A remark on the theorem of Simmons.

By A. Rényi in Budapest.

The theorem of Simmons in question [1] can be formulated as follows:

If \( n \) and \( h \) are positive integers, and if we put for \( 0 \leq p \leq 1, q = 1 - p \)

\[
(1) \quad f_{n, h}(p) = \sum_{r=0}^{h-1} \binom{n}{r} p^r q^{n-r} - \sum_{r=h+1}^{n} \binom{n}{r} p^r q^{n-r},
\]

then we have

\[
(2) \quad f_{n, h} \left( \frac{h}{n} \right) > 0 \quad \text{if} \quad p = \frac{h}{n} < \frac{1}{2}.
\]

An ingenious and simple proof of this theorem has been given by E. Feldheim ([2] and [3]; the proof is reproduced also in the text book [4], p. 171—172).

The generalization of the inequality of Simmons, for the case when \( np \)

is not an integer, has been considered in this journal by Ch. Jordan\(^1\) [5] and recently by I. B. Ház [6].

Ház tried to generalize the inequality of Simmons in that he has shown that for fixed values of \( n \) and \( h \)

\[
(3) \quad f_{n, h}(p) > 0 \quad \text{if} \quad 1 \leq h \leq \frac{n+1}{2} \quad \text{and} \quad \frac{h-1}{n} \leq p < \min \left( \frac{1}{2}, \frac{h}{n} \right).
\]

The aim of this note is to show that the apparent generalization given by Ház is really a consequence of the original inequality of Simmons if \( \frac{h}{n} < \frac{1}{2} \), and for the remaining cases \( n = 2h \) resp. \( n = 2h-1 \) it follows

---

\(^1\) One of Jordan's results expressed by the notations of the present paper runs as follows:

\[
(4) \quad f_{n, h}(p) > \binom{n}{h} p^h q^{n-h} \quad \text{if} \quad p < \frac{1}{2} \quad \text{and} \quad \frac{h-1}{n} \leq p \leq \frac{h-1}{n};
\]

further for \( \frac{h}{n+1} \leq p \leq \frac{h}{n} \) and \( p < \frac{1}{2} \) the reversed inequality is valid.
from the evident relations

\[ f_{2h, h} \left( \frac{1}{2} \right) = 0 \text{ and } f_{2h-1, h} \left( \frac{1}{2} \right) > 0. \]

To prove our assertions we need nothing else than the well known formula

\[ \sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r} = (n-s) \binom{n}{s} \int_{0}^{1} t^{s} (1-t)^{n-s-1} \, dt \]

(see e.g. [2] p. 110 or [4] p. 133). It follows from (1) and (5) that

\[ f_{n, h}(p) = \binom{n}{h} \int_{0}^{1} (h(1-t) + (n-h)t) t^h (1-t)^{n-h-1} \, dt - 1. \]

It can be seen from (6) without any calculations that \( f_{n, h}(p) \) is a decreasing function of \( p \) (0 ≤ p ≤ 1). Thus it follows from (2) that

\[ f_{n, h}(p) > 0 \text{ for } p \leq \frac{h}{n} \text{ if } \frac{h}{n} < \frac{1}{2}, \]

further it follows from (4) resp. (5) that

\[ f_{2h, h}(p) > 0 \text{ and } f_{2h-1, h}(p) > 0 \text{ for } p < \frac{1}{2}. \]

Evidently (7) and (8) contain (3) which is thus shown to be a consequence of (2) resp. (4).

We have at the same time shown that for \( \frac{h}{n} < \frac{1}{2} \) (3) can be replaced by the stronger inequality

\[ f_{n, h}(p) > f_{n, h} \left( \frac{h}{n} \right) \text{ for } p < \frac{h}{n} < \frac{1}{2}. \]

References.


(Received February 12, 1957.)