is large "enough".

In the proof above we could use an arbitrary countable elementary submodel M of V_λ with $\mathcal{A} \in M$. However, in the next proof we need elementary submodels with some extra properties.

Theorem 4.2. *If $\mathcal{A}$ is a family of finite sets such that $\kappa = |\mathcal{A}|$ is an uncountable regular cardinal, then $\mathcal{A}$ contains a Δ-system of size κ.*

Proof. We can assume that $\mathcal{A} \subset \left[\kappa\right]^{<\omega}$.

Let Σ be a large enough finite set of formulas. By Corollary 2.6(2) there is a set M with $|M| < \kappa$ such that $\mathcal{A} \in M \prec_\Sigma V$ and $M \cap \kappa \in \kappa$.

Since $|\mathcal{A}| = \kappa$, we can pick $A \in \mathcal{A} \setminus M$. Let $D = M \cap A$. Since $\left[M\right]^{<\omega} \subset M$ we have $D \in M$ by Claim 3.6. Then

(19)
$$\exists \mathcal{B}\ (\mathcal{B} \subset \mathcal{A} \text{ is } \subset\text{-maximal among the } \Delta\text{-systems with kernel } D).$$

Since $M \prec_\Sigma V$, and the parameters $\mathcal{A}$ and D are in M, there is $\mathcal{B} \in M$ such that

(20)
$$M \models (\mathcal{B} \subset \mathcal{A} \text{ is } \subset\text{-maximal among the } \Delta\text{-systems with kernel } D).$$

Since $M \prec_\Sigma V$,

(21) $\quad \mathcal{B} \subset \mathcal{A}$ is $\subset$-maximal among the Δ-systems with kernel D.

Claim: $|\mathcal{B}| = \kappa$.
Assume on the contrary that $|\mathcal{B}| < \kappa$. Since $\mathcal{B} \in M$ we have $|\mathcal{B}| \in M \cap \kappa$. Thus $|\mathcal{B}| \subset M$ and so $\mathcal{B} \subset M$ by Claim 3.7.

Let $\mathcal{C} = \mathcal{B} \cup \{A\}$. If $B \in \mathcal{B}$, then $B \in M$ and so $B \subset M$ by $M \prec_\Sigma V$. Thus $B \cap A = D$. So $\mathcal{C} \supsetneq \mathcal{B}$ is a Δ-system with kernel D. Contradiction. □

To proof the next theorem we need elementary submodels one more additional property.

Theorem 4.3. *If $\kappa^\omega = \kappa$ then every family $\mathcal{A} = \{A_\alpha : \alpha < \kappa^+\} \subset \left[\kappa^+\right]^\omega$ contains a Δ-system of size κ^+. Especially, every family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}^+\} \subset \left[\mathfrak{c}^+\right]^\omega$ contains a Δ-system of size $\mathfrak{c}^+$.*

Proof. Let Σ be a large enough finite set of formulas. By Corollary 2.6(3) there is a set M with $|M| = \kappa$ such that $\mathcal{A} \in M \prec_\Sigma V$, $M \cap \kappa^+ \in \kappa^+$ and $\left[M\right]^\omega \subset M$.

Since $|\mathcal{A}| = \kappa^+ > |M|$, we can pick $A \in \mathcal{A} \setminus M$. Let $D = M \cap A$. Since $\left[M\right]^\omega \subset M$, we have $D \in M$.

Then

(22)

$\exists \mathcal{B}$ ($\mathcal{B} \subset \mathcal{A}$ is $\subset$-maximal among the Δ-systems with kernel D).

Since $M \prec_\Sigma V$ and $\left[M\right]^\omega \subset M$, the parameters $\mathcal{A}$ and D are in M, so there is $\mathcal{B} \in M$ such that

(23)

$M \models$ ($\mathcal{B} \subset \mathcal{A}$ is $\subset$-maximal among the Δ-systems with kernel D).

Since $M \prec_\Sigma V$,

(24) $\mathcal{B} \subset \mathcal{A}$ is $\subset$-maximal among the Δ-systems with kernel D.

Claim: $|\mathcal{B}| = \kappa^+$.

Assume on the contrary that $|\mathcal{B}| \leq \kappa$. Thus $|\mathcal{B}| \subset \kappa \subset M$ and so $\mathcal{B} \subset M$ by Claim 3.7.

Let $\mathcal{C} = \mathcal{B} \cup \{A\}$. If $B \in \mathcal{B}$, then $B \in M$ and so $B \subset M$ and $A \cap B = D$ by $M \prec_\Sigma V$. So $\mathcal{C} \supsetneq \mathcal{B}$ is a Δ-system with kernel D. Contradiction. □

Next we prove two classical partition theorems.

Theorem 4.4 (Erdős–Dusnik–Miller)**.** *If $\kappa = \mathrm{cf}(\kappa) > \omega$ then $\kappa \to (\kappa, \omega + 1)^2$.*

Proof. Fix a coloring $f : \left[\kappa\right]^2 \to 2$.

Let Σ be a large enough finite set of formulas. By Corollary 2.6(2) there is a set M with $|M| < \kappa$ such that $f \in M \prec_\Sigma V$ and $M \cap \kappa \in \kappa$.

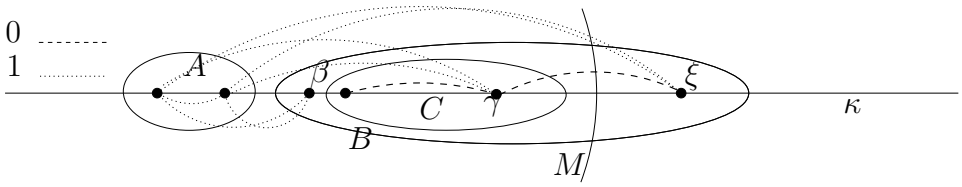

FIGURE 1

Fix $\xi \in \kappa \setminus M$. Let A be a $\subset$-maximal subset of $M \cap \kappa$ such that $A \cup \{\xi\}$ is 1-homogeneous. If A is infinite, then we are done.

Assume that A is finite. Let

(25) $$B = \{\beta \in \kappa \setminus A : \forall \alpha \in A \ f(\beta, \alpha) = 1\}.$$

Clearly $\xi \in B$. Since $f, A \in M$ we have $B \in M$. Let $C \subset B$ be a $\subset$-maximal 0-homogeneous subset.

Claim: $|C| = \kappa$.
Assume on the contrary that $|C| < \kappa$. Then $|C| \in M \cap \kappa$ and so $|C| \subset M$ because $M \cap \kappa \in \kappa$. Thus $C \subset M$ by Claim 3.7. Let $\gamma \in C$. Since $\gamma \in M \setminus A$ we have that $A \cup \{\gamma\} \cup \{\xi\}$ is not 1-homogeneous. But $A \cup \{\xi\}$ is 1-homogeneous and $\gamma \in B$, so $f(\gamma, \xi) = 0$. Thus $C \cup \{\xi\}$ is 0-homogeneous. Since $\xi \in B$, we have $\xi \in C$ by the maximality of C, which contradicts $B \subset M$. □

Theorem 4.5 (Erdős–Rado)**.** $\mathfrak{c}^+ \to (\mathfrak{c}^+, \omega_1 + 1)^2$.

Proof. Fix a function $f : \left[\mathfrak{c}^+\right]^2 \to 2$.

Let Σ be a large enough finite set of formulas. By Corollary 2.6(3) there is a set M with $|M| = \mathfrak{c}$ such that $f \in M \prec_\Sigma V$, $M \cap \mathfrak{c}^+ \in \mathfrak{c}^+$ and $\left[M\right]^\omega \subset M$.

Pick $\xi \in \mathfrak{c}^+ \setminus M$.

Let A be a $\subset$-maximal subset of $M \cap \kappa$ such that $A \cup \{\xi\}$ is 1-homogeneous. If A is uncountable, then we are done.

Assume that A is countable. Since $\left[M\right]^\omega \subset M$, we have $A \in M$.

Let

$$B = \{\beta \in \kappa \setminus A : \forall \alpha \in A \ f(\beta, \alpha) = 1\}. \tag{26}$$

Since $f, A \in M$ we have $B \in M$. Let $C \subset B$ be a $\subset$-maximal 0-homogeneous subset.
Claim: $|C| = \mathfrak{c}^+$.
Assume on the contrary that $|C| \leq \mathfrak{c}$. Then $|C| \subset \mathfrak{c} \subset M$ and so $C \subset M$ by Claim 3.7. Let $\gamma \in C$. Since $\gamma \in M \setminus A$ we have that $A \cup \{\gamma\} \cup \{\xi\}$ is not 1-homogeneous. But $A \cup \{\xi\}$ is 1-homogeneous and $\gamma \in B$, so $f(\gamma, \xi) = 0$. Thus $C \cup \{\xi\}$ is 0-homogeneous. Since $\xi \in B$, we have $\xi \in C$ by the maximality of C, which contradicts $B \subset M$. □

Given a *set-mapping* $F : X \to \mathcal{P}(X)$ we say that a subset $Y \subset X$ is a *F-free* iff $y' \notin F(y)$ for $y \neq y' \in Y$.

Theorem 4.6. *If $\kappa = \mathrm{cf}(\kappa) > \omega$ and $F : \kappa \to \left[\kappa\right]^{<\omega}$ then there is an F-free subset C of size κ.*

Proof. Let Σ be a large enough finite set of formulas. By Corollary 2.6(2) there is a set M with $|M| < \kappa$ such that $F \in M \prec_\Sigma V$ and $M \cap \kappa \in \kappa$.

Let $\xi \in \kappa \setminus M$ and $A = F(\xi) \cap M$. Let C be a $\subset$-maximal F-free subset of $\kappa \setminus A$. Since $F, A \in M$ we can assume that $C \in M$.
Claim: $|C| = \kappa$.
Assume on the contrary that $|C| < \kappa$. Then $C \subset M$ by Claim 3.7.

Since $F(\gamma) \subset M$ for $\gamma \in C$ and $F(\xi) \cap C \subset A \cap C = \emptyset$ we have that $C \cup \{\xi\}$ is also F-free. So C was not $\subset$-maximal. Contradiction. □

First we prove a weak form of Fodor's Pressing Down Lemma.

Theorem 4.7. *If $\kappa = \mathrm{cf}(\kappa) > \omega$, $f : \kappa \to \kappa$ is a regressive function then there is $\eta < \kappa$ such that $f^{-1}\{\eta\}$ is unbounded in κ.*

Proof. Let Σ be a large enough finite set of formulas. By Corollary 2.6(2) there is a set M with $|M| < \kappa$ such that $f \in M \prec_\Sigma V$ and $M \cap \kappa \in \kappa$.

Let $\xi = \sup(M \cap \kappa)$ and consider $\eta = f(\xi)$. We claim that $T = f^{-1}\{\eta\}$ is unbounded in κ. Since $\eta \in \xi = M \cap \kappa$ we have $T \in N$. If T is bounded, then $\sup T \in M \cap \kappa = \xi$. However $\xi \in T$ which is a contradiction. □

Theorem 4.8. *(Fodor's Pressing Down Lemma) If $\kappa = \mathrm{cf}(\kappa) > \omega$, $S \subset \kappa$ is stationary, and $f : S \to \kappa$ is a regressive function then there is an ordinal $\eta < \kappa$ such that $f^{-1}\{\eta\}$ is stationary.*

Proof. Let Σ be a large enough finite set of formulas. By Corollary 2.6(4) there is a set M with $|M| < \kappa$ such that $S, f \in M \prec_\Sigma V$ and $\xi = M \cap \kappa \in S$.

Let $\eta = f(\xi)$. We show that $T = f^{-1}\{\eta\}$ is stationary. Clearly $T \in M$. If T is not stationary then there is a closed unbounded set $C \in N$ such that $C \cap T = \emptyset$.

Claim: $\sup(M \cap \kappa) \in C$ *if $C \in M$ is a club subset of κ.*

Since C is closed, if $\sup(M \cap \kappa) \notin C$ then there is $\eta < \sup(M \cap \kappa)$ such that $(C \setminus \eta) \cap M = \emptyset$. Then $M \models$ "$C \setminus \eta = \emptyset$". Thus $V \models$ "$C \setminus \eta = \emptyset$", i.e. $C \subset \eta$. Contradiction, $T = f^{-1}\{\eta\}$ is stationary.

So by the claim $\xi \in C \cap T$. Contradiction. □

5. Decomposition theorems

In the previous section we proved theorems which claimed that *"Given a large enough structure $\mathcal{A}$ we can find a large enough nice substructure of $\mathcal{A}$."* In this section we proved results which have a different flavor: *Every large structure having certain properties can be partitioned into "nice" small pieces.*

In [9] the following statements were proved:

Theorem 5.1 (Nash-Williams)**.** *G is decomposable into cycles if and only if it has no odd cut.*

We give a new proof which illustrates how one can use "*chains of elementary submodels*". To do so we need two lemmas. The first one was proved in in section 3:

Lemma 3.5. *If $G = \langle W, E\rangle$ is an NW-graph, $G \in M \prec_\Sigma V$ for some large enough finite set Σ of formulas, then $G \restriction M$ is also an NW-graph.*

The second one is following statement.

Lemma 5.2. *If $G = \langle W, E\rangle$ is an NW-graph, $G \in M \prec_\Sigma V$ for some large enough finite set Σ of formulas, then $G \setminus M$ is also an NW-graph.*

Lemma 5.2 above follows easily from the next one.

Lemma 5.3. *Assume that $M \prec_\Sigma V$ with $|M| \subset M$ for some large enough finite set Σ of formulas. If $G \in M$ is a graph, $x \neq y \in V(G)$ and $F \subset E(G \setminus M)$, such that*

(27) $|F| \leq |M|$, $\gamma_{G\setminus M}(x, y) > 0$ *and F separates x and y in $G \setminus M$*

then

(28) F *separetes x and y in G.*

Proof of Lemma 5.2 from Lemma 5.3. Assume on the contrary that $G\setminus M$ has an odd cut F. Since any cut is the disjoint union of bonds we can assume that F is a bond.

Pick $c_1c_2 \in F$. Then clearly $\gamma_{G\setminus M}(c_1, c_2) > 0$. Moreover F separates c_1 and c_2 in $G \setminus M$, so F separates them in G by by lemma 5.3, i.e. c_1 and c_2 are in different connected components of $G \setminus F$

However F can not be a bond in G, so by Proposition 2.3 there is a connected component D of $G \setminus F$ such that $F \subset \left[D\right]^2$. i.e. c_1 and c_2 are the same connected component of $G \setminus F$. This contradiction proves the lemma. □

Proof of Lemma 5.3. Assume that G, M, x, y and F form a counterexample.

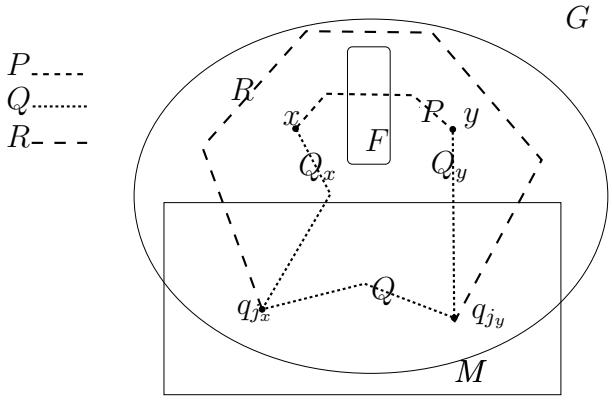

Figure 2

Fix a path $P = p_0 p_1 \dots p_n$ from x to y in $G \setminus M$ which witnesses that $\gamma_{G\setminus M}(x,y) > 0$, i.e. $p_0 = x$, $p_n = y$ and $p_i p_{i+1} \in E(G) \setminus M$ for $i < n$.

We assumed that F does not separate x and y in G, so there is a path $Q = q_0 \dots q_m$ from x to y witnessing this fact, i.e. $q_0 = x$, $q_m = y$ and $q_j q_{j+1} \in E(G) \setminus F$ for $j < m$. Since F separates x and y in $G \setminus M$ there is at least one $j^* < m$ such that $q_{j^*} q_{j^*+1} \in M$.

Let $j_x = \min\{j : q_j \in M\}$ and $j_y = \max\{j : g_j \in M\}$. Since $j_x \leq j^*$ and $j_y \geq j^* + 1$ we have $j_x < j_y$. Let $x' = q_{j_x}$ and $y' = q_{j_y}$. Let $Q_x = q_{j_x} q_{j_x - 1} \dots q_1 q_0$ and $Q_y = q_m q_{m-1} \dots q_{j_y}$. Then $Q_x P Q_y$ is a walk from x' to y' in $G \setminus M$. Hence $\gamma_{G\setminus M}(x', y') > 0$.

Claim: $\gamma_G(x', y') > |M|$.

Indeed, assume that $\lambda = \gamma_G(x', y') \leq |M|$. Since $M \prec_\Sigma V$ and $x', y' \in M$ there is $A \in M \cap \big[V(G)\big]^\lambda$ such that A separates x' and y' in G. Since $|A| = \lambda \subset M$ we have $A \subset M$. So M separates x' and y', i.e. $\gamma_{G\setminus M}(x', y') = 0$. This contradiction proves the claim.

By the weak Erdős-Menger Theorem there are $\gamma_G(x', y')$ many edge disjoint paths between x' and y' in G. Since $|M \cup F| = |M| < \gamma_G(x', y')$ there is a path $R = r_0 \dots r_k$ from x' to y' which avoids $M \cup F$. Then $Q_x^{-1} R Q_y^{-1}$ is walk from x to y in $G \setminus M$ which avoid F. Contradiction. □

Proof of theorem 5.1. We prove the theorem by induction on $|V(G)|$.

If G is countable infinite then for each $e \in E(G)$ there is a cycle C in G with $e \in E(C)$ because e is not a cut in G. Moreover, $G \setminus C$ is also an NW-graph, i.e. it does not have odd cuts. Using this observation we can construct a sequence $\{C_i : i < \omega\}$ of edge disjoint cycles in G with $E(G) = \cup\{E(C_i) : i < \omega\}$.

Assume now that $\kappa = V(G) > \omega$ and we prove the statement for graphs of cardinality $< \kappa$.

Let Σ be a large enough finite set of formulas. By the Reflection Principle 2.5 there is a cardinal λ such that $V_\lambda \prec_\Sigma V$ and $\big[V_\lambda\big]^\kappa \subset V_\lambda$. Then $G \in V_\lambda$.

We will construct a sequence $\langle M_\alpha : \alpha < \kappa \rangle \subset V_\lambda$ of elementary submodels of V_λ with

$$(*_\alpha) \qquad |M_\alpha| = \omega + |\alpha|,\ \alpha \subset M_\alpha \text{ and } M_\alpha \in M_{\alpha+1}$$

as follows:

(i) Let M_0 be a countable elementary submodel of V_λ with $G \in M_0$.
(ii) if $\beta < \kappa$ is limit then let $M_\beta = \cup\{M_\alpha : \alpha < \beta\}$. Since $|M_\beta| \leq \omega + |\beta| < \kappa$ and $M_\beta \subset V_\lambda$ we have $M_\beta \in V_\lambda$.

(iii) If $\beta = \alpha + 1$ then $|M_\alpha \cup \{M_\alpha\} \cup \beta| = \omega + |\beta|$ so by Löwenheim-Skolem Theorem there is $M_\beta \prec V_\lambda$ with $M_\alpha \cup \{M_\alpha\} \cup \beta \subset M_\beta$ and $|M_\beta| = \omega + |\beta|$.

The construction clearly guarantees $(*_\alpha)$. Using the chain $\langle M_\alpha : \alpha < \kappa\rangle$ decompose G as follows:

- for $\alpha < \kappa$ let $G_\alpha = (G \setminus M_\alpha) \restriction M_{\alpha+1}$.

By lemma 5.2 the graph $G'_\alpha = G \setminus M_\alpha$ is NW. Moreover, since $M_\alpha \in M_{\alpha+1}$ we have $G \setminus M_\alpha \in M_{\alpha+1}$. So we can apply lemma 3.5 for $M_{\alpha+1}$ and G'_α to deduce that G_α is NW.

So we decomposed the graph G into NW-graphs $\{G_\alpha : \alpha < \kappa\}$. Moreover, $|V(G_\alpha)| \leq |M_{\alpha+1}| \leq \omega + |\alpha| < \kappa$, so by the inductive hypothesis, every G_α is the union of disjoint cycles. So G itself is the union of disjoint cycles which was to be proved. □

5.1. **General framework.** If Φ is a graph property then we write $G \in \Phi$ to mean that the graph G has property Φ.

We say that a graph property Φ is *well-reflecting* iff for each graph $G \in \Phi$ whenever $G \in M \prec_\Sigma V$ with $|M| \subset M$ for some large enough finite set Σ of formulas, we have both $G \restriction M \in \Phi$ and $G \setminus M \in \Phi$.

Theorem 5.4. *Let Φ be a well-reflecting graph property. Then every graph $G \in \Phi$ can be decomposed into a family $\{G_i : i \in I\} \subset \Phi$ of countable graphs.*

To prove this theorem we need to introduce the following notion. Let κ and λ be cardinals. We say that $\langle M_\alpha : \alpha < \kappa\rangle$ is a *κ-chain of submodels of V_λ* iff

(1) the sequence $\langle M_\alpha : \alpha < \kappa\rangle \subset V_\lambda \cap \left[V_\lambda\right]^{<\kappa}$ is strictly increasing and continuous (i.e. $M_\beta = \cup\{M_\alpha : \alpha < \mu\}$ for limit β),
(2) $M_\alpha \prec V_\lambda$, $\alpha \subset M_\alpha$ and $M_\alpha \in M_{\alpha+1}$ for $\alpha < \kappa$,

Fact 5.5. *If $\left[V_\lambda\right]^{<\kappa} \subset V_\lambda$ then for each $x \in V_\lambda$ there is a κ-chain of elementary submodels $\langle M_\alpha : \alpha < \kappa\rangle$ of V_λ with $x \in M_0$.*

Proof. Actually such a chain was constructed in the proof of Theorem 5.1. □

Proof of Theorem 5.4. By induction on $|G|$. If $|G|$ is countable then there is nothing to prove.

Assume that $G = \langle \kappa, E\rangle$ and $\kappa > \omega$. By the Reflection Principle 2.5 there is a cardinal λ such that $V_\lambda \prec_\Sigma V$ and $\left[V_\lambda\right]^\kappa \subset V_\lambda$. Then, by Fact 5.5 there is a κ-chain of elementary submodels of V_λ with $G \in M_0$. For $\alpha < \kappa$ let $G_\alpha = (G \setminus M_\alpha) \restriction M_{\alpha+1}$. Since Φ is well-reflecting, the graph $G'_\alpha = G \setminus M_\alpha$ is in Φ. Moreover, since $M_\alpha \in M_{\alpha+1}$

we have $G \setminus M_\alpha \in M_{\alpha+1}$. So applying once more the fact that Φ is well reflecting for $M_{\alpha+1}$ and G'_α we obtain that G_α is in Φ.

So we decomposed the graph G into graphs $\{G_\alpha : \alpha < \kappa\} \subset \Phi$. However $|V(G_\alpha)| \leq |M_{\alpha+1}| \leq \omega + |\alpha| < \kappa$, so by the inductive hypothesis, every G_α has a decomposition $\mathcal{G}_\alpha$ into countable elements of Φ. Then $\mathcal{G} = \cup\{\mathcal{G}_\alpha : \alpha < \kappa\}$ is the desired decomposition of G. □

Theorem 5.6. *Let Φ and Ψ be graph properties. Assume that*
(1) Φ is well-reflecting,
(2) if $H \in \Phi$ is a countable graph then $H \in \Psi$,
(3) if G has a decomposition $\{G_i : i \in I\}$ with $G_i \in \Psi$ then $G \in \Psi$.
Then $G \in \Phi$ implies $G \in \Psi$.

Proof. Theorem 5.4 and (1) yield that G has a decomposition into countable graphs $\{G_i : i \in I\} \subset \Phi$. By (2) , $\{G_i : i \in I\} \subset \Psi$. Finally, by (3), this implies $G \in \Psi$ which was to be proved. □

In lemmas 3.5 and 5.2 we proved that the graph property "*there is no odd cut*" is well-reflecting.

As we will see, Theorem 5.6 can be applied as a "black box" principle in many proofs.

5.2. **Applications of Theorem 5.6.** First we give a new proof of a result of Laviolette.

Theorem 5.7 ([8, Corollary 1])**.** *Every bridgeless graph can be partitioned into countable bridgeless graphs.*

Proof. We need the following lemma:

Lemma 5.8. *The "*bridgeless*" property is well-reflecting.*

Proof of lemma 5.8. Assume that G is a graph and $G \in M \prec_\Sigma V$ for some large enough finite family Σ of formulas.
(1) Assume that an edge $e = xy$ is a bridge in $G \restriction M$. Then

$$M \models e \text{ separates } x \text{ and } y, \tag{29}$$

so, by $M \prec_\Sigma V$

$$V \models e \text{ separates } x \text{ and } y, \tag{30}$$

i.e. e is a bridge in G.
(2) Assume that an edge $e = xy$ is a bridge in $G \setminus M$. Then e separates x and y in $G \setminus M$, so by lemma 5.3, e separates x and y in G, i.e. e is a bridge in G. □

By lemma 5.8, we can apply theorem 5.4 to get the statement of this theorem. □

Let us formulate two corollaries.

Corollary 5.9 (Laviolette, [8, Theorem 1]). *Every bridgeless graph has a cycle ω-cover.*

Proof. Every countable bridgeless graph clearly has a cycle ω-cover, and by the previous theorem every bridgeless graph can be partitioned into countable bridgeless graphs. □

It is worth to mention that in [8] Theorem 5.8 was a corollary of Corollary 5.9.

Before formulation of the second corollary let us recall the following conjecture of Seymour and Szekeres.

Double Cover Conjecture . *Every bridgeless graph has a cycle double cover.*

Since every bridgeless graph can be partitioned into countable bridgeless graphs, we yield

Corollary 5.10 (Laviolette, [8]). *If the Double Circle Conjecture holds for all countable graphs then it holds for each graphs.*

Next we sketch two more applications.

In [9] the following statements were also proved:

Theorem 5.11 (Nash-Williams). *(1) A graph G can be decomposed into cycles and endless chains if and only if it has no vertex of odd valency. (2) G is decomposable into endless chains if and only if it has no vertex of odd valency and no finite non-trivial component.*

Let us recall that a connected component is *non-trivial* if it has at least two elements.

Proof of 5.11. For $i = 1, 2$ we say that a graph G is NW_i iff G satisfies the assumption of statement (i) from 5.1.

Lemma 5.12. *The statements of Theorem 5.1 hold for countable graphs.*

The proof of Lemma 5.12 is left to the reader.

Lemma 5.13. *The following graph properties are well-reflecting:*
(1) there is no vertex of odd valency.
(2) there is no finite non-trivial component.

Proof of lemma 5.13. (1) Assume that in G there is no vertex of odd valency. Let $G \in M \prec_\Sigma V$ with $|M| \subset M$ for some large enough finite set Σ of formulas.

Claim *There is no vertex of odd valency in $G \restriction M$*

Indeed, let $x \in V$, and assume that the set $A = \{v \in V(G) : yx \in E(G \restriction M)\}$ is finite. Then $A \in M$, and for each $v \in V(G) \cap M$ we have $v \in A$ iff $vx \in E(G) \cap M$. Thus

$$M \models A = \{v \in V(G) : vx \in E(G)\}, \tag{31}$$

so, by $M \prec_\Sigma V$, we have

$$M \models A = \{v \in V(G) : vx \in E(G)\}, \tag{32}$$

i.e. $A = \{v \in V(G) : vx \in E(G)\}$. $V \models A = G(x)$, i.e. $A = G(x)$. Thus $d_G(x) = d_{G\restriction M}(x)$.
Let $G \in M \prec_\Sigma V$, $|M| \subset M$, and $x \in V$. If $x \notin M$, then $G(x) = (G \setminus M)(x)$, so $d_{G\restriction M}(x) = d_G(x)$ is even.
Claim *There is no vertex of odd valency in* $G \setminus M$
Assume $x \in M$. If $d_G(x) \leq |M|$ then $\{v \in V(G) : vx \in E(G)\} \subset M$, and so $d_{G\setminus M}(x) = 0$. If $d_G(x) > |M|$ then $d_G(x) = d_{G\setminus M}(x)$.
If $x \in M$ then clearly $d_G(x) = d_{G\setminus M}(x)$.

(2) Assume that in G there is no finite component Let $G \in M \prec_\Sigma V$ with $|M| \subset M$ for some large enough finite set Σ of formulas.
Claim *There is no finite non-trivial component in* $G \restriction M$
Let $x \in V(G) \cap M$ and assume that x has a finite component C in $G \restriction M$. Then $C \in M$ and

$$M \models C \text{ is the component of } x, \tag{33}$$

so

$$V \models C \text{ is the component of } x, \tag{34}$$

i.e. G has finite component.
Claim *There is no finite non-empty component in* $G \setminus M$
Assume that there is a finite non-trivial component C in $G \setminus M$. Since C is not a component in M there is an edge $cd \in E(G) \cap M$ with $c \in C$. Since C is non-trivial there is $c' \in C$ such that cc' is an edge in $G \setminus M$. Then $c \in M$ and $c' \notin M$.
Since $d_G(c) \leq |M|$ would imply $c' \in \{c^* : cc^* \in E(G)\} \subset M$ we have $d_G(x) > |M|$. However $\{c^* : cc^* \in E(G)\} \setminus M \subset C$, and so $|C| > |M|$. Contradiction. □

We want to apply theorem 5.6. Let Φ_i be the property NW_i for $i = 1, 2$, and Ψ_1 be "*decomposable into cycles and endless chains* ", and Ψ_2 be "*decomposable into endless chains* ".
Then condition 5.6.(1) holds by lemma 5.13, 5.6.(2) is true by lemma 5.12. 5.6.(3) is trivial from the definition. Putting together these things we obtain the theorem. □

6. Bond faithful decompositions

In this section we prove a decomposition theorem in which we can not apply Theorem 5.6.

Definition 6.1. Let κ be an infinite cardinal. A decomposition $\mathcal{H}$ of a graph G is *κ-bond faithful* iff $|E(H)| \leq \kappa$ for each $H \in \mathcal{H}$,

(i) any bond of G of cardinality $\leq \kappa$ is contained in some member of the decomposition,

(ii) any bond of cardinality $< \kappa$ of a member of the decomposition is a bond of G.

Theorem 6.2. *For any cardinal κ every graph has a κ-bond faithful decomposition.*

Laviolette [8] proved that the theorem above holds for $\kappa = \omega$ in ZFC, and for any $\kappa > \omega$ under GCH. We eliminate GCH from the proof.

The following lemma is the key of the proof.

Lemma 6.3. *If G is a graph, $G \in M \prec_\Sigma V$ with $|M| \subset M$ for some large enough finite set Σ of formulas.*

(I) If $F \subset E(G \restriction M)$ is a bond of $G \restriction M$ with $|F| < |M|$ then F is a bond in G.

(II) If $F \subset E(G)$ is a bound of $G \setminus M$ with $|F| < |M|$ then F is a bond in G.

Proof of 6.3. (I) Assume on the contrary that F is not a bound in G. Pick $xx' \in F$. Then by Proposition 2.3 x and x' are in the same connected component D of $G \setminus F$, and so there is a path $P = x_1 x_2 \dots x_n$, in $G \setminus F$, $x_1 = x$, $x_n = x'$. Choose the path in such a way that the cardinality of the finite set

$$I_P = \{i : x_i x_{i+1} \notin M\} \tag{35}$$

is minimal. Since F is a cut in $G \restriction M$ we have $I_P \neq \emptyset$. Let $i = \min I_p$. Then $x_i \in M$. Let $j = \min\{j > i : x_j \in M\}$. Then $j > i + 1$, $x_i, x_j \in M$, and moreover $\gamma_{(G \setminus M) \setminus F}(x_i, x_j) > 0$.

Claim 6.4. *If $x, y \in M$, $\gamma_{G \setminus M}(x, y) > 0$ then $\gamma_{G \restriction M}(x, y) = |M|$.*

Proof of the Claim. Write $\mu = |M|$. If $\gamma_G(x, y) \leq \mu$ then there is $A \in \big[V(G)\big]^\mu$ such that A separates x and y in G. Since $G, x, y, \mu \in M \prec_\Sigma V$ we can find such an A in M. Since $\mu \subset M$ we have $A \subset M$, and so M separates x and y in G. Thus $\gamma_{G \setminus M}(x, y) = 0$.

But $\gamma_{G \setminus M}(x, y) > 0$, so we have $\gamma_G(x, y) > |M|$. So, by the weak Erdős-Menger Theorem there is a family $\mathcal{P}$ of μ many edge disjoint paths between x and y in G. Since $G, x, y, \mu \in M$ we can find such a

$\mathcal{P}$ in M. But $|\mathcal{P}| = \mu \subset M$, and so $\mathcal{P} \subset M$. Thus there are μ-many edge disjoint paths between x and y in M, i.e. $\gamma_{G\restriction M}(x,y) = \kappa$. □

By the Claim $\gamma_{G\restriction M}(x_i, x_j) = \kappa$. So, by the weak infinite Menger Theorem, there are κ many edge disjoint path in $G \setminus M$ between x and y. Since $|F| < \kappa$, there is a path $Q = x_i y_1 \dots y_k x_j$ in $G \restriction M$ which avoid F. Then $P' = x_1 \dots x_j y_1 \dots y_k x_j \dots x_n$ is a path between x_1 and x_n in $G \setminus F$ with $|I_{P'}| < |I_P|$. Contradiction.
(II) Let $c_1 c_2 \in F$. Then $\gamma_{G\setminus M}(c_1, c_2) > 0$, F separates c_1 and c_2 in $G \setminus M$, so F also separates c_1 and c_2 in G by lemma 5.3. In other words, c_1 and c_2 are in different connected component of $G \setminus F$, and so F should be a bond in G by Proposition 2.3. □

Proof of theorem 6.2. By induction on $|V(G)|$. If $|V(G)| \leq \kappa$ then the one element decomposition $\{G\}$ works.

Assume that $G = \langle \mu, E\rangle$, and $\mu > \kappa$. Let Σ be a large enough finite set of formulas. By the Reflection Principle 2.5 there is a cardinal λ such that $V_\lambda \prec_\Sigma V$ and $\left[V_\lambda\right]^\mu \subset V_\lambda$.

By Fact 5.5 there is a μ-chain of elementary submodels $\langle M_\alpha : \alpha < \mu\rangle$ of V_λ with $G \in M_0$.

Using the chain $\langle M_\alpha : \alpha < \mu\rangle$ partition G as follows:

- for $\alpha < \mu$ let $G_\alpha = (G \setminus M_\alpha) \restriction M_{\alpha+1}$.

Let $G'_\alpha = G \setminus M_\alpha$. By lemma 6.3(II)

- any bond of cardinality $< \kappa$ of G'_α is a bond of G.

Moreover, since $M_\alpha \in M_{\alpha+1}$ we have $G \setminus M_\alpha \in M_{\alpha+1}$. So we can apply lemma 6.3(I) for $M_{\alpha+1}$ and G'_α to derive that

- any bond of cardinality $< \kappa$ of G_α is a bond of G'_α.

Putting together

- any bond of cardinality $< \kappa$ of G_α is a bond of G.

Moreover $|V(G_\alpha)| \leq |M_{\alpha+1}| \leq \omega + |\alpha| < \kappa$, so by the inductive hypothesis, every G_α has a κ-bond faithful decomposition $\mathcal{H}_\alpha$. Let $\mathcal{H} = \cup\{\mathcal{H}_\alpha : \alpha < \mu\}$. $\mathcal{H}$ clearly satisfies 6.1(ii): if F is a bond of some $H \in \mathcal{H}_\alpha$ with $|F| < \kappa$, then F is a bond of G_α, and so F is a bond of G.

Finally we check 6.1(i). We recall one more result of Laviolette: a result from

Theorem 6.5 ([8, Proposition 3])**.** *For any cardinal κ every graph has an decomposition which satisfies 6.1(i).*

Let us remark that GCH was assumed in [8, Proposition 3], but in the proof it was not used.

Since $M_0 \prec_\Sigma V$, we have a decomposition $\mathcal{K} \in M_0$ of G which witnesses 6.1(i). Assume that A is a bond of G with $|A| \leq \kappa$. Then there is $K \in \mathcal{K}$ such that $A \subset E(K)$. Let α be minimal such that $E(K) \cap M_{\alpha+1} \neq \emptyset$. Then $K \in M_{\alpha+1}$. Thus $A \subset E(H) \subset E(G_\alpha)$. Since, by the inductive assumption, the decomposition $\mathcal{H}_\alpha$ satisfies 6.1(i) there is $H \in \mathcal{H}_\alpha$ with $A \subset E(H)$. But $H \in \mathcal{H}$, so we are done. □

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