THERE IS NO FINITE-INFINITE DUALITY PAIR - FORMING ANTICHAINS - IN THE DIGRAPH POSET

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Abstract. Let $\mathbb{D}$ denote the partially ordered sets of homomorphism classes of finite directed graphs, ordered by the homomorphism relation. Order theoretic properties of this poset have been studied extensively, and have interesting connections to familiar graph properties and parameters. This paper studies the generalized duality pairs in $\mathbb{D}$: it gives a new, short proof for the Foniok - Nešetřil - Tardif theorem (characterizing all finite-finite duality pairs in $\mathbb{D}$), and shows, that there is no finite-infinite duality pair - where the pairs form antichains - in the digraph-poset.

1. Introduction

Let $\vec{G}$ and $\vec{H}$ be two directed graphs and write $\vec{G} \le \vec{H}$ or $\vec{G} \to \vec{H}$ provided that there is a homomorphism from $\vec{G}$ to $\vec{H}$, that is, a map $f : V(\vec{G}) \to V(\vec{H})$ such that for all $\langle x, y\rangle \in E(\vec{G})$, $\langle f(x), f(y)\rangle \in E(\vec{H})$. (Since in this paper we deal mainly with directed graphs - or *digraphs* for short - therefore, with some abuse of notation, we do not use the arrow notation when we can do it without danger of misunderstanding.) When we have $G \to H$ then we say that G admits an *H-coloration*. (The origin of this notion is the fact that an undirected graph G has a homomorphism into the ℓ-vertices (undirected) complete graph iff G is ℓ-colorable in the usual sense.)

The relation $\le$ is a quasi-order and so it induces an equivalence relation: we say that G and H are *homomorphism-equivalent* or *hom-equivalent* and write $G \sim H$ if and only if $G \le H$ and $H \le G$. The *homomorphism poset* $\mathbb{D}$ is the partially ordered set of all equivalence classes of finite directed graphs, ordered by the $\le$. We will often abuse notation by replacing the classes that comprise $\mathbb{D}$ with their members.

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This partially ordered set is of significant intrinsic interest and is useful tool in the study of directed graph properties. For instance, it is easily seen that it is a countable distributive lattice: the supremum, or *join*, of any pair is their disjoint sum, and the infimum, or *meet*, is their categorical or relational product. Poset $\mathbb{D}$ is "predominantly" dense – it is shown by Nešetřil and Tardif [17]. Furthermore it embeds all countable partially ordered sets – see [20] for a presentation. The latter statement holds for the posets of all directed trees or paths, respectively, see [13] and [14].

The maximal chains and antichains of an ordered set are well known objects of interest. In this case, maximal antichains are particularly relevant because of their relationship to the notion of a *homomorphism duality*, introduced by Nešetřil and Pultr [15]: say that an ordered pair $\langle F, D\rangle$ of directed graphs, is a *duality pair* if

$$F\rightarrow \;=\; \nrightarrow D \tag{1}$$

where $F\rightarrow \;=\; \{G : F \rightarrow G\}$ and $\nrightarrow D \;=\; \{G : G \nrightarrow D\}$. Equivalently, the set of all structures is partitioned by the *upset* [or *final segment*] $F\rightarrow$ and the *downset* [or *initial segment*] $\rightarrow D$. [Here we also use the other common notation $F^{\uparrow}$ and $D^{\downarrow}$ for upsets and downsets, respectively.]

In the lattice $\mathbb{D}$ of directed graphs, each tree can play the role of F in (1). In fact, in [17], Nešetřil and Tardif obtain a correspondence between duality pairs and *gap*s in the homomorphism order for general relational structures. They note, among others, in [18] that the 2-element maximal antichains in $\mathbb{D}$ are exactly the duality pairs $\langle F, D\rangle$ where F is a tree and D is its *dual*, where the dual can be directly constructed.

Foniok, Nešetřil and Tardif [9, 10] are concerned with the most general circumstance. Let $\mathcal{F}$ and $\mathcal{D}$ both be finite antichains in the poset $\mathbb{D}$. Call $\langle \mathcal{F}, \mathcal{D}\rangle$ a *generalized finite duality* if

$$\bigcup_{F\in\mathcal{F}} F\rightarrow \;=\; \bigcap_{D\in\mathcal{D}} \nrightarrow D. \tag{2}$$

Here one can assume - without extra cost - that both classes are antichains. Indeed, since $\mathcal{F}$ is finite therefore the set $\mathcal{F}'$ of minimum elements is an antichain, and $\mathcal{F}^{\uparrow} = (\mathcal{F}')^{\uparrow}$. Similarly, the maximal elements of $\mathcal{D}$ form the antichain $\mathcal{D}'$ for which $\mathcal{D}^{\downarrow} = (\mathcal{D}')^{\downarrow}$. Furthermore an easy consequence of (2) is that any element of $\mathcal{F}$ is incomparable with each element of $\mathcal{D}$.

Another consequence of this definition is the following formula:

$$\mathbb{D} = \left(\bigcup_{F\in\mathcal{F}} F\!\rightarrow \right) \;\cup\; \left(\bigcup_{D\in\mathcal{D}} \rightarrow\! D \right). \tag{3}$$

(Here, of course, the incomparability of the elements of $\mathcal{F}$ and $\mathcal{D}$ is not an immanent requirement.) The generalized finite dualities are characterized in [9, 10]. Foniok, Nešetřil and Tardif also show that all maximal finite antichains A (with three exceptional cases) yields the generalized finite duality $\langle \mathcal{F}, A \setminus \mathcal{F} \rangle$.

It is quite natural to ask, in more general circumstances, if maximal antichains possess these sorts of partitions.

Indeed, Ahlswede, P.L. Erdős and N. Graham [1] introduced the notion of "splitting" a maximal antichain. Say that a maximal antichain A of a poset P *splits* if A can be partitioned into two subsets B and C such that $P = B^{\uparrow} \cup C^{\downarrow}$; and say that P has the *splitting property* if all of its maximal antichains split. They obtained sufficient conditions for the splitting property in finite posets, from which they proved, in particular, that all finite Boolean lattices possess it. It is also a natural notion for infinite posets; see [5, 6].

On one hand side the correspondence between generalized dualities and maximal antichains obtained in [9, 10] and the partition in (3) demonstrate that for $\mathbb{D}$ essentially all finite maximal antichains split. On the other side: dropping the finiteness condition in the definition (2) one may have the notion of *generalized duality*. Using techniques from [6], Duffus, P.L. Erdős, Nešetřil and Soukup studied this further generalized notion. They proved, that under rather mild conditions, any finite, non-maximal antichain of $\mathbb{D}$ can be extended, on one hand side, into a generalized duality pair of antichains or, on the other side, into a maximal infinite antichain what does not split.

In case of general duality one has to take special care for the antichain properties since the procedure described after formula (2) can not be done in case of infinite classes.

In this paper we study the remaining cases, where at least one class of $\langle \mathcal{F}, \mathcal{D} \rangle$ is finite. We include a simplified short proof of the Foniok - Nešetřil - Tardif's theorem on generalized finite duality pairs (Section 3). In Section 2 we give the technical prerequisites for the proofs. In Section 4 we show the somewhat surprising fact, that there exists no generalized duality pair in $\mathbb{D}$ where $\mathcal{F}$ is a finite while $\mathcal{D}$ is an infinite antichain. Finally in Section 5 we introduce the problem of the infinite - finite duality pairs.

In addition to the selected individual papers, we refer the reader to the book [13] by Hell and Nešetřil that is devoted to graph homomorphisms. Chapter 3 of it gives a thorough introduction and many of the key results on maximal antichains and dualities in posets of (undirected or directed) finite graphs.

A preliminary, extended abstract version of this paper was published in 2008 ([7]).

2. Prerequisites

Given a poset $\mathcal{P} = (P, \le)$ and $A \subset P$ let

$$A^{\uparrow} = \{p \in P : \exists a \in A \ a \le_P p\}; \qquad A^{\downarrow} = \{p \in P : \exists a \in A \ p \le_P a\}.$$

Let A be a maximal antichain in P. A partition (B, C) of A is a *split of* A iff $P = B^{\uparrow} \cup C^{\downarrow}$. We say that A *splits* if A has a split.

The equivalence classes of finite directed forests and directed trees in $\mathbb{D}$ will be denoted by $\mathbb{F}$ and by $\mathbb{T}$, respectively. Given a directed graph D denote by $\operatorname{Comp}(D)$ the set of connected components of D.

For a given oriented *walk* (directed walk where the edges are not necessarily are directed consecutively) its *net-length* is the (absolute) difference between the numbers of edges oriented in one way or the other. Given $D \in \mathbb{D}$ let the net-length $\ell(D)$ of D be the supremum of the net-length of the oriented walks in D. Clearly $\ell(D) = \infty$ iff D contains an unbalanced circle.

Write $\mathbb{B} = \{D \in \mathbb{D} : \ell(D) < \infty\}$ and $\mathbb{U} = \{D \in \mathbb{D} : \ell(D) = \infty\}$. The graphs in $\mathbb{B}$ are the *balanced* ones. Clearly $\mathbb{F} \subsetneq \mathbb{B}$.

We need a result of J. Nešetřil and C. Tardif ([17]) which shows that each directed tree has a unique dual:

Theorem 2.1. *For each $T \in \mathbb{T} \backslash \{\vec{P}_0, \vec{P}_1, \vec{P}_2,\}$ there is a unique $D_T \in \mathbb{D}$ such that $\langle T, D_T \rangle$ is a duality, i.e. T and D_T are incomparable and $T \to = \not\to D_T$.*

We will use the Directed Sparse Incomparability Lemma in the following form (see [2]):

Theorem 2.2 (Directed Sparse Incomparability Lemma)**.** *For each directed graph $H \in \mathbb{D} \setminus \mathbb{F}$ and for all integers $m, k \in \mathbb{N}$ there is a directed graph H' such that*

(1) $k < \operatorname{girth}(H') < \infty$, (*this is the* girth *of the underlying undirected graph, the girth of a tree is* ∞),
(2) *for each directed graph G with $|V(G)| < m$ we have $H' \to G$ if and only if $H \to G$.*
(3) $H \not\to H'$.

3. The Foniok-Nešetřil-Tardif Theorem

In this Section we give a simplified short proof for the Foniok-Nešetřil-Tardif Theorem using only the Directed Sparse Incomparability Lemma (Theorem 2.2) and the Nesetril-Tardif Theorem (Theorem 2.1).

Lemma 3.1. *Let G be a balanced kernel with $G \not\to \vec{P}_2$. Then it contains no component C with $C \not\to \vec{P}_2$.*

Proof. Assume the contrary. Due to the condition the kernel contains component C' of net-length ≥ 3 and $C \to \vec{P}_2 \to C'$ would happen, a contradiction. $\square$

Lemma 3.2. *If $\mathcal{B} \subset \mathbb{D}$ is finite and $X \in \mathcal{B}$ with $X \notin \mathcal{B}\setminus\{X\}^{\downarrow}\setminus\mathcal{B}\setminus\{X\}$ then $(\{X\}^{\downarrow} \setminus \{X\}) \not\subset (\mathcal{B} \setminus \{X\})^{\downarrow} \setminus (\mathcal{B} \setminus \{X\})$ for $X \in \mathcal{B} \setminus \mathbb{F}$.*

Proof. Let $n = \max\{|Q| : Q \in \mathcal{B}\}$. Since $X \notin \mathbb{F}$ we can apply the Directed Sparse Incomparability Lemma for X and $k = m = n + 1$ to obtain a graph Y. Then $Y \to Q$ implies $X \to Q$ for $Q \in \mathcal{B}$ because $|Q| < m$. Thus $Y \in \big[\{X\}^{\downarrow} \setminus \{X\}\big] \setminus \big[(\mathcal{B} \setminus \{X\})^{\downarrow} \setminus (\mathcal{B} \setminus \{X\})\big]$. $\square$

Theorem 3.3 (Foniok, Nesetril Tardif, [8])**.**
(A) *If $\mathcal{F} \subset \mathbb{F}$ is a finite antichain of forests, $\mathcal{F} \neq \{\vec{P}_0\}$, $\{\vec{P}_1\}$, $\{\vec{P}_2\}$, then it has a "finite dual" $D_{\mathcal{F}}$, i.e. there is a finite antichain $D_{\mathcal{F}} \subset \mathbb{D}$ such that $\mathcal{F} \cup D_{\mathcal{F}}$ is an antichain and $(\mathcal{F}^{\uparrow} \setminus \mathcal{F}, D_{\mathcal{D}}^{\downarrow} \setminus D_{\mathcal{D}})$ is a partition of $\mathbb{D}$.*
(B) *If $\mathcal{F}$ and $\mathcal{D}$ are finite antichains in $\mathbb{D}$, $\mathcal{F} \neq \{\vec{P}_0\}$, $\{\vec{P}_1\}$, $\{\vec{P}_2\}$, and $(\to \mathcal{F}, \mathcal{D} \to)$ is a partition of $\mathbb{D}$, then $\mathcal{F} \cup \mathcal{D}$ is an antichain.*
(C) *Let $\mathcal{A} \subset \mathbb{D}$ be a finite maximal antichain in $\mathbb{D}$, $\mathcal{A} \neq \{\vec{P}_0\}$, $\{\vec{P}_1\}$, $\{\vec{P}_2\}$. Then $\mathcal{A}$ splits and $\langle \mathcal{A} \cap \mathbb{F}, \mathcal{A} \cap \mathbb{U}\rangle$ is the unique split of $\mathcal{A}$. Especially $\mathcal{A} = (\mathcal{A} \cap \mathbb{F}) \cup D_{\mathcal{A}\cap\mathbb{F}}$. Moreover $\mathcal{A} \cap (\mathbb{B} \setminus \mathbb{F}) = \emptyset$.*

Proof of Theorem 3.3. (A) Write $\mathcal{F} = \{F^i : i < n\}$ and $\mathrm{Comp}(F_i) = \{F^i_j : j < k_i\}$ for $i < n$. (In this paper the indices run from 0 to upper-bound-1.) By Theorem 2.1 we have $\mathbb{D} \setminus (F^i_j \to) = \to D_{F^i_j}$. Then

$$\mathbb{D} \setminus \bigcup_{i<n} (F^i \to) = \bigcap_{i<n} (\mathbb{D} \setminus (F^i \to)) = \bigcap_{i<n} (\mathbb{D} \setminus \bigcap_{j<k_i} (F^i_j \to)) =$$

$$\bigcap_{i<n} (\bigcup_{j<k_i} (\mathbb{D} \setminus (F^i_j \to))) = \bigcap_{i<n} \bigcup_{j<k_i} (\to D_{F^i_j}) = \bigcup_{f \in \prod_{i<n} k_i} \bigcap_{i<n} (\to D_{F^i_{f(i)}}) =$$

$$\bigcup_{f \in \prod_{i<n} k_i} (\to \prod_{i<n} D_{F^i_{f(i)}}).$$

So $\mathcal{D}_{\mathcal{F}}$ is just the $\mathbb{D}$-maximal elements of $\left\{\prod_{i<n} D_{F^i_{f(i)}} : f \in \prod_{i<n} k_i\right\}$, which proves (A).

Next we prove (B).
Let $\mathcal{B} = \mathcal{A} \cup \mathcal{D}$. If $X \in \mathcal{F}$ then $X \notin \mathcal{B} \setminus \{X\}^{\downarrow} \setminus \mathcal{B} \setminus \{X\}$. So, by Lemma 3.2, if $X \notin \mathbb{F}$ then there is $Y \in \mathbb{D}$ such that $D \notin \mathcal{F}^{\uparrow} \setminus \mathcal{F} \cup \mathcal{D}^{\downarrow} \setminus \mathcal{D}$, which is a contradiction. So $\mathcal{F} \subset \mathbb{F}$.

Hence, by (A), the finite antichain $\mathcal{F} \subset \mathbb{D}$ has the finite dual $D_{\mathcal{A}}$ such that $(\mathcal{A}^{\uparrow} \setminus \mathcal{A}, D^{\downarrow}_{\mathcal{A}} \setminus D_{\mathcal{A}})$ is a partition of $\mathbb{D}$. Hence $D^{\downarrow}_{\mathcal{A}} \setminus D_{\mathcal{A}} = \mathcal{D}^{\downarrow} \setminus \mathcal{D}$. Since $\mathcal{D}$ and $D_{\mathcal{A}}$ are antichains we have $\mathcal{D} = D_{\mathcal{A}}$. Thus $\mathcal{F} \cup \mathcal{D} = \mathcal{F} \cup D_{\mathcal{F}}$ is an antichain which was to be proved.

Finally we prove (C).

Let $\vec{P}(2,n)$ be the following oriented path: $\vec{P}_2$ then n zigzag steps, then $\vec{P}_2$ again, then n zigzag steps, etc, up to n recursion, that is

$$\vec{P}(2,n) = 1\Big((10)^n 1\Big)^n 1.$$

Clearly $\ell(\vec{P}(2,n)) = n+1$.

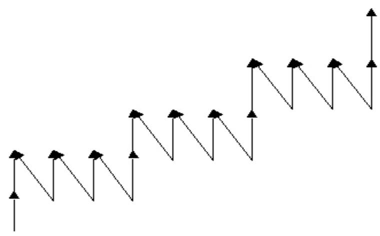

FIGURE 1. Graph $\vec{P}(2,3)$

Let $\ell = \max\{\ell(Q) : Q \in \mathcal{A} \cap \mathbb{B}\}$ and $n = \max\{|Q| : Q \in \mathcal{A}\}$.

Lemma 3.4. $\mathcal{A}^{\downarrow} \setminus \mathcal{A} = (\mathcal{A} \cap \mathbb{U})^{\downarrow} \setminus (\mathcal{A} \cap \mathbb{U})$.

Proof of the Lemma. Assume on the contrary that there exists $X \in (\mathcal{A}^{\downarrow} \setminus \mathcal{A}) \setminus \big[(\mathcal{A} \cap \mathbb{U})^{\downarrow} \setminus (\mathcal{A} \cap \mathbb{U})\big]$. Let $Y = X + \vec{P}(2, \ell+1)$.

For $Q \in \mathcal{A}$ if $Q \to Y$ then some component C of Q maps into $\vec{P}(2,\ell)$ and so $C \to \vec{P}_2$ because ℓ is large enough. But this is excluded by Lemma 3.1, a contradiction.

Moreover $Y \not\to Q$ for $Q \in \mathcal{A} \cap \mathbb{B}$ because $\ell(Y) > \ell(Q)$. For $Q \in \mathcal{A} \cap \mathbb{U}$ we have $Y \not\to Q$ because $X \to Y$ and $X \not\to Q$.

Hence Y is incomparable to any element of the maximal antichain $\mathcal{A}$. Contradiction. $\square$

Since $\mathcal{A}$ is an antichain, o applying Lemma 3.2 for $\mathcal{B} = \mathcal{A}$ we obtain

Lemma 3.5. $(\{X\}^{\downarrow} \setminus \{X\}) \not\subset (\mathcal{A} \setminus \{X\})^{\downarrow} \setminus (\mathcal{A} \setminus \{X\})$ *for* $X \in \mathcal{A} \setminus \mathbb{F}$.

Lemmas 3.4 and 3.5 together yield

$$\mathcal{A} \cap \mathbb{F} = \mathcal{A} \cap \mathbb{B}, \tag{4}$$

that is every balanced element in $\mathcal{A}$ is a directed forest.

Lemma 3.6. $(\mathcal{A} \cap \mathbb{F})^{\uparrow} \setminus (\mathcal{A} \cap \mathbb{F}) = \mathcal{A}^{\uparrow} \setminus \mathcal{A}$.

Proof. Assume on the contrary that

$$X \in \left[\mathcal{A}^{\uparrow} \setminus \mathcal{A}\right] \setminus \left[(\mathcal{A} \cap \mathbb{F})^{\uparrow} \setminus (\mathcal{A} \cap \mathbb{F})\right]. \tag{5}$$

Then $Q \to X$ for some $Q \in \mathcal{A} \cap \mathbb{U}$ and so $X \in \mathbb{U}$. Hence we can apply the Directed Sparse Incomparability Lemma for X and $k = m = \max\{n+1, |X|+1\}$ to obtain a graph Y.

Assume first that $Q' \to Y$ for some $Q' \in \mathcal{A}$. Then the image of Q' is a forest in Y because $girth(Y) > |Q'|$. Hence $Q' \in \mathbb{B}$. Thus $Q' \in \mathbb{F}$ by (4). But $Q' \to Y \to X$ which is not possible by our assumption (5).

Hence $Y \to Q$ for some $Q \in \mathcal{A}$. But then $X \to Q$ because $|Q| < n+1$. Contradiction because $X \in (\mathcal{A}^{\uparrow} \setminus \mathcal{A})$. □

Lemma 3.7. $(\{F\}^{\uparrow} \setminus \{F\}) \not\subset (\mathcal{A} \setminus \{F\})^{\uparrow} \setminus (\mathcal{A} \setminus \{F\})$ *for* $F \in \mathcal{A} \cap \mathbb{F}$.

Proof. Let $F \in \mathcal{A} \cap \mathbb{F}$. Let $Y = F + \vec{P}(2, \max(\ell, n)+1)$. Then $F \to Y$ but $\ell(Y) > \ell(F)$ so $Y \not\to F$. So $Y \in F^{\uparrow} \setminus F$.

Assume now that there exists a $Q \in \mathcal{A} \setminus \{F\}$ with $Q \to Y$. Then every connected component C of Q can be mapped either into F or into $\vec{P}(2, \ell)$. In the latter, however, $C \to \vec{P}_2$ because ℓ is large enough. But this contradicts to Lemma 3.1. □

By equation (4) $(\mathcal{A} \cap \mathbb{F}, \mathcal{A} \cap \mathbb{U})$ is a partition of $\mathcal{A}$. Hence Lemmas 3.4 and 3.6 imply that $\langle \mathcal{A} \cap \mathbb{F}, \mathcal{A} \cap \mathbb{U} \rangle$ is a split in $\mathbb{D}$.

If $\langle \mathcal{B}, \mathcal{C} \rangle$ is a split of $\mathcal{A}$ then $\mathcal{C} \supset \mathcal{A} \cap \mathbb{U}$ by Lemma 3.5, and $\mathcal{B} \supset \mathcal{A} \cap \mathbb{F}$ by Lemma 3.7. Hence $\langle \mathcal{B}, \mathcal{C} \rangle = \langle \mathcal{A} \cap \mathbb{F}, \mathcal{A} \cap \mathbb{U} \rangle$. This proves Theorem 3.3. □

4. There is no finite-infinite duality pair

In the remaining part of this paper we will study those generalized dualities, where one class is finite while the other one is infinite.

In this Section we will show that there exists no generalized duality pair $\langle \mathcal{F}, \mathcal{D} \rangle$ in $\mathbb{D}$ such that $\mathcal{F}$ is finite while $\mathcal{D}$ is infinite.

In the coming proof we will apply the following week version of a theorem of Neštřil and Tardif:

Proposition 4.1 ([17])**.** *Assume that $F, G \in \mathbb{D}$ are incomparable, G is connected, and there exists no directed graph H such that $F < H < F + G$ (in this case the pair $F, F + G$ is called a* gap*). Then G is a directed tree.*

Here we give an easy and short proof of it: Assume that $G \notin \mathbb{T}$. Then the Directed Sparse Incomparability Lemma is applicable: let $k = m = \max\{|G|, |F|\} + 1$. Then Theorem 2.2 supplies an $H' \in \mathbb{D}$ such that $H' \to G$ but not vice versa, and H' and F are incomparable. Then $F + H'$ would be an H denied in the Proposition 4.1. □

Theorem 4.2. *There is no generalized duality pair $\langle \mathcal{F}, \mathcal{D} \rangle$ forming an antichain where $|\mathcal{F}| < \omega$ and $|\mathcal{D}| = \omega$,*

Proof. Assume on the contrary that $\langle \mathcal{F}, \mathcal{D} \rangle$ is such a pair. If $\mathcal{F} \subset \mathbb{F}$ then $\mathcal{D} = D(\mathcal{F})$ is finite, so we can assume that $F_1 \in \mathcal{F} \setminus \mathbb{F}$.

Let $\mathfrak{T} = \bigcup\{\text{Comp}(F) : F \in \mathcal{F}\} \cap \mathbb{T}$ and $\mathfrak{C} = \bigcup\{\text{Comp}(F) : F \in \mathcal{F}\} \setminus \mathbb{T}$ and let $\mathcal{A} = \{D_T : T \in \mathfrak{T}\}$, where D_T is the dual of the directed tree T. (Let's recall, that $\text{Comp}(F)$ denotes the set of the connected components of F.) Let

$$\mathcal{A}^* = \left\{\prod \mathcal{B}' : \mathcal{B}' \subset \mathcal{A}\right\}.$$

Let $\mathcal{D}_1 = \mathcal{D} \cap \mathcal{A}^*$.

Let $n = \max\{|F| : F \in \mathcal{F} \cup \mathcal{D}_1\}$ and apply the Sparse Incomparability Lemma (Theorem 2.2) for the graph F_1 with parameters $k = m = n + 1$ to get the graph X_1. Then $F_1 \not\to X_1$ by Theorem 2.2 (3) and $X_1 \to F_1$ by Theorem 2.2 (2). Furthermore for each $G \in (\mathcal{F} \cup \mathcal{D}_1) \setminus \{F_1\}$ we have $X_1 \not\to G$ by Theorem 2.2 (2). Furthermore if $G \to X_1$ then $G \to F_1$ would happen, a contradiction. So X_1 is incomparable with each $G \in (\mathcal{F} \cup \mathcal{D}_1) \setminus \{F_1\}$.

Now, since $\langle \mathcal{F}, \mathcal{D} \rangle$ is a duality pair, therefore there is an $X_1' \in \mathcal{D} \setminus \mathcal{D}_1$ such that $X_1 \to X_1'$ and X_1' is incomparable with all elements of $\mathcal{F}$ by definition. We are going to find an $X_1'' \in \{X_1'\}^{\uparrow} \setminus \{X_1'\}$ such that it is incomparable with $\mathcal{F}$. The existence of such X_1'' contradicts the assumption that $\langle \mathcal{F}, \mathcal{D} \rangle$ is a split in $\mathbb{D}$.

Let $\mathcal{B} = \{D_T : T \in \mathfrak{T} \text{ and } T \not\to X_1'\}$. If this is empty, then let $T \in \mathbb{T}$ such that $T \not\to X_1'$. Now the choice $X_1'' = X_1' + T$ is satisfactory, the Theorem is proved.

Otherwise $X_1' \to D_T$ for each $D_T \in \mathcal{B}$. Thus $X_1' \to \prod \mathcal{B}$. But $\prod \mathcal{B} \in \mathcal{A}^*$ and $X_1' \notin \mathcal{A}^*$, so $\prod \mathcal{B}$ and X_1' are not equivalent, i.e. $\prod \mathcal{B} \not\to X_1'$. Due to Proposition 4.1, the pair $(X_1', \prod \mathcal{B})$ is not a gap, there are elements in this interval. We will choose our X_1'' within this interval.

Then for each $F \in \mathcal{F}$, which $F \not\to X_1'$ due to one of its tree-component, will not map into X_1'' as well.

Enumerate $\mathfrak{C}$ as $\{C_i : i < m\}$. By a finite induction on $i \leq m$ define graphs $G_0, G_1, .., G_m$ such that

(1) $G_0 = \prod \mathcal{B}$,
(2) $G_0 \leftarrow G_1 \leftarrow \ldots \leftarrow G_m \leftarrow X_1'$,
(3) if $C_i \not\to X_1'$ then $C_i \not\to G_{i+1}$.

Assume that we have constructed G_i. If $C_i \not\to X_1'$, but $C_i \to G_i$ then consider the pair $(X_1', X_1' + C_i)$. Since C_i is not a tree, this pair is not a gap (Proposition 4.1). Let G_{i+1} be any element strictly between X_1' and $X_1' + C_i$.

We claim that $X_1'' = G_m$ is incomparable with any $F \in \mathcal{F}$.

Assume on the contrary that $F \to G_m$ for some $F \in \mathcal{F}$. By definition there is $Y \in \mathrm{Comp}(F)$ such that $Y \not\to X_1'$.

If $Y \in \mathfrak{T}$ then $D_Y \in \mathcal{B}$, so $\prod \mathcal{B} \to D_Y$, and so $G_m \to D_Y$. Thus $Y \not\to G_m$. Thus $F \not\to G_m$ as well.

Assume now that $Y \in \mathfrak{C}$. Then $Y = C_i$ for some i and so $Y \not\to G_i$ and so $Y \not\to G_m$. Thus $F \not\to G_m$.

Contradiction, the Theorem is proved. □

5. Does exist infinite-finite duality pairs?!

In this Section we discuss the last open case: the existence of infinite-finite duality pairs in $\mathbb{D}$ where the classes form antichains. This case has a huge theoretical interest due to its close connection to the constrain satisfaction problems.

More specifically let $\langle \mathcal{F}, \mathcal{D} \rangle$ be a generalized duality pair where $\mathcal{F} \subset \mathbb{T}$ but it can be infinite, while $\mathcal{D}$ consists of preciously one element D. Then we say that D has the *tree duality*. The following theorem is a seminal result in the constrain satisfaction problem:

Theorem 5.1 (Hell - Nešetřil - Zhu [12]). *If digraph D has the tree duality then the D-colorability of each directed graph G can be decided in polynomial time.*

(Here the tree-duality can be strengthened to *bounded treewidth duality.*) The basic tool to prove this statement is the so-called *consistency check procedure.* This procedure is always finite and it succeeds if and only if $G \to D$. An even stronger result applies as well:

Theorem 5.2 (Hell - Nešetřil - Zhu [12]). *The* "tree duality of H" *is equivalent to the following property:* $\forall G$: ($G \to H$ *if and only if the consistency check for G with respect to H succeeds*).

As far as these authors are aware, it is still an open problem whether there exists duality pair $\langle \mathcal{F}, \mathcal{D} \rangle$ with infinite antichain $\mathcal{F} \subset \mathbb{T}$ and $|\mathcal{D}| = 1$.

6. Acknoledgement

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