HEREDITARILY LINDELÖF SPACES OF SINGULAR DENSITY

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Abstract. A cardinal λ is called ω-inaccessible if for all $\mu < \lambda$ we have $\mu^\omega < \lambda$. We show that for every ω-inaccessible cardinal λ there is a CCC (hence cardinality and cofinality preserving) forcing that adds a hereditarily Lindelöf regular space of density λ. This extends an analogous earlier result of ours that only worked for regular λ.

In [1] we have shown that for any cardinal λ a natural CCC forcing notion adds a hereditarily Lindelöf 0-dimensional Hausdorff topology on λ that makes the resulting space X_λ left-separated in its natural well-ordering. It was also shown there that the density $d(X_\lambda) = \mathrm{cf}(\lambda)$, hence if λ is regular then $d(X_\lambda) = \lambda$. The aim of this paper is to show that a suitable extension of the construction given in [1] enables us to generalize this to many singular cardinals as well.

Note that the existence of an L-space, that we now know is provable in ZFC (see [3]), is equivalent to the existence of a hereditarily Lindelöf regular space of density ω_1. Since the cardinality of a hereditarily Lindelöf T_2 space is at most continuum, just in ZFC we cannot replace in this ω_1 with anything bigger. The following problem however, that is left open by our subsequent result, can be raised naturally.

Problem 1. *Assume that $\omega_1 < \lambda \le \mathfrak{c}$. Does there exist then a hereditarily Lindelöf regular space of density λ ?*

We should emphasize that this problem is open for all cardinals λ, regular or singular, in particular for $\lambda = \omega_2$.

Before describing our new construction, let us recall that the one given in [1] is based on simultaneously and generically "splitting into two" the complements $\lambda \setminus \alpha$ for all proper initial segments α of λ. The novelty in the construction to be given is that we shall perform the same simultaneous splitting for the complements of the members of a

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family $\mathcal{A}$ of subsets of λ that is, at least when λ is singular, much larger than the family of its proper initial segments (that is just λ if we are considering von Neumann ordinals). The following definition serves to describe the properties of such a family of subsets of λ.

Definition 2. Let λ be an infinite cardinal. A family $\mathcal{A}$ of *proper* subsets of λ is said to be *good* over λ if it satisfies properties (i)-(iii) below:

(i) $\lambda \subset \mathcal{A}$ that is, all proper initial segments of λ belong to $\mathcal{A}$;
(ii) for every subset $S \subset \lambda$ with $|S| < \lambda$ there is $A \in \mathcal{A}$ with $S \subset A$;
(iii) for every subset $S \subset \lambda$ with $|S| = \omega_1$ there is $T \in [S]^{\omega_1}$ such that if $A \in \mathcal{A}$ then either $|A \cap T| \leq \omega$ or $T \subset A$.

If λ is regular then $\mathcal{A} = \lambda$, the family of all proper initial segments of λ, is a good family over λ. Indeed, (i) and (ii) are obviously valid and if $S \in [\lambda]^{\omega_1}$ then any subset T of S of order type ω_1 satisfies (iii). If, however, λ is singular then this $\mathcal{A}$ definitely does not satisfy condition (ii). Actually, we do not know if it is provable in ZFC that for any (singular) cardinal λ there is a good family over λ. But we know that they do exist if λ is ω-inaccessible, that is $\mu^\omega < \lambda$ holds whenever $\mu < \lambda$.

Theorem 3. *If λ is an ω-inaccessible cardinal then there exists a good family $\mathcal{A} \subset [\lambda]^{<\lambda}$ over λ.*

Proof. It is well-known that there is a map $G : [\omega]^\omega \to \omega$ with the property that for every $a \in [\omega]^\omega$ we have $G\big[[a]^\omega\big] = \omega$. In other words: we may color the infinite subsets of ω with countably many colors so that on the subsets of any infinite set all the colors are picked up. Such a coloring may be constructed by a simple transfinite recursion.

Next we fix a *maximal almost disjoint* family $\mathcal{F}$ of subsets of order type ω of our underlying set λ and then we "transfer" the coloring G to each member F of $\mathcal{F}$. More precisely, this means that for every $F \in \mathcal{F}$ we fix a map $G_F : [F]^\omega \to F$ such that $G_F\big[[a]^\omega\big] = F$ whenever $a \in [F]^\omega$. Then we "fit together" these colorings G_F to obtain a coloring $H : [\lambda]^\omega \to \lambda$ of all countable subsets of λ as follows: For any $S \in [\lambda]^\omega$ we set $H(S) = G_F(S)$ if there is an $F \in \mathcal{F}$ with $S \subset F$ and $H(S) = 0$ otherwise. The coloring H is well-defined because, as $\mathcal{F}$ is almost disjoint, for every $S \in [\lambda]^\omega$ there is at most one $F \in \mathcal{F}$ with $S \subset F$.

Now, a set $C \subset \lambda$ is called H-closed if for every $S \in [C]^\omega$ we have $H(S) \subset C$. Clearly, for every set $A \subset \lambda$ there is a smallest H-closed set including A that will be denoted by $cl_H(A)$ and is called the H-closure of A.

Let us set $A^+ = A \cup H\big[[A]^\omega\big]$ for any $A \subset \lambda$. It is obvious that then we have

$$cl_H(A) = \bigcup_{\alpha<\omega_1} A^\alpha,$$

where the sets A^α are defined by the following transfinite recursion: $A^0 = A,\ A^{\alpha+1} = (A^\alpha)^+$, and $A^\alpha = \cup_{\beta<\alpha} A^\beta$ for α limit. Since

$$H\big[[A]^\omega\big] \subset \bigcup \{F \in \mathcal{F} : |F \cap A| = \omega\} \cup \{0\}\,,$$

it is also obvious that we have $|A^+| \le |A|^\omega$ for all $A \subset \lambda$ and consequently

$$|cl_H(A)| \le |A|^\omega$$

as well. In particular, $|A| < \lambda$ implies $|cl_H(A)| < \lambda$ because λ is ω-inaccessible.

Now we claim that the family $\mathcal{A}$ of all H-closed sets of cardinality less than λ is good over λ. Indeed, first notice that because each $F \in \mathcal{F}$ has order type ω, for every set $S \in [F]^\omega$ we have

$$H(S) = G_F(S) < \sup F = \sup S,$$

implying that every initial segment α of λ is H-closed and so $\mathcal{A}$ satisfies condition (i) of definition 2. Condition (ii) is satisfied trivially.

To see (iii) we first show that there is no infinite strictly descending sequence of H-closed subsets of λ, or in other words: the family of H-closed sets is well-founded with respect to inclusion. Assume, reasoning indirectly, that $\{C_n : n < \omega\}$ is a strictly decreasing sequence of H-closed sets and for each $n < \omega$ we have $\alpha_n \in C_n \setminus C_{n+1}$. By the maximality of $\mathcal{F}$ then there is some $F \in \mathcal{F}$ such that the set $S = F \cap \{\alpha_n : n < \omega\}$ is infinite. Then, for any $k < \omega$, the set $S \cap C_k$ is also infinite and consequently we have

$$H\big[[S \cap C_k]^\omega\big] = G_F\big[[S \cap C_k]^\omega\big] = F \subset C_k$$

because C_k is H-closed. But for any $m < \omega$ such that $\alpha_m \in S$ this would imply

$$\alpha_m \in F \subset C_{m+1}\,,$$

which is clearly a contradiction.

Now let $S \subset \lambda$ with $|S| = \sigma$. Our previous result clearly implies that there is a set $T \in [S]^\sigma$ such that we have $cl_H(U) = cl_H(T)$ whenever $U \subset T$ with $|U| = \sigma$. In other words, this means that for every H-closed set C we have either $|C \cap T| < \sigma$ or $T \subset C$. In particular, for $\sigma = \omega_1$ this shows that our family $\mathcal{A}$ satisfies condition (iii) of definition 2 as well, hence it is indeed good over λ. □

Problem 4. *Is it provable in ZFC that for every (singular) cardinal λ there is a good family over λ?*

Next we present our main result that, in view of theorem 3, immediately implies the consistency of the existence of hereditarily Lindelöf regular spaces of density λ practically for any singular cardinal λ. (Of course, this has to be in a model in which $\lambda \le \mathfrak{c}$.) We shall follow [2] in our notation and terminology concerning forcing.

Theorem 5. *Let $\mathcal{A}$ be a good family over λ. Then there is a complete (hence CCC) subforcing $\mathbb{Q}$ of the Cohen forcing $Fn(\mathcal{A} \times \lambda, 2)$ such that in the generic extension $V^{\mathbb{Q}}$ there is a hereditarily Lindelöf 0-dimensional Hausdorff topology τ on λ that has density λ. If we also have $\mathcal{A} \subset [\lambda]^{<\lambda}$ (as in theorem 3) then every subset of λ of size $< \lambda$ is even τ-nowhere dense.*

Proof. We start by defining the the subforcing $\mathbb{Q}$ of $Fn(\mathcal{A} \times \lambda, 2)$: $\mathbb{Q}$ consists of those $p \in Fn(\mathcal{A}\times\lambda, 2)$ for which $\langle A, \alpha\rangle \in \operatorname{dom} p$ with $\alpha \in A$ implies $p(A, \alpha) = 0$ and $\langle A, \gamma_A\rangle \in \operatorname{dom} p$ implies $p(A, \gamma_A) = 1$, where $\gamma_A = \min(\lambda \setminus A)$. It is straight-forward to check that $\mathbb{Q}$ is a complete suborder of $Fn(\mathcal{A} \times \lambda, 2)$.

For any condition $p \in \mathbb{Q}$ and any set $A \in \mathcal{A}$ we define

$$U^p_A = \{\alpha : p(A, \alpha) = 1\},$$

and if $G \subset \mathbb{Q}$ is generic then, in $V[G]$, we set

$$U_A = \bigcup\{U^p_A : p \in G\}.$$

Next, let $U^1_A = U_A$ and $U^0_A = \lambda \setminus A$ and τ be the topology on λ generated by the sets $\{U^i_A : i < 2, A \in \mathcal{A}\}$. Note that then the family $\mathcal{B} = \{B_\varepsilon : \varepsilon \in Fn(\mathcal{A}, 2)\}$ is a base for τ, where $B_\varepsilon = \bigcap_{A\in\operatorname{dom}\varepsilon} U^{\varepsilon(A)}_A$. It is clear from the definition that each B_ε is clopen, hence τ is 0-dimensional. Now, if $\beta < \alpha < \lambda$ then we have $\alpha \in \mathcal{A}$ by (i) and hence $\beta \in \alpha \subset U^0_\alpha$ while $\alpha = \gamma_\alpha \in U^1_\alpha$, which shows that τ is also Hausdorff. It is also immediate from (ii) that no set $S \in [\lambda]^{<\lambda}$ is τ-dense, hence the space $\langle \lambda, \tau\rangle$ has density λ. Indeed, if $S \subset A \in \mathcal{A}$ then we have $S \cap U^1_A = \emptyset$, while $U^1_A \ne \emptyset$. Thus it only remains for us to prove that the topology τ is hereditarily Lindelöf.

Assume, reasoning indirectly, that some condition $p \in \mathbb{Q}$ forces that τ is not hereditarily Lindelöf, i. e. there is a right separated ω_1-sequence in λ. More precisely, this means that there are $\mathbb{Q}$-names $\dot{s}$ and $\dot{e}$ such that p forces "$\dot{s} : \omega_1 \to \lambda,\ \dot{e} : \omega_1 \to Fn(\mathcal{A}, 2),\ \dot{s}(\alpha) \in B_{\dot{e}(\alpha)}$, and $\dot{s}(\beta) \notin B_{\dot{e}(\alpha)}$ whenever $\alpha < \beta < \lambda$." Then, in the ground model V, for each $\alpha < \omega_1$ we may pick a condition $p_\alpha \le p$, an ordinal $\nu_\alpha < \lambda$, and a finite function $\varepsilon_\alpha \in Fn(\mathcal{A}, 2)$ such that

$$p_\alpha \Vdash \dot{s}(\alpha) = \nu_\alpha \wedge \dot{e}(\alpha) = \varepsilon_\alpha.$$

Since $\mathbb{Q}$ is a complete suborder of $Fn(\mathcal{A} \times \lambda, 2)$ it has property K, hence we may assume without any loss of generality that the conditions p_α are pairwise compatible. By extending the conditions p_α, if necessary, we may assume that $\operatorname{dom} p_\alpha = I_\alpha \times a_\alpha$ with $I_\alpha \in [\mathcal{A}]^{<\omega}$ and $a_\alpha \in [\lambda]^{<\omega}$, moreover $\operatorname{dom} \varepsilon_\alpha \subset I_\alpha$ and $\nu_\alpha \in a_\alpha$ whenever $\alpha < \omega_1$. With an appropriate thinning out (and re-indexing) we can achieve that if $\alpha < \beta < \omega_1$ then

$$\nu_\beta \notin a_\alpha \cup \{\gamma_A : A \in I_\alpha\}.$$

Using standard counting and delta-system arguments, we may assume that each ε_α has the same size $n < \omega$, moreover the sets

$$\operatorname{dom} \varepsilon_\alpha = \{A_{i,\alpha} : i < n\} \in [\mathcal{A}]^n$$

form a delta-system, so that for some $m < n$ we have $A_{i,\alpha} = A_i$ if $i < m$ for all $\alpha < \omega_1$, and the families $\{A_{m,\alpha}, ..., A_{n-1,\alpha}\}$ are pairwise disjoint. We may also assume that for every $i < n$ there is a fixed value $l_i < 2$ such that $\varepsilon_\alpha(A_i) = l_i$ for all $\alpha < \omega_1$. With a further thinning out we may achieve to have

$$\operatorname{dom} \varepsilon_\alpha \cap I_\beta = \{A_i : i < m\}$$

whenever $\alpha < \beta < \omega_1$.

Finally, by property (iii) of the good family $\mathcal{A}$, we may also assume that the set $T = \{\nu_\alpha : \alpha < \omega_1\} \in [\lambda]^{\omega_1}$ satisfies either $|A \cap T| \le \omega$ or $T \subset A$ whenever $A \in \mathcal{A}$.

Now, after all this thinning out, we claim that there is a countable ordinal $\alpha > 0$ such that, for every $i < n$, if $\nu_\alpha \in A_{i,0}$ then $l_i = 0$. Indeed, arguing indirectly, assume that for every $0 < \alpha < \omega_1$ there is an $i_\alpha < n$ with $\nu_\alpha \in A_{i_\alpha,0}$ and $l_{i_\alpha} = 1$. Then there is a fixed $j < n$ such that the set $\{\alpha : i_\alpha = j\}$ is uncountable and $l_j = 1$. But the first part implies $|A_{j,0} \cap T| = \omega_1$, hence $\nu_0 \in T \subset A_{j,0} \subset U^0_{A_{j,0}}$ that would imply $\varepsilon_0(A_{j,0}) = l_j = 0$, a contradiction.

So, let us choose $\alpha > 0$ as in our above claim. We then define a finite function $q \in Fn(\mathcal{A} \times \lambda, 2)$ by setting $q \supset p_0 \cup p_\alpha$,

$$\operatorname{dom} q = \operatorname{dom} p_0 \cup \operatorname{dom} p_\alpha \cup \{\langle A_{i,0}, \nu_\alpha \rangle : m \le i < n\},$$

and finally

$$q(A_{i,0}, \nu_\alpha) = l_i$$

for all $m \le i < n$. We have $\nu_\alpha \notin a_0$, and also $A_{i,0} \notin I_\alpha$ for $m \le i < n$ by our construction, hence this definition of q is correct. Moreover, by the above claim if $\nu_\alpha \in A_{i,0}$ then $l_i = 0$ and if $\nu_\alpha \notin A_{i,0}$ then $\nu_\alpha \neq \gamma_{A_{i,0}}$, consequently we actually have $q \in \mathbb{Q}$.

Let us observe, however, that we have $q(A_{i,0}, \nu_\alpha) = l_i$ for all $i < n$. Indeed, if $i < m$ then this holds because $p_\alpha(A_{i,0}, \nu_\alpha) = p_\alpha(A_i, \nu_\alpha) = l_i$. But this implies that $q \Vdash \nu_\alpha \in B_{\varepsilon_0}$ and hence $q \Vdash \dot{s}(\alpha) \in B_{\dot{e}(0)}$ that is clearly a contradiction because q extends p.

Now assume that we also have $\mathcal{A} \subset [\lambda]^{<\lambda}$ (in V). Since $\mathbb{Q}$ is CCC, every subset of λ in $V^{\mathbb{Q}}$ is covered by a ground model set of the same size, hence it suffices to show that any ground model member Y of $[\lambda]^{<\lambda}$ is τ-nowhere dense. To see this, we first note that it follows from a straight-forward density argument that for every $\varepsilon \in Fn(\mathcal{A}, 2)$ we have $|B_\varepsilon| = \lambda$. (Actually, this only uses the assumption that $|\lambda \setminus \cup\mathcal{A}_0| = \lambda$ for every $\mathcal{A}_0 \in [\mathcal{A}]^{<\omega}$ which is weaker than $\mathcal{A} \subset [\lambda]^{<\lambda}$.)

Next, consider any set $Y \in [\lambda]^{<\lambda} \cap V$ and a fixed $\varepsilon \in Fn(\mathcal{A}, 2)$. Since $\mathcal{A}$ satisfies condition (ii) of definition 2, we may clearly find an $A \in \mathcal{A}$ such that $Y \subset A$ and $A \notin \operatorname{dom} \varepsilon$. Let $\varepsilon' = \varepsilon \cup \{\langle A, 1\rangle\}$, then $B_{\varepsilon'} = B_\varepsilon \cap U^1_A$ is a non-empty open subset of B_ε that is clearly disjoint from A and hence from Y as well. This shows that Y is indeed τ-nowhere dense. □

For a singular cardinal λ of cofinality ω the results of [1] did imply the existence of hereditarily Lindelöf regular spaces of density λ, by taking the topological sum of those of density λ_n with λ_n regular and $\lambda = \sum_{n<\omega} \lambda_n$. It should be emphasized, however, that the spaces obtained in this way clearly do not have the stronger property we obtained in theorem 5 that all subsets of size less than λ are nowhere dense. So, we do have here something new even in the case of singular cardinals of cofinality ω.

Finally, we would like to point out that the forcing construction given in [1] may be considered as a particular case of that in theorem 5, where the good family $\mathcal{A}$ over λ happens to be equal to the family of all proper initial segments of λ.

References

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