INFINITE COMBINATORICS: FROM FINITE TO INFINITE

LAJOS SOUKUP

Abstract. We investigate the relationship between some theorems in finite combinatorics and their infinite counterparts: given a "finite" result how one can get an "infinite" version of it? We will also analyze the relationship between the proofs of a "finite" theorem and the proof of its "infinite" version.

Besides these comparisons, the paper gives a proof of a theorem of Erdős, Grünwald and Vázsonyi giving the full descriptions of graphs having one/two-way infinite Euler lines. The last section contains some new results: an infinite version of a multiway-cut theorem is included.

1. Introduction

The introduction should be started with a negative statement: this paper is not a survey of the most important results of infinite combinatorics. Some surveys can be found in [15], [7] or in [8].

In this paper we intend to investigate the relationship between some theorems in finite combinatorics and their infinite counterparts: given a "finite" theorem how one cat get a "infinite" version of it? So we study the methods of generalizations. We will survey some problems from finite combinatorics and we will analyze the relationship between their proofs and the proofs of their "infinite" versions.

Although this paper is not a guide how to get new "infinite" results we will give examples of applications of some basic proof methods from infinite combinatorics.

Beside the investigation of these connections, in section 3.1 we will recall some "forgotten" results of Erdős, Grünwald and Vázsonyi, (see [9] and [10]) with full proof because these theorems are not easily accessible in the literature (originally they were published in Hungarian,

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[9], then in German, [10]). Moreover, in section 4, we give an infinite version of a theorem from [11] concerning the minimal size of multi-way cuts.

The results from section 1-3 are folklore if no references are given. Only section 4 contains a new result of the author.

Our notation is standard. See e.g. [4].

The set of neighbouring vertices of a vertex v in a graph G is denoted by $\Gamma_G(v)$. If A is a set of vertices then $\Gamma_G(A) = \cup\{\Gamma_G(v) : v \in A\}$. The degree $d_G(v)$ of a vertex v is $|\Gamma_G(v)|$.

A *trail* T in a graph G is a sequence $T = \langle x_0, x_1 \dots, x_n\rangle$ of vertices such that $E(T) = \langle x_i x_{i+1} : i < n\rangle$ is a family of pairwise different edges of G. The vertices x_0 and x_n are the *end-vertices* of the trail. A *circuit* is a trail whose end-vertices coincide. A *path* is a trail with distinct vertices.

A graph is *connected* iff there is a path between any two of its vertices. The maximal connected subgraphs of a graph are the *components* of the graph.

Directed trails and *directed paths* are defined similarly in directed graphs (digraphs, in short).

If G is a directed graph and $p = \langle x_0, x_1, \dots, x_n\rangle$ is a directed path in G then we write $\mathrm{first}(p) = x_0$, $\mathrm{last}(p) = x_n$ and $\mathrm{E}(p) = \{x_0x_1, \dots, x_{n-1}x_n\}$.

If $G = (V, E)$ is a directed graph and $A \subset V$ then

$$\mathrm{In}(A) = \{v : \exists a \in A \ va \in E\}$$

and $\mathrm{in}(A) = |\,\mathrm{In}(A)|$; similarly,

$$\mathrm{Out}(A) = \{v : \exists a \in A \ av \in E\}$$

and $\mathrm{out}(A) = |\,\mathrm{Out}(A)|$.

Since we will discuss theorems in finite combinatorics and their infinite counterparts side by side we introduce the following terminology: theorems in finite combinatorics will be enumerated as *Finite Theorem 1, Finite Theorem 2*, etc, and the corresponding results from infinite combinatorics will be enumerated as *Infinite Theorem 1, Infinite Theorem 2*, etc.

2. Method of proofs, transfer principles.

The first example illustrates the simplest case: there is no difference between the finite and infinite theorems, moreover the same proof works in both cases, all we should do is to remove the word “finite” from both the theorem and from its proof.

2.1. **Connectedness.**

Finite Theorem 1. *A finite graph $G = (V, E)$ is connected iff given any partition (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.*

Proof. A connected graph clearly has this property.

To see the other direction let $x \in V$ and put

$$A = \{z \in V : \text{there is an } x\text{-}z\text{-path in } G\}.$$

Since there is no edge between A and $V \setminus A$ and $a \in A$, we have $A = V$. Thus from x there is a path to each vertex of G. □

Infinite Theorem 1. *A graph $G = (V, E)$ is connected iff given any partition (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.*

The same proof works.

The next example is - at least for the first sight - very similar.

2.2. **Spanning trees.**

Finite Theorem 2. *Every finite connected graph $G = (V, E)$ has a spanning tree.*

We "know" that the same statement holds for arbitrary graphs:

Infinite Theorem 2. *Every connected graph $G = (V, E)$ has a spanning tree.*

But, as we will see soon, the relationship between theirs proofs is more delicate. The "finite theorem" has (at least) two different proofs:

First Proof. Let $T = (V, F)$ be a minimal connected subgraph of G. Then T can not contain a circle, so it is a spanning tree. □

The method of this proof can not be applied to get the "infinite" version because it is not easy to guarantee that there is a minimal connected subgraph of an infinite graph: an infinite graph G may contain a decreasing chain $G_0, G_1, \ldots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\cap_{i\in\mathbb{N}} E(G_i) = \emptyset$.

Now consider the second proof of the finite theorem.

Second Proof. Let $T = (V', E')$ be a maximal subtree of G. Since there is no edge between V' and $V \setminus V'$ we have $V' = V$. Hence T is a spanning tree. □

This proof can be modified to get the infinite theorem:

Proof. Let $\mathcal{T}$ be the family of subtrees of G. For $T, T' \in \mathcal{T}$ write $T \prec T'$ iff T is a subtree of T'.

Since $\mathcal{T}$ is closed under increasing union, $\langle \mathcal{T}, \prec \rangle$ has a maximal element $T = (V', E')$ by Zorn's Lemma. Since there is no edge between V' and $V \setminus V'$ we have $V' = V$. Hence T is a spanning tree. □

So almost the same proof works, but we used Zorn's Lemma (i.e. Axiom of Choice) It is a natural question whether we really need the Axiom of Choice? The next theorem gives the answer:

Theorem 2.1. *(ZF) If every connected graph has a spanning tree then Axiom of Choice holds.*

Proof. Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of non-empty sets. We want to find a choice function.

First we can assume that the elements of $\mathcal{A}$ are pairwise disjoint.

Construct a graph $G = (V, E)$ as follows. Let

$$V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \cup\{A_i : i \in I\},$$

where $\{x\} \cup \{y_i, z_i : i \in I\}$ are new, pairwise different vertices, and put

$$E = \{xy_i : i \in I\} \cup \cup_{i \in I}\{x_i a, ay_i : a \in A_i\}.$$

Then G is connected, so, by the assumption, it has a spanning tree $T = (V, F)$. Then

(i) $\{xy_i : i \in I\} \subset F$,
(ii) for each $i \in I$ there is exactly one $a_i \in A_i$ such that $x_i a_i, a_i y_i \in F$,
(iii) for each $a \in A_i \setminus \{a_i\}$ we have $x_i a \in F$ iff $ay_i \notin F$.

Thus $f(i) = a_i$ is a choice function for $\mathcal{A}$ and f is definable using T. □

So it was a case when we have the same theorem for finite and infinite. Even the proofs are almost the same, but in the infinite case we should use Axiom of Choice to get some maximal structure.

Next we will see an example when the finite case has a straightforward generalization for the countable case, but there is no way to get some similar result for uncountable graphs.

2.3. **Normal spanning tree.** A *normal spanning tree* (or *depth-first search tree*) of a connected graph $G = (V, E)$ is a rooted subtree T of G such that for each edge $xy \in E$ the endpoints x and y are comparable in the rooted tree order.

Finite Theorem 3. *Every finite connected graph has a normal spanning tree.*

Proof. Apply the depth-first algorithm to construct a normal spanning tree. □

What about the infinite graphs? Well, the complete graph on $\aleph_1$, $K_{\aleph_1}$, does not have a normal spanning tree. On the other hand, we have

Infinite Theorem 3. *Every countable connected graph has a normal spanning tree.*

The simple greedy depth-first algorithm may not work even in $K_{\aleph_0}$ because it may find an infinite path which does not contain all the vertices. However an inductive algorithm may works: using a carefully chosen ordering we can guarantee that all the vertices is included in some finite step into the spanning tree.

So far we have seen examples when either we have the same statement for finite and infinite graph, or it was clear that certain statements simply fail for uncountable graphs.

2.4. **Pseudo-winners in tournaments.** Given a directed graph $G = (V, E)$ for $A \subset V$ let $\mathrm{Out}_1(A) = A \cup \mathrm{Out}(A)$ and $\mathrm{Out}_n(A) = \mathrm{Out}_1(\mathrm{Out}_{n-1}(A))$ for $n > 1$, i.e. $v \in \mathrm{Out}_n(A)$ iff there is a path of length at most n which leads from some elements of A to v. Similarly, let $\mathrm{In}_1(A) = A \cup \mathrm{In}(A)$ and $\mathrm{In}_n(A) = \mathrm{In}_1(\mathrm{In}_{n-1}(A))$ for $n > 1$. If $A = \{v\}$ write $\mathrm{Out}_n(v)$ for $\mathrm{Out}_n(\{v\})$, and $\mathrm{In}_n(v)$ for $\mathrm{In}_n(\{v\})$.

Let $T = (V, E)$ be a tournament and let $t \in V$. We say that t is a *pseudo-winner* iff $\mathrm{Out}_2(t) = V$.

Finite Theorem 4. *Every finite tournament has a pseudo-winner.*

Proof. If t has maximal out-degree then t is a pseudo-winner.

Indeed, Let $v \in V$. If $tv \in E$ then $v \in \mathrm{Out}(t) \subset \mathrm{Out}_2(t)$.

If $vt \in E$ then $t \in \mathrm{Out}(v) \setminus \mathrm{Out}(t)$, so there is $s \in \mathrm{Out}(t) \setminus \mathrm{Out}(v)$ because $|\mathrm{Out}(t)|$ was maximal. But then tsv is a directed path of length 2, and so $v \in \mathrm{Out}_2(t)$. □

Now consider the infinite case. The simplest generalization fails because

Observation 2.2. *There is no pseudo-winner in the tournament* $(\mathbb{Z}, <)$.

However, in $(\mathbb{Z}, <)$ we have $\mathbb{Z} = \mathrm{In}(0) \cup \mathrm{Out}(1)$. As it turns out, this behavior of $\mathbb{Z}$ is not exceptional.

Infinite Theorem 4. *A tournament $T = (V, E)$ contains a pseudo-winner or there are $x \neq y \in V$ such that $V = \mathrm{Out}(x) \cup \mathrm{In}(y)$.*

Proof. Indeed, if y is not a pseudo-winner witnessed by x, i.e. $x \notin \mathrm{Out}_2(y)$, then $V = \mathrm{Out}(x) \cup \mathrm{In}(y)$. □

One can find other interesting results concerning the structure of infinite tournaments, see e.g. [13]:

Theorem 2.3. *(1) Let $T = (V, E)$ be an infinite tournament. If $V = \mathrm{Out}_n(v)$ for some $n \geq 3$ and $v \in V$ then $V = \mathrm{Out}_3(w)$ for some $w \in V$.*
(2) There is an infinite tournament $T = (V, E)$ such that $V = \mathrm{Out}_3(v)$ for some $v \in V$ but $V \neq \mathrm{Out}_2(w)$ for any $w \in V$.

This was an example when the finite and the infinite theorems are quite different. But the infinite case is also easy provided you know what you have to prove.

The next subsection contains an example when the infinite theorem is still open.

2.5. **Quasi-kernels in digraphs.** Let $G = (V, E)$ be a digraph, $A, B \subset V$. An independent set $A \subset V$ is a *quasi-kernel (quasi-sink)* iff $V = \mathrm{Out}_2(A)$ ($V = \mathrm{In}_2(A)$).

Finite Theorem 5 (Chvátal, Lovász, [5])**.** *Every finite digraph has a quasi-kernel.*

The simple generalization fails even for infinite tournaments: the tournament $(\mathbb{Z}, <)$ is a counterexample.

For infinite tournament it was easy to find an infinite version of this theorem but what is the right generalization for infinite digraphs?

Theorem 4 implies that *if $G = (V, E)$ is an infinite tournament then there are point $x \neq y \in V$ s.t. $V = \mathrm{Out}_2(\{x\}) \cup \mathrm{In}_2(\{y\})$.* One can guess that this formulation gives us the right infinite version of the theorem of Chvátal and Lovász, namely we conjectured that *every digraph contains two disjoint independent sets, A and B such that $V = \mathrm{Out}_2(A) \cup \mathrm{In}_2(B)$.*

For a while we tried to find a counterexample, but at some point we've found a quite easy way to show that :

Infinite Theorem 5 (P. L. Erdős, A. Hajnal, –, [13])**.** *Every digraph contains two disjoint independent sets, A and B such that $V = \mathrm{Out}_2(A) \cup \mathrm{In}_2(B)$.*

However, during hunting counterexamples we realized that all the digraphs we could construct have a much stronger property which led us to the formulation of the following conjecture.

Kernel-Sink Conjecture: *Given a directed graph $G = (V, E)$ there is a partition (W_0, W_1) of V such that $G[W_0]$ has a quasi-kernel and $G[W_1]$ has a quasi-sink.*

Let us remark that theorem 4 implies this statement for infinite tournaments. In [13] we prove that this conjecture holds for different classes of infinite graphs, but the conjecture is still open.

In the next subsection we will see a problem when the finite case is trivial, the general infinite case is hard but solved, however the countable case is completely open.

2.6. **Unfriendly partitions.** Let $G = (V, E)$ be a graph. A partition (A, B) of V is called *unfriendly* iff every vertex has at least as many neighbor in the other class as in its own.

Finite Theorem 6. *Every finite graph has an unfriendly partition.*

Proof. Take a partition having maximal number of edges between the classes of the partition. This partition should be unfriendly. □

After proving that large classes of infinite graphs have unfriendly partitions it was natural to formulate the following conjecture, [2]
Unfriendly Partition Conjecture. *Every graph has an unfriendly partition.*

However, this conjecture was refuted:

Infinite Theorem 6.1 (Shelah, [18])**.** *There is an uncountable graph without unfriendly partitions.*

Having disproved the plain generalization what are the other possibilities?

Infinite Theorem 6.2 (Shelah[18])**.** *Every graph has a partition into three pieces such that every vertex has at least as many neighbor in the two other classes as in its own.*

Or you can get a positive theorem for infinite graphs provided you consider only graphs which are similar to a finite graph. A graph is called *locally finite* iff every vertex has finite degree.

Infinite Theorem 6.3. *Every locally finite graph has an unfriendly partition.*

Proof. We will apply Gödel's Compactness Theorem below.

Gödel's Compactness Theorem. *A first order theory T has a model iff every finite subset of T has a model.*

In many cases (including this one) you can substitute Gödel's Compactness Theorem by other results, e.g. by König's Lemma, but I think that the familiarity with Gödel's Compactness Theorem is very useful if one wants to do infinite combinatorics.

So let $G = (V, E)$ a locally finite graph.

Consider the following first order language $\mathcal{L}$: $\{c_v : v \in V\}$ is the set of constant symbols, and R_A and R_B are unary relation symbols.

Define the following formulas:
ψ: $\forall x\ \big(R_A(x) \leftrightarrow \neg R_B(x)\big)$,
for all $v \in V$ write $\mathcal{F}_v = \{F \subset E(v) : |F| \geq d(v)/2\}$ and put
$\varphi_{v,A}$: $R_A(c_v) \to \bigvee_{F\in\mathcal{F}_v} \bigwedge_{x\in F} R_B(c_x)$,
$\varphi_{v,B}$: $R_B(c_v) \to \bigvee_{F\in\mathcal{F}_v} \bigwedge_{x\in F} R_A(c_x)$.
Now define our theory T as follows: $T = \{\psi, \varphi_{v,A}, \varphi_{v,B} : v \in V\}$

Claim 1. *Every $T' \in \big[T\big]^{<\omega}$ has a model.*

Indeed, let $W = \{v : c_v \text{ occurs in } T'\}$. Then $G[W]$ has an unfriendly partition (A, B). Let M be the following model: the underlying set M is W, c_v is interpreted as v for $v \in W$, and R_A is interpreted as A and R_B is interpreted as B. Then $M \models T'$.

Using the Claim and Gödel's Compactness Theorem we obtain that T has a model M. Let $A = \{v \in V : M \models R_A(c_v)\}$ and $B = \{v \in V : M \models R_B(c_v)\}$.

Then (A, B) is a partition because ψ holds in M. Moreover if $v \in A$ then v has at least as many neighbor in b as in A because $\varphi_{v,A}$ holds.

Hence (A, B) is an unfriendly partition of G. □

Shelah's counterexample is uncountable which led to the following reformulation of the refuted conjecture:
Unfriendly Partition Conjecture, Revised: *Every countable graph has an unfriendly partition.*

Let us remark that if $G = (V, E)$ is countable and every $v \in V$ has infinite degree then G clearly has an unfriendly partition. We have seen that G has an unfriendly partition if every vertex has finite degree. So the hard case is the "mixed" countable case.

So far the revised unfriendly partition conjecture is completely open.

2.7. **Splitting antichains.** Given a poset P an element $y \in P$ is a *cutting point* iff $\exists x, z \in P$ such that $x <_P y <_P z$ and $[x, z] = [x, y] \cup [y, z]$. P is *cut-free* if there is no cutting point in P.

A maximal antichain $A \subset P$ *splits* iff A has a partition $A = B \cup^* C$ such that $P = B^{\uparrow} \cup C^{\downarrow}$, i.e for each $p \in P$ we have either $b \leq p$ for some $b \in B$ or $p \leq c$ for some $c \in C$.

Finite Theorem 7 (Ahlswede, R ; Erdős, P. L.; Graham, Niall, [3])**.** *In a finite cut-free poset every finite maximal antichain splits.*

The plain generalization fails for infinite posets. In fact, in [12] it was proved that if P is an infinite cut-free poset whose order structure is "rich enough" then there are both splitting and non-splitting maximal antichains in P.

As usual, the method of a successful generalization for infinite posets was to keep finite certain key structures as follows.

An antichain A in a poset P is *locally finite* iff every element of P is comparable to only finitely many elements of the antichain.

Infinite Theorem 7 (P. L. Erdős, –, [12])**.** *In a cut-free poset every locally finite maximal antichain splits.*

The interesting point in this generalization is that we do not know how to prove this infinite theorem from the finite one just using Gödel Compactness Theorem! The problem is that if P is cut-free and $Q \subset P$ is finite then there is no way to find a cut-free finite $Q' \supset Q$.

In the next section we will see a problem when we have theorems for the uncountable infinite case but the finite case is harder than even the countable infinite.

2.8. **Chromatic number of product of graphs.**

Hedetniemi's Conjecture: *If G and H are finite graphs then $\chi(G\times H) = \min\{\chi(G), \chi(M)\}$.*

There are only partial results, e.g.

Finite Theorem 8 (El-Sahar, Sauer)**.** *If $\min\{\chi(G), \chi(H)\} \geq 4$ then $\chi(G\times H) \geq 4$.*

Consider first the countable infinite case.

Infinite Theorem 8.1 (Hajnal)**.** *If $\chi(G), \chi(H) \geq \omega$ then $\chi(G\times H) \geq \omega$.*

On the other hand, there are counterexamples for uncountable cardinalities:

Infinite Theorem 8.2 (Hajnal, [16])**.** *There are two ω_1-chromatic graphs G and H on ω_1 such that $\chi(G\times H) = \omega$.*

The construction is based on the existence of disjoint stationary subsets of ω_1.

Infinite Theorem 8.3 (–, [19])**.** *It is consistent with GCH that there are two ω_2-chromatic graphs G and H on ω_2 s. t. $\chi(G\times H) = \omega$.*

The proof is a forcing construction.

However, there are open problems even for the uncountable cases, e.g.:

Problem 2.4. *Is it consistent with GCH that there are two ω_3-chromatic graphs G and H on ω_3 s. t. $\chi(G\times H) = \omega$?*

3. Classical theorems

In this section we investigate the relation between four classical theorems and their infinite versions.

3.1. **Euler trails and Euler circles.** In a graph G an *Euler circuit* is a circuit containing all the edges of G. An *Euler trail* is a trail containing all the edges of G.

Finite Theorem 9. *(1) A finite connected graph has an Euler-circle iff the graph is* Eulerian, *i.e. each vertex has even degree.*
(2) A finite connected graph has an Euler-trail with end-vertices $v \neq w$ iff v and w are the only vertices of odd degree.

First one should find an infinite version of the notion of Euler trails (Euler circuits).

A *one-way infinite Euler trail* T in a graph G is a one-way infinite sequence $T = (x_0, x_1 \dots,)$ of vertices such that $E(T) = \{x_i x_{i+1} : i \in \mathbb{N}\}$ is a 1–1 enumeration of the edges of G. x_0 is the *end-vertex* of the trail. A *two-way infinite Euler trail* T in a graph G is a two-way infinite sequence $T = (\dots, x_{-2}, x_{-1}, x_0, x_1 \dots,)$ of vertices such that $\{x_i x_{i+1} : i \in \mathbb{Z}\}$ is a 1–1 enumeration of the edges of G.

Problem 3.1 (König)**.** *When does a countable infinite graph G contain a one/two-way infinite Euler trail?*

The plain generalizations of the finite theorems fail for infinite graphs (see Figure 1 below): in the first graph G each vertex has even degree, but there is no two-way infinite Euler trail, in the second graph H there is exactly one vertex with odd degree but there is no one-way infinite Euler trail.

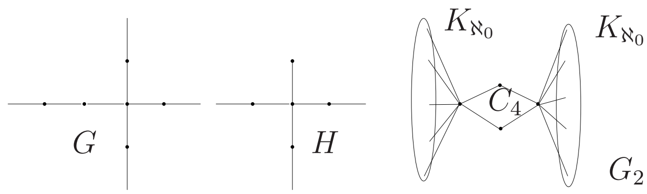

Figure 1

Infinite Theorem 9 (Erdős, P.; Grünwald, T.; Vázsonyi, E., 1938, [9] and [10])**.** *A graph $G = (V, E)$ has a one-way infinite Euler trail with end-vertex $v \in V$ iff (o1)-(o4) below hold:*
(o1) G is connected, $|E(G)| = \aleph_0$,
(o2) $d_G(v)$ is odd or infinite,

(o3) $d_G(v')$ *is even or infinite for each* $v' \in V(G) \setminus \{v\}$,
(o4) $G \setminus E'$ *has one infinite component for each finite* $E' \subset E$.

To simplify our notation we will write $owit(G, v)$ to mean that (o1)-(o4) above hold for G and v.

If $G = (V, E)$ is a graph and T is a trail in G define the graph $G \setminus T = (V', E')$ as follows: $E' = E \setminus E(T)$ and $V' = \{v \in V : d_{E \setminus E(T)}(v) > 0\}$, i.e. remove the isolated vertices from the graph $(V, E \setminus E(T))$.

Proof. The assumptions (o1)–(o4) are clearly necessary.

Assume now that $owit(G, v)$ holds. The key step of the proof is the following lemma:

Lemma 3.2. *Assume that* G *is a graph,* $v \in V(G)$, $e \in E(G)$ *and* $owit(G, v)$ *holds. Then there is a trail* T *with endpoints* v *and* v' *such that* $e \in E(T)$ *and* $owit(G \setminus T, v')$ *holds.*

Proof. Since G is connected, there is an endpoint v^* of e and a trail T' in G from v to v^* such that e is the last edge of T.

Let $G^* = G \setminus T'$. In T' two vertices, v and v^* have odd degree. Hence, by (o2) and (o3), in G^* only one vertex, v^* may have odd degree, and the degree of v^* in G^* is either infinite or odd. So the component G' of v^* in G^* should be infinite because a finite component can not contain exactly one vertex with odd degree. By (o4), all the other components of G^* are finite. Moreover, all these finite components should be Eulerian because in G^* only one vertex, v^* may have odd degree. Let H be the union of T and the finite components of G^*. This is a connected finite graph in which exactly two vertices, v and v^* have odd degrees. Hence in H there is an Euler-trail T from v to v^*. Then $G \setminus T = G'$. We show that $owit(G \setminus T, v^*)$ holds. (o1) holds because G' is a component of G^* so it is connected. Since $d_G(x) = d_{G \setminus T}(x) + d_T(x)$ for each $x \in V$, and $d_T(x)$ is odd iff $x = v$ or $x = v^*$, an easy computation gives that (o2) and (o3) also hold for $G \setminus T$. If F is a finite set of edges of $G \setminus T$ then $(G \setminus T) \setminus F = G \setminus (E(T) \cup F)$ so, applying (o4) for G, we obtain that $(G \setminus T) \setminus F$ has only one infinite component. Hence (o4) also holds for $G \setminus T$. Hence T satisfies the requirements. $\square$

Using this lemma an easy inductive construction gives a one-way infinite Euler trail in G because G has just countably many edges. $\square$

Infinite Theorem 10 (Erdős, P; Grünwald, T.; Vázsonyi, E., 1938, [9] and [10])**.** *A graph* G *has a two-way infinite Euler trail iff (t1)–(t4) below hold:*

(t1) G *is connected,* $|E(G)| = \aleph_0$,
(t2) $d_G(v)$ *is even or infinite for each* $v' \in V(G)$

(t3) *$G \setminus E'$ has at most two infinite components for each finite $E' \subset E$.*
(t4) *$G \setminus E'$ has one infinite component for a finite $E' \subset E$ provided that every degree is even in (V, E').*

We will write $twit(G)$ to mean that the stipulations (t1)-(t4) above hold for G.

The third graph G_2 on figure 1 shows that we really need to assume (t4): G_2 satisfies (t1)-(t3) but it does not have a two-way infinite Euler trail.

Proof. The assumptions (t1)–(t3) are clearly necessary. To check (t4) assume that $E' \subset E$ is finite such that every degree is even in (V, E'). Let $T = (\dots, x_{-2}, x_{-1}, x_0, x_1 \dots,)$ be a two-way infinite Euler line in G. Fix $n \in \mathbb{N}$ such that $E' \subset E_n$, where $E_n = \{x_i x_{i+i} : -n \le i < n\}$. Consider the graph $G_n = (V_n, E_n \setminus E')$ where $V_n = \{x_i : -n \le i \le n\}$. Then in G_n only the vertices x_{-n} and x_n have odd degree, hence they are in the same connected component. Hence in $G \setminus E'$ the connected component of x_{-n} and x_n contains $V \setminus V_n$, and so there is only one infinite component.

Assume now that $twit(G)$ holds. We should distinguish two cases.
Case 1:

$(*)$ *For each finite trail T the graph $G \setminus T$ has one infinite component.*

Lemma 3.3. *Let G be a graph, $v \in V(G)$ and $e \in E(G)$. If $twit(G)$ and $(*)$ hold then there is a circuit T in G such that $v \in V(T)$, $e \in E(T)$ and $twit(G \setminus T)$.*

Proof of the Lemma. Since G is connected, there is a trail T' in G from v to some endpoint v' of e such that e is the last edge of that trail.

Then in $G \setminus T'$ at most two vertices, v and v' may have odd degree. The vertices v and v' can not be in different connected component of $G \setminus T'$. Otherwise one of that components would be finite and would contain exactly one vertex with odd degree, which is impossible. So there is a path S from v to v' in $G \setminus T'$. Then $T'' = S \cup T'$ is a circuit. Let G' be the infinite component of $G \setminus T''$. Clearly all the finite components of $G \setminus T''$ should be Eulerian. Let H is the union of T'' and the finite components of $G \setminus T''$. This is a connected Eulerian finite graph. Let T be an Euler circle of H. Since $G \setminus T'$ had exactly one infinite component, the graph $G \setminus T$ is just that component. Hence $twit(G \setminus T)$ holds. Hence T satisfies the requirements of the lemma. □

By the lemma above there are a sequence $\{v_i : i < \omega\}$ of vertices and edge-disjoint circuits $\{T_i : i < \omega\}$ in G such that
(a) $x_i, x_{i+1} \in V(T_i)$ for $i < \omega$,

(b) $E(G) = \bigcup\{E(T_i) : i < \omega\}$.

Using these circuits we can easily put together a two-way infinite Euler trail.

Case 2:

($*$) *There is a finite trail T such that the graph $G \setminus T$ has two infinite components.*

Let v_1 and v_2 be the endpoints of T. These vertices should be in different components of $G \setminus T$ otherwise there were a circuit $T' \supset T$ containing v_1 and v_2 and so $G \setminus T'$ would have two infinite components, which contradicts (t4).

Let G_1 and G_2 be the components of v_1 and v_2, respectively, in $G \setminus T$. G_1 and G_2 should be infinite since a finite graph can not contain exactly one vertex with odd degree.

Hence all the finite components of $G \setminus T$ should be Eulerian. Let H be the union of T and these finite components. This is a connected finite graph in which exactly two vertices, v and v' have odd degree. Hence in H there is an Euler-trail T' from v_1 to v_2. Then the graphs G_1, G_2 and T' are edge disjoint,

(A) $E(G) = E(G_1) \cup E(G_2) \cup E(T')$,
(B) $owit(G_1, v_1)$ and $owit(G_2, v_2)$ hold.

Hence in G_i there is a one way infinite Euler trail T_i with end-vertex v_i, for $i = 1, 2$. Thus the concatenation of T_1, T' and T_2 is a two-way infinite Euler trail in G. □

3.2. **Covering and matching.** Given a graph $G = (V, E)$ a set of edges is *independent* if no two elements are adjacent. If an edge e is incident with a vertex x we say that x *covers* e and e *covers* x. Given a graph $G = (V, E)$ a set $A \subset V$ is *matchable* into $B \subset V$ iff there is a set F of independent edges between A and B such that F covers A, i.e. every $a \in A$ is covered by some $e \in F$.

If G is bipartite with bipartition $V = W \cup^* M$ we will write $G = (M, W, E)$.

Hall's Theorem. *In a finite bipartite graph $G = (M, W, E)$ the set M is matchable into W iff $|\Gamma_G(A)| \geq |A|$ for each $A \subset M$.*

König's Theorem. *In a finite bipartite graph $G = (M, W, E)$*

$$\max\{|F| : F \subset E \textit{ is independent }\} = \min\{|C| : C \subset M \cup W \textit{ and } C \textit{ covers } E\}.$$

Menger's Theorem. *If $G = (V, E)$ is a finite graph, and v and w are non-adjacent vertices in G then*

$$\min\{|X| : X \subset V \text{ separates } v \text{ and } w\} = \max\{|\mathcal{P}| : \mathcal{P} \text{ is a family of vertex-disjoint } v\text{-}w\text{-paths}\}.$$

The plain generalizations of König's and Menger's Theorems hold for infinite graphs, but as Erdős observed these "generalizations" says almost nothing about infinite graphs. Indeed, consider the infinite version of König's Theorem: if a maximal independent family F of edges of G is infinite then let C be just the set of end-points of the elements of F. Then C clearly covers all the edges by the maximality of F and $|C| = 2|F| = |F|$ because $|F|$ was infinite.

However, (finite) König's and Menger's Theorems can be reformulated in such a way that the plain infinite versions of the reformulated theorems are deep results.

König's Theorem, reformulated. *In a finite bipartite graph $G = (M, W, E)$ there is an independent set $F \subset E$ and a set $C \subset M \cup W$ which covers E such that $|e \cap C| = 1$ for each $e \in F$.*

In 1984 Aharoni proved the infinite version of this reformulation:

Infinite König's Theorem (R. Aharoni)**.** *Every bipartite graph $G = (M, W, E)$ has an independent set F of edges and a set $C \subset M \cup W$ which covers E such that $|e \cap C| = 1$ for each $e \in F$.*

Menger's Theorem, reformulated. *If $G = (V, E)$ is a finite graph, and v and w are non-adjacent vertices in G, then there is a v-w-separating set X and there is a family $\mathcal{P}$ of vertex-disjoint v-w-paths such that $|P \cap X| = 1$ for each $P \in \mathcal{P}$.*

Based on this reformulation Erdős formulated the Erdős-Menger conjecture, which was proved by Aharoni and Berger in 2005:

Infinite Menger's Theorem (Aharoni, Berger, [1])**.** *If $G = (V, E)$ is an arbitrary graph, and v and w are non-adjacent vertices in G, then there is a v-w-separating set X and there is a family $\mathcal{P}$ of vertex-disjoint v-w-paths such that $|P \cap X| = 1$ for each $P \in \mathcal{P}$.*

There is a different problem with the plain generalization of Hall's Theorem: namely it fails! Indeed, consider the following "playboy" example: $M = \{m_i : i \geq 0\}$ $W = \{w_i : i \geq 1\}$ $E = \{(m_i, w_i) : i \geq 1\} \cup \{(m_0, w_i) : i \geq 1)\}$, and let $G = (M, W, E)$. Then $|\Gamma_G(A)| \geq |A|$ for each $A \subset M$, but M is not matchable into W. The problem is that $A = \{m_i : i \geq 1\} = \subsetneq M$ has the property that every matching of A covers W.

But as it turned out, this is the only possible problem:

Infinite Hall's Theorem (Aharoni, 1984). *If in a bipartite graph $G = (M, W, E)$ the set M does not have a matching then there is $X \subset M$ such that X is unmatchable but $\Gamma_G(X)$ is matchable into X.*

It is worth to note that for finite graphs Aharoni's theorem above is just the classical Hall's Theorem. Indeed, if $\Gamma_G(X)$ is matchable into X then $|\Gamma_G(X)| \le |X|$. So since the matching is not a bijection and $|X|$ is finite we have $|\Gamma_G(X)| < |X|$.

4. Multi-way cuts

In this section we will see that some plain generalization holds, the countable case is not harder than the finite. However, the uncountable case will demand a model-theoretic method.

Given a graph $G = (V, E)$ and $S \subset V$ let $G - S = G[V \setminus S]$, i.e. the induced subgraph on $V \setminus S$. An *S-colouring* of G is a function $f : V \longrightarrow S$ with $f \restriction S = \mathrm{id}_S$, i.e. f is the identity on S. The *value* $e_G(f)$ of an S-colouring f is the number of *bi-chromatic* edges, i.e. the number of edges whose endpoints have different colours.

If G is finite, let

$$\pi_{G,S} = \min\{e_G(f) : f \text{ is an } S\text{-colouring.}\}$$

Multiway Cut Problem. *Given a finite graph $G = (V, E)$ and a nonempty set $S \subset V$ determine $\pi_{G,S}$!*

This problem is NP-complete, [6]. However, there are some lower bounds for $\pi_{G,S}$.

The lower bound $\nu_{G,S}$ was introduced and studied in [14] and in [11].

Let $\vec{G}$ be a directed graph obtained by an orientation of the edges of G. For each $s \in S$ let $\mathcal{P}_s$ be a family of edge-disjoint directed paths from s into some element of $S \setminus \{s\}$ in $\vec{G}$. Put $\mathcal{P} = \cup\{\mathcal{P}_s : s \in S\}$. Let f be an arbitrary S-colouring. Then

($\bullet$) *there is an injection $e_{\vec{G}}$ from $\mathcal{P}$ into the set of f-bi-chromatic edges.*

Indeed, for each $P \in \mathcal{P}_s \subset \mathcal{P}$ let $e_{\vec{G}}(P)$ be the first f-bi-chromatic edge of the path P.

Hence, if we define $\nu_{G,S}$ as the maximum of $|\mathcal{P}|$ where the maximum is taken over all orientations $\vec{G}$ of G, then we have

$$\nu_{G,S} \le \pi_{G,S}.$$

Finite Theorem 10 (Erdős, P. L,; Frank, A; Székely, L [11]). *If $G = (V, E)$ is a finite graph and $S \subset V$ has at least two elements such that $G - S$ is a tree then $\nu_{G,S} = \pi_{G,S}$.*

If you want to find an infinite version of this theorem you can easily recognize that cardinality is "too coarse" invariant, see just the argument after the first infinite version of König's Theorem.

However, ($\bullet$) holds even for an infinite graphs, and the finite theorem can be reformulated as follows: *if $G = (V, E)$ is a finite graph and $S \subset V$ has at least two elements such that $G - S$ is a tree then*

- *there is an orientation $\vec{G}$ of G, and*
- *there is a family $\mathcal{P} = \cup\{\mathcal{P}_s : s \in S\}$, where $\mathcal{P}_s$ is a family of edge-disjoint directed paths from s into some element of $S \setminus \{s\}$ in $\vec{G}$,*

such that $e_{\vec{G}}$ (as it was defined above) is a bijection between $\mathcal{P}$ and the f-bi-chromatic edges.

This is the version of the theorem which is meaningful and non-trivial even for infinite graphs.

Infinite Theorem 11. *Assume that $G = (V, E)$ is an infinite graph and $S \subset V$ is a finite subset having at least two elements such that $G - S$ is a tree which does not contain infinite paths. Then*

- *there is an orientation $\vec{G}$ of G, and*
- *there is a family $\mathcal{P} = \cup\{\mathcal{P}_s; s \in S\}$, where $\mathcal{P}_s$ is a family of edge-disjoint directed paths from s into some element of $S \setminus \{s\}$ in $\vec{G}$,*

such that $e_{\vec{G}}$ is a bijection between $\mathcal{P}$ and the f-bi-chromatic edges.

Proof. The proof tries to imitate the arguments from [11].

Consider the tree $T = G - S$.

We can assume that if ws is an edge for some $s \in S$ and $w \in T$ then w is a leaf of T and $d_G(w) = 2$ because we can subdivide the edge ws by a new node.

We can assume that every leaf of T is connected to some element of S.

Fix a vertex r as the root of T, and let $\vec{T}$ be the rooted tree order of T. Since T does not contain infinite paths, we have that $\vec{T}^*$, the inverse order of $\vec{T}$ is well-founded. Hence we can define a function $L : T \longrightarrow P(S) \setminus \{\emptyset\}$ by the following well-founded induction.

Assume that $L(w')$ is defined for $w <_{\vec{T}} w'$. If w is a leaf, let $L(w) = \{s \in S : (s, w) \in E\}$. By our assumption, we have $|L(w)| = 1$.

Assume that w is not a leaf. For each $s \in S$ let

$$K(w, s) = \{w' : (w, w') \in E, w <_{\vec{T}} w', s \in L(w')\},$$

then put

$$\kappa_w = \max\{|K(w, s)| : s \in S\}$$

and

$$L(w) = \{s \in S : |K(w, s)| = \kappa_w\}.$$

Since S is finite, κ_w is always defined and so $L(w) \neq \emptyset$.

Since a rooted tree order is always well-founded we can define the S-colouring f of G as follows.

For the root $r \in T$ let $f(r) \in L(r)$ be arbitrary. Assume that $f(w')$ is defined for the immediate $\vec{T}$-predecessor w' of w.

If $f(w') \in L(w)$ then let $f(w) = f(w')$. If $f(w') \notin L(w)$ then let $f(w) \in L(w)$ be arbitrary.

Next we determine the orientation of the edges of G in $\vec{G}$. We will say that an edge uv in $\vec{G}$ is an *up-edge* iff $u <_{\vec{T}} v$, it is a *down-edge* otherwise.

The bi-chromatic edges are defined to be down-edges. Now for each bi-chromatic edge uw, $w <_{\vec{T}} u$, fix an $f(u)$-monochromatic, $<_{\mathcal{T}}$-increasing path Q_u from u to $f(u)$ in G. Let the edges of Q_u be all down edges. So Q_u is a directed path from $f(u)$ into u in $\vec{G}$ and the edges in Q_u are all $f(u)$-monochromatic. All the other edges of T are defined to be up-edges.

If us is an edge in G for some $s \in S$ and $u \in T$ then orient us such that $in_{\vec{G}}(u) = out_{\vec{G}}(u) = 1$.

In this way we obtained an orientation $\vec{G}$ of G. Let us denote the families of up-edges and the down-edges by $\vec{E}^{up}$ and $\vec{E}^{down}$, respectively.

For each $s \in S$ let

$$F_s = \{uw \in \vec{G} : w <_{\vec{T}} u, s = f(u) \neq f(w)\}.$$

and

$$A_s = \{u \in V : \exists w \in V \ uw \in F_s\}.$$

Then $\mathcal{Q}_s = \{Q_u : u \in A_s\}$ is a family of edge disjoint directed paths in $\vec{G}$. It is enough to find a family $\mathcal{R}_s = \{R_u : u \in A_s\}$ of directed paths in $\vec{G}$ such that

(A) R_u is a directed path from u to some element of $S \setminus \{s\}$ and the first edge in R_u is just $uw \in F_s$,

(B) the paths $\mathcal{R}_s \cup \mathcal{Q}_s$ are pairwise edge-disjoint.

Indeed, let $\mathcal{P}_s = \{Q_u{}^\frown R_u\}$ for $s \in S$. Then $e_{\vec{G}}(Q_u{}^\frown R_u) = uw$ where w is the $<_{\vec{T}}$-predecessor of u. Hence $e''_{\vec{G}}\mathcal{P}_s = F_s$ and so $e''_{\vec{G}}\mathcal{P}$ is just the family of bi-chromatic edges.

Let $V_s = A_s \cup \{v \in T : f(v) \neq s\}$,

$$E_s = (\{yw \in \vec{E}^{up} : s \notin L(w)\} \cup \{yw \in \vec{E}^{down} : s \in L(y) \wedge f(w) \neq s\}),$$

and

$$B_s = \{y \in T : yt \in \vec{E} \text{ for some } t \in S \setminus \{s\}\}.$$

Let $\vec{H}_s = (V_s, E_s)$. We want to use Theorem 4.1 below for $\vec{H}$, A_s and B_s to get the desired family $\mathcal{R}_s$ of directed paths.

Theorem 4.1. *Assume that $G = (V, E)$ is a directed graph which does not contain directed infinite walks, and A and B are disjoint vertex sets such that*

(1) $\text{in}(a) = 0$ *and* $\text{out}(a) = 1$ *for each* $a \in A$*,*
(2) $\text{in}(b) = 1$ *and* $\text{out}(b) = 0$ *for each* $b \in B$*,*
(3) $\text{in}(x) \leq \text{out}(x)$ *for each* $x \in V \setminus (A \cup B)$*.*

Then there is a family $\mathcal{R}$ of edge-disjoint paths such that

(4) $\{\text{first}(p) : p \in \mathcal{R}\} = A$*,*
(5) $\{\text{last}(p) : p \in \mathcal{R}\} \subset B$*.*

We postpone the proof of this theorem.

It is clear that (1) and (2) hold for $\vec{H}_s$, A_s and B_s. To check (3) let $u \in V_s \setminus (A_s \cup B_s)$.

If $f(u) = s$ then $in_{\vec{H}_s}(u) = 0$.

Assume that $f(u) = t \neq s$.

Let

$$C = K(u, s) \setminus K(u, t) \text{ and } D = K(u, t) \setminus K(u, s).$$

Since $f(u) \in L(u)$ we have

$(*)$ $|D| \geq |C|$ and $|C| = |D|$ implies $s \in L(u)$.

Let x be the predecessor of u in $\vec{T}$ provided $u \neq r$.

Case 1: $u = r$ or $xu \in \vec{G}$, i.e. xu is an up-edge.

Then $\text{In}_{\vec{H}_s}(u) \setminus \{x\} = C$. Indeed, if $y \in K(u, t)$ then $f(y) = t$. So yu can not be an edge from some path Q_z for some $z \in A_t$ because xu is an up-edge.

Hence $\text{In}_{\vec{H}_s}(u) \leq |C| + 1$ and $\text{In}_{\vec{H}_s}(u) = |C|$ if $u = r$.

Moreover $\text{Out}_{\vec{H}_s}(u) \supset D$. Hence $\text{Out}_{\vec{H}_s}(u) \geq |D|$.

If $u = r$ or $|C| < |D|$ then we are done. If $u \neq r$ and $|C| = |D|$ then $s \in L(u)$ and so $xu \notin E_s$. Thus $\text{in}_{\vec{H}_s}(u) \leq |C| \leq |D| \leq \text{out}_{\vec{H}_s}(u)$.

Case 2: $ux \in \vec{G}$, i.e. ux is a down-edge.

Let us start with an observation:

$(*)$ If $|C| = |D|$ then $ux \in E_s$.

Indeed, if $|C| = |D|$ then $s \in L(u)$. Now $f(x) \neq s$ because $f(u) \neq s$ and $s \in L(u)$. Hence $ux \in E_s$.

Now for some $u <_{\vec{T}} z$ we have that zu is a down-edge from some path $Q \in \mathcal{Q}_t$.

If $z \in D$ then $s \notin L(z)$ hence $zu \notin E_s$. Hence $\text{In}_{\vec{H}_s}(u) = C$. Moreover $D \setminus \{z\} \supset \text{Out}_{\vec{H}_s}(u)$.

If $|D| > |C|$ then $\mathrm{Out}_{\vec{H}_s}(u) \geq |D| - 1 \geq |C| \geq \mathrm{In}_{\vec{H}_s}(u)$.

If $|D| = |C|$ then $ux \in E_s$ by $(*)$ and so $\mathrm{Out}_{\vec{H}_s}(u) \geq (|D| - 1) + 1 = |D| \geq |C| \geq \mathrm{In}_{\vec{H}_s}(u)$.

Finally, if $z \notin D$ then $\mathrm{In}_{\vec{H}_s}(u) \leq |C| + 1$. If $|D| \geq |C| + 1$ then $\mathrm{Out}_{\vec{H}_s}(u) \geq |D| \geq |C| + 1 \geq \mathrm{In}_{\vec{H}_s}(u)$.

If $|D| = |C|$ then $ux \in E_s$ by $(*)$ and so $\mathrm{Out}_{\vec{H}_s}(u) \geq |D| + 1 \geq |C| + 1 \geq \mathrm{In}_{\vec{H}_s}(u)$.

Hence 4.1(3) also holds, so we apply theorem 4.1 to get a family $\mathcal{R}'_s = \{R'_u : u \in A_s\}$ of edge-disjoint paths from A_s into B_s. The only problem is that the end-points of these paths are leaves of T instead of elements of S. This problem is cured in the next step.

Let $\mathcal{R}_u = \{R'_u{}^\frown f(last(R_u)) : u \in U\}$. Then $\mathcal{R}_u$ satisfies our requirements. $\square$

Proof of Theorem 4.1. We prove the theorem by transfinite induction on $|A|$.

Assume first that A is countable, $A = \{a_n : n < \omega\}$.

Let p' be a maximal directed walk from a_0. Since G does not contain infinite walks it follows that p' is finite. Conditions (1)–(3) imply $b_0 = \mathrm{last}(p') \in B$. Thus there is a directed path $p_0 \subset p'$ with $\mathrm{first}(p_0) = a_0$ and $\mathrm{last}(p_0) = b_0$.

Let $V' = V \setminus \{a_0, b_0\}$, $A' = A \setminus \{a_0\}$, $B' = B \setminus \{b_0\}$ and $E' = E \setminus E(p_0)$. Then the directed graph $G' = (V', E')$ and the disjoint vertex sets A' and B' satisfy (1)-(3), so we can repeat the procedure above to find a directed path p_1 in G' with $\mathrm{first}(p_1) = a_1$ and $\mathrm{last}(p_1) \in B'$.

Repeating this procedure we obtain a family $\mathcal{R} = \{p_n : n \in \omega\}$ of edge-disjoint paths with $\mathrm{first}(p_n) = a_n$. Thus $\mathcal{R}$ satisfies the requirements.

Assume now that $|A| = \kappa > \omega$. The natural idea is just to fix an enumeration $\{a_\xi : \xi < \kappa\}$ of A and try to simulate the procedure above.

However, in this case we can stuck even in the case $\kappa = \omega_1$.

Consider the following example: $A = \{a_\xi : \xi < \omega_1\}$, $B = \{b_\xi : \xi < \omega_1\}$, $V = A \cup B \cup \{v\}$, and

$$E = \{(a_n, v), (v, b_n) : n < \omega\} \cup \{(a_\omega, v)\} \cup \{(a_\xi, b_\xi) : \omega + 1 \leq \xi < \omega_1\}$$

If for each $n < \omega$ in the n^{th} step we pick the path $p_n = a_n v b_n$ then in the ω^{th} step there is no edge-disjoint path from a_ω into B.

So instead of the direct approach we use some induction.

Let $|A| = \kappa > \omega$ and assume that the theorem holds for all triples (G', A', B') with $|A'| < \kappa$.

We will partition A into some pieces and will use the inductive hypothesis inside the pieces. However we need some tool to find the right partition.

Let θ be a large regular cardinal, typically $\theta = (2^{|G|})^+$ is enough.

The *transitive closure,* $TC(x)$, of a set x is the set

$$x \cup (\cup(x) \cup (\cup \cup x) \dots,$$

i.e. the smallest transitive set containing x as a subset.

Let $H(\theta)$ be the family of sets whose transitive closure has cardinality less than θ. Put $\mathcal{H}(\theta) = \langle H(\theta), \in, \triangleleft \rangle$, where $\triangleleft$ is a well-ordering of $H(\theta)$.

Lemma 4.2. *If $G, A, B \in M \prec N \prec \mathcal{H}(\theta)$, $M \in N$, $|M| \subset M$, $|N| \subset N$ and $|N| < \kappa$ then there is a family $\mathcal{R}$ of edge disjoint paths such that*

(i) $\{\mathrm{first}(p) : p \in \mathcal{R}\} = A \cap (N \setminus M)$,
(ii) $\{\mathrm{last}(p) : p \in \mathcal{R}\} \subset B \cap (N \setminus M)$,
(iii) $\mathrm{E}(p) \subset N \setminus M$ *for* $p \in \mathcal{R}$.

Proof of the lemma. Let

$$V' = (V \cap (N \setminus M)) \cup \{w \in (V \cap M) : \exists v \in V \cap (N \setminus M)\ vw \in E\},$$

$E' = E \cap (N \setminus M)$ and $G' = (V', E')$

Claim 4.2.1. *The graph G' and the vertex sets $A \cap (N \setminus M)$ and $B \cap (N \setminus M)$ satisfy (1)-(3).*

Proof of the Claim. If $a \in A \cap (N \setminus M)$ then there is exactly one edge $av \in E$ for some $v \in V$. Then $e \in N$ but $e \notin M$ because a is definable from e. Hence (1) holds. Similarly, we can obtain (2).

To check (3) let $x \in V' \setminus (A \cup B)$.

Assume first that $x \in N \setminus M$. If $\mathrm{out}_G(x) > |N|$ then $\mathrm{out}_{G'}(x) = |N|$ because $N \models$"$|\,\mathrm{Out}_G(x) \setminus (\mathrm{Out}_G(x) \cap M)| = \mathrm{out}_G(x)$." Since $\mathrm{in}_{G'}(x) \leq |N|$ we have (3).

If $\mathrm{out}_G(x) \leq |N|$ then $\mathrm{in}_G(x) \leq |N|$ and so $\mathrm{Out}_G(x) \cup \mathrm{In}_G(x) \subset N$. But $xy \notin M$ for $y \in \mathrm{Out}_G(x) \cup \mathrm{In}_G(x)$ because $xy \in M$ implies $x \in M$. Hence $\mathrm{Out}_{G'}(x) = \mathrm{Out}_G(x)$ and $\mathrm{In}_{G'}(x) \subseteq \mathrm{In}_G(x)$ and so (3) holds.

Assume finally that $x \in V' \cap M$. Then $\mathrm{In}_G(x) \not\subset M$ and so $\mathrm{in}_G(x) > |M|$. If $\mathrm{out}_G(x) > |N|$ then $\mathrm{out}_{G'}(x) = |N|$ and so $\mathrm{in}_{G'}(x) \leq |N| = \mathrm{out}_{G'}(x)$. If $\mathrm{out}_G(x) \leq |N|$ then $\mathrm{Out}_G(x) \subset N$. Since $|M| < \mathrm{in}_G(x) \leq \mathrm{out}_G(x) \leq |N|$ we have $\mathrm{out}_{G'}(x) = \mathrm{out}_G(x) \geq \mathrm{in}_G(x) \geq \mathrm{in}_{G'}(x)$.

$\square$

Since $|A \cap (N \setminus M)| \leq |N| < \kappa$ we can apply the inductive hypothesis to find a family $\mathcal{R}$ of edge disjoint directed G'-paths satisfying (4) and (5), i.e. (i) and (ii) hold for $\mathcal{R}$. But the elements of $\mathcal{R}$ are G'-paths, so

$\mathrm{E}(p) \subset E \cap (N \cap M)$ and so (iii) also holds. This completes the proof of Lemma 4.2. □

Let $\langle M_\alpha : \alpha < \kappa \rangle$ be an increasing continuous chain of elementary submodels of $\langle H(\theta), \in \rangle$, i.e.,

(1) M_α is a elementary submodel of $\mathcal{H}(\theta)$ for $\alpha < \kappa$
(2) $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$ for $\alpha < \kappa$,
(3) $M_\alpha = \bigcup \{M_\beta : \beta < \alpha\}$ for limit α,

such that $\alpha \subset M_\alpha$, $|M_\alpha| = \alpha + \omega$ and $G, A, B \in M_0$.

For each $\alpha < \kappa$ apply the Lemma above for $M = N_\alpha$ and $N = N_{\alpha+1}$ to obtain a family $\mathcal{R}_\alpha$ of edge-disjoint paths such that

(i) $\{\mathrm{first}(p) : p \in \mathcal{R}_\alpha\} = A \cap (N_{\alpha+1} \setminus N_\alpha)$,
(ii) $\{\mathrm{last}(p) : p \in \mathcal{R}_\alpha\} \subset B \cap (N_{\alpha+1} \setminus N_\alpha)$,
(iii) $\mathrm{E}(p) \subset N_{\alpha+1} \setminus N_\alpha$ for $p \in \mathcal{R}$.

Then $\mathcal{R} = \cup\{\mathcal{R}_\alpha : \alpha < \kappa\}$ satisfies the requirements. (4) is clear because because $A = \cup\{A \cap (N_{\alpha+1} \setminus N_\alpha) : \alpha < \kappa\}$ by the continuity. (5) is trivial. Finally the elements of $\mathcal{R}$ are edge-disjoint because if $p \in \mathcal{R}_\alpha$ and $q \in \mathcal{R}_\beta$ for some $\alpha < \beta < \kappa$ then $\mathrm{E}(p) \subset N_{\alpha+1} \subset N_\beta$ and $\mathrm{E}(q) \cap N_\beta = \emptyset$. □

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Alfréd Rényi Institute of Mathematics
E-mail address: soukup@renyi.hu