ON SINGULAR RADII OF POWER SERIES

by

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Let \( \mathcal{R}_u \) denote the class of analytic functions

\[
(1a) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n
\]

which are regular and unbounded in \(|z| < 1\). According to D. GAILER and W. MEYER—KÖNIG [1] we call the radius \( R_v \) defined by \( z = re^{\iota \varphi}, 0 \leq r < 1 \) singular for \( f(z) \), if \( f(z) \) is unbounded in any sector \(|z| < 1, \varphi - \varepsilon < \arg z < \varphi + \varepsilon \) with \( \varepsilon > 0 \). A radius which is not singular for \( f(z) \) is called regular for \( f(z) \). In [1] it has been shown that if \( f(z) \) belongs to the class \( \mathcal{R}_u \) and the power series of \( f(z) \) has HADAMARD-gaps, i. e.

\[
(1b) \quad f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}
\]

with

\[
(2a) \quad \frac{n_{k+1}}{n_k} \geq q > 1 \quad (k = 0, 1, \ldots)
\]

then every radius is singular for \( f(z) \). Clearly for every \( f(z) \in \mathcal{R}_u \) there is at least one singular radius. It is easy to see that if we suppose only that the power series (1b) has FABRY-gaps, i. e. if instead of (2a) we suppose only

\[
(2b) \quad \lim_{x \to +\infty} \frac{1}{x} \sum_{n_k < x} 1 = 0 ,
\]

then it is possible that there is only one singular radius for \( f(z) \). A simple example is furnished by

\[
(3a) \quad f_1(z) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=0}^{k^2-1} z^{N_k+j}
\]

where \( N_{k+1} \geq N_k + k^2 \ (k = 1, 2, \ldots) \). Clearly \( f_1(z) \) is regular in \(|z| < 1\) and if \( x \) is real, we have

\[
\lim_{x \to 1^-} f_1(x) = +\infty
\]
thus $f_1(z)$ belongs to the class $Q_u$ and $R_0$ is a singular radius for $f_1(z)$. On the other hand we have by (3a)

\[(3b) \quad |f_1(z)| \leq \frac{\pi^2}{3|1-z|} \quad \text{for } |z| < 1 ; \]

thus every radius $R_y$ with $0 < y < 2\pi$ is regular for $f_1(z)$.

It is also clear from this example that to ensure that every radius should be singular for $f(z)$ it is not sufficient to prescribe the rate in which the ratio

$$
\frac{1}{x} \sum_{n_k < x} 1
$$

tends to 0 for $x \to +\infty$. As a matter of fact, for $f_1(z)$ defined by (3a) we have

$$
\frac{1}{x} \sum_{n_k < x} 1 \leq \frac{s^2}{N_s}
$$

where $s$ is defined by the inequality $N_s \leq x < N_{s+1}$ and thus we can choose the sequence $N_s$ so that

$$
\frac{1}{x} \sum_{n_k < x} 1 < \varepsilon(x)
$$

holds, where $\varepsilon(x)$ ($x = 1, 2, \ldots$) is a sequence of positive numbers, tending to 0 arbitrary rapidly.

P. ERDŐS [2] has shown — answering a question of GAIER and MEYER—KÖNIG — that to ensure that every radius should be singular for $f(z)$, it is not even sufficient to suppose that the exponents $n_k$ of the lacunary power series (1b) of $f(z) \in Q_u$ satisfy the condition

\[(2c) \quad \lim_{k \to \infty} (n_{k+1} - n_k) = +\infty . \]

The question arises, for which sequences $n_k$ does there exist a function $f(z)$ belonging to the class $Q_u$ and having the power series expansion (1b), which has only one singular radius? Clearly it is impossible to give a criterion, which depends only on the rate of growth of the sequence $n_k$, because the number-theoretical properties of the sequence $n_k$ come in. As a matter of fact let the sequence $n_k$ satisfy the following condition:

D) for every $m$ ($m = 1, 2, \ldots$) there exists an integer $k_m$ such that for $k \geq k_m$ $n_k$ is divisible by $2^m$.

In this case if $R_y$ is a singular radius for $f(z)$ then $R_{y'}$, where $y' = y + 2\pi l/2^m$ is also singular for any pair of positive integers $l$ and $m$; as a matter of fact, if $z_j$ ($j = 1, 2, \ldots$) is a sequence of complex numbers with $|z_j| < 1$, $q - \varepsilon < \arg z_j < q + \varepsilon$ and

$$
\lim_{j \to +\infty} |f(z_j)| = +\infty ,
$$

then putting $q' = q + 2\pi l/2^m$ and $z'_j = z_j \exp(2\pi il/2^m)$ we have $q' - \varepsilon < \arg z'_j < q' + \varepsilon$ and as the series for $f(z'_j)$ differs from that for $f(z_j)$ only in a finite number of terms, we have also

$$
\lim_{j \to +\infty} |f(z'_j)| = +\infty .
$$
As the set of values of \( q > \) for which \( R_q \) is singular for \( f(z) \) is clearly closed (see [1]), it follows that every radius \( R_q \) is singular for \( f(z) \). Now the divisibility condition D) implies (2c), but (except for this) is compatible with every possible order of growth of \( n_k \); by other words if \( \omega_k \) is an increasing sequence of positive integers, tending arbitrarily slowly to \( +\infty \), then there exists a sequence \( n_k \) of integers having the property D) and satisfying the condition \( n_{k+1} - n_k < \omega_k \). Thus our question has to be modified to some extent. We ask for which sequences \( n_k \) does there exist a sequence \( n'_k \) such that \( 0 \leq n'_k - n_k < \omega_k \) where \( \omega_k \) is a sequence tending arbitrarily slowly to \( +\infty \), and a function

\[
(1c) \quad f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}
\]

belonging to the class \( (R_q) \), which has \( R_0 \) as its only singular radius? We shall prove, by using standard methods of probability theory, that if \( n_k \) satisfies the condition

\[
(2d) \quad \lim \inf_{(k-j) \to +\infty} \frac{1}{(n_k - n_j)^{k-j}} = 1
\]

then there exists always such a function.

Thus we prove the following

**Theorem 1.** Let \( n_k \) be an increasing sequence of natural numbers, satisfying the condition (2d). Then for any sequence \( \omega_k \) of natural numbers for which

\[
\lim_{k \to +\infty} \omega_k = +\infty,
\]

there exists a sequence \( n'_k \) of natural numbers such that \( 0 \leq n'_k - n_k < \omega_k \) and an analytic function \( f(z) \), which is regular in the unit circle has the power series\(^1\) (1c), is unbounded in \( |z| < 1 \), but is bounded in the domain \( |z| < 1, |\arg z| > \epsilon \) for any \( \epsilon > 0 \).

Our proof of the above Theorem is not constructive; we prove only by using probabilistic methods, the existence of a suitable function \( f(z) \), but can not give it explicitly.

The condition (2d) plays a role in other problems of a similar kind too; e. g. P. Erdős has proved [3] that if (2d) is satisfied, there exists a power series (1b) which converges uniformly but not absolutely for \( |z| = 1 \).

**Proof of theorem 1.** We shall need the following

**Lemma.**\(^2\) Let \( m_1 < m_2 < \ldots < m_d \) be natural numbers, \( v_1, v_2, \ldots, v_d \) independent random variables, each of which takes on the values 0, 1, \ldots, \( s - 1 \) with the same probability \( 1/s \). Let \( z \) be a complex number such that \( |z| \leq 1 \) and \( 2s|1 - z| \geq 1 \). Let us consider the random variable

\[
(4a) \quad Z = \sum_{j=1}^{d} z^{m_j + v_j}.
\]

\(^1\) \( f(z) \) can be chosen so that its power series has nonnegative coefficients.

\(^2\) A similar lemma has been used in a previous paper [4] of the authors of the present paper.
Then we have\(^3\)

\[
P \left( \left| Z \right| \geq \frac{4d}{s|1-z|} \right) \leq 4e^{-\frac{d}{32s^2}}
\]

**Proof of the Lemma.** Let us put \( z = re^{\varphi} \) and denote by \( C \) resp. \( S \) the real resp. imaginary part of \( Z \), i.e. we put

\[(4b)\]

\[C = \sum_{j=1}^{d} r^{m_j+v_j} \cdot \cos (m_j + v_j) \varphi\]

and

\[(4c)\]

\[S = \sum_{j=1}^{d} r^{m_j+v_j} \cdot \sin (m_j + v_j) \varphi\]

As

\[|Z| \leq 2 \max(|C|, |S|)\]

we have evidently

\[(6)\]

\[P \left( \left| Z \right| \geq \frac{4d}{s|1-z|} \right) \leq P \left( \left| C \right| \geq \frac{2\sqrt{2}d}{s|1-z|} \right) + P \left( \left| S \right| \geq \frac{2\sqrt{2}d}{s|1-z|} \right)\]

Now let us calculate the mean value of \( e^{tC} \) where we shall choose the value of the real number \( t \) later. We have

\[
\mathbb{M} \{e^{tC}\} = \prod_{j=1}^{d} \left[ \mathbb{M} \left( e^{tr^{m_j+v_j}\cos(m_j+v_j)\varphi} \right) \right]
\]

As

\[\left| \frac{1}{s} \sum_{n=0}^{s-1} r^{m_j+h} \cos(m_j + h) \varphi \right| \leq \left| \frac{1}{s} \sum_{n=0}^{s-1} z^{m_j+h} \right| \leq \frac{2}{s|1-z|}\]

and

\[\left| \frac{1}{s} \sum_{n=0}^{s-1} r^{N(m_j+h)} \cos(N(m_j + h) \varphi \right| \leq 1 \quad (N = 2, 3, \ldots)
\]

we have for \( 0 < |t| < 1/2 \)

\[(7)\]

\[\mathbb{M} \{e^{tC}\} \leq \left( 1 + \frac{2 |t|}{s|1-z|} + t^2 \right)^d\]

Evidently

\[P \left( \left| C \right| \geq \frac{2 \sqrt{2}d}{s|1-z|} \right) = P \left( C \geq \frac{2 \sqrt{2}d}{s|1-z|} \right) + P \left( C \leq -\frac{2 \sqrt{2}d}{s|1-z|} \right)\]

\(^3\) Here and in what follows \( P \{ \ldots \} \) denotes the probability of the event in the brackets and \( \mathbb{M} \{ \xi \} \) the mean value of the random variable \( \xi \).
further if $t < 0$, then

$$P \left( C \geq \frac{2 \sqrt{2d}}{s |1 - z|} \right) \leq M \{e^{iC}\} e^{-\frac{2|2td}{s|1 - z|}}$$

and

$$P \left( C \leq -\frac{2 \sqrt{2d}}{s |1 - z|} \right) \leq M \{e^{-iC}\} e^{-\frac{2|2td}{s|1 - z|}}$$

By choosing in (7)

$$t = \frac{1}{4s|1 - z|}$$

we obtain, taking into account that $8\sqrt{2} - 9 > 2$ and that $|1 - z|^2 \leq 4$,

$$P \left( |C| \geq \frac{2 \sqrt{2d}}{s |1 - z|} \right) \leq 2e^{-\frac{d}{32e^2}}$$

In the same way it can be shown that

$$P \left( |S| \geq \frac{2 \sqrt{2d}}{s |1 - z|} \right) \leq 2e^{-\frac{d}{32e^2}}$$

Clearly (6), (9a) and (9b) imply (5). Thus our Lemma is proved.

Let us choose now a subsequence $n_{k_p}$ of the sequence $n_k$ such that

$$k_1 < k_2 < \ldots < k_p < \ldots ,$$

$$\lim_{p \to +\infty} (k_{2p+1} - k_{2p}) = +\infty$$

and

$$\lim_{p \to +\infty} (n_{k_{2p+1}} - n_{k_{2p}})^{\frac{1}{k_{2p+1} - k_{2p}}} = 1$$

By (2d) this is possible. As a matter of fact, if $0 < \varepsilon < \frac{1}{4}$ and

$$(n_k - n_j)^{\frac{1}{k-j}} < 1 + \varepsilon$$

then either $j > [k\varepsilon]$ or $j \leq [k\varepsilon]$; in the latter case we have

$$(n_k - n_{[k\varepsilon]})^{\frac{1}{k-[k\varepsilon]}} \leq (n_k - n_j)^{\frac{1}{k-j}} \leq \left(1 + \varepsilon\right)^{\frac{1}{1-j}} \leq 1 + 3\varepsilon$$

Thus we may suppose that there exists a sequence of pairs $(k, j)$ such that

$$k \to +\infty, \ j \to +\infty, \ (k - j) \to +\infty$$

and $(n_k - n_j)^{\frac{1}{k-j}} \to 1$. This implies the existence of a sequence $k_p$ having the required properties.

Clearly we may rarify the sequence $k_p$ as much as we want; thus it can be supposed that besides (10a) and (10b) the following three conditions are also satisfied:

$$\lim_{p \to +\infty} (n_{k_{2p+1}} - n_{k_{2p}})^{\frac{1}{k_{2p+1} - k_{2p}}} = 1$$

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and $(n_k - n_j)^{\frac{1}{k-j}} \to 1$. This implies the existence of a sequence $k_p$ having the required properties.
and

(10e) \[ k_{2p+1} - k_{2p} > 64 p^{10} \]

Now let us put

(11a) \[ d_p = k_{2p+1} - k_{2p} \]

and

(11b) \[ m_{pj} = n_{k_{2p+1}} - n_{k_{2p}} \quad (j = 1, 2, \ldots, d_p) \]

further put

(11c) \[ \delta_p = \frac{1}{p} \]

(11d) \[ s_p = p^4 \]

and

(11e) \[ N_p = (m_{pd_p} + s_p) s_p \delta_p^2 \quad (p = 1, 2, \ldots) \]

Let us put

(12a) \[ z_{ph} = e^{N_p^{2\sin h}} \quad (h = 0, 1, \ldots, N_p - 1) \]

further

(12b) \[ z^*_{ph} = \begin{cases} z_{ph} & \text{for } \delta_p N_p \leq h \leq (1 - \delta_p) N_p \\ 2 \cos 2\pi \delta_p - z_{ph} & \text{for } 0 \leq h < \delta_p N_p \text{ and } (1 - \delta_p) N_p < h < N_p \end{cases} \]

(clearly in the second case \( z^*_{ph} \) is obtained by reflecting \( z_{ph} \) on the line \( \Re(z) = \cos 2\pi \delta_p \)).

Evidently

(13) \[ |z^*_{ph} - 1| \geq 1 - \cos 2\pi \delta_p \geq 8 \delta_p^2 \quad \text{for } \ p \geq 4, \quad h = 1, 2, \ldots, \ N_p \]

Let us denote by \( L_p \) the contour consisting of the arc \( 2\pi \delta_p \leq \varphi \leq 2\pi (1 - \delta_p) \) of the unit circle \( z = e^{i\varphi} \) and of the arc \( |\varphi| < 2\pi \delta_p \) of the circle \( z = 2 \cos 2\pi \delta_p - e^{i\varphi} \); clearly the points \( z^*_{ph} \ (h = 1, 2, \ldots, N_p) \) divide the line \( L_p \) into arcs of the length \( 2\pi/N_p \). By our lemma we have, denoting by \( v_{pj} \) \( (j = 1, 2, \ldots, d_p) \) independent random variables, each of which takes on the values 0, 1, \ldots, \( s_p - 1 \) with the probability \( 1/s_p \),

(14) \[ P \left( \max_{1 \leq h \leq N_p} \left| \sum_{j=1}^{d_p} z^*_{ph} m_{pj} + v_{pj} \right| > \frac{4 d_p}{8 s_p \delta_p^2} \right) \leq 4 N_p e^{- \frac{d_p}{32 s_p^2}} \]

Now putting

(15) \[ Q_p(z) = \sum_{j=1}^{d_p} z^{m_{pj} + v_{pj}} \]

we have

(16) \[ |Q_p'(z)| \leq d_p (m_{pd_p} + s_p) \quad \text{for } |z| \leq 1 \]
and thus for any two points \( z, z' \) of the closed unit circle

\[
|Q_p(z) - Q_p(z')| \leq d_p(m_{pd_p} + s_p)|z - z'|.
\]

Thus we obtain

\[
\max_{z \in \mathcal{L}_p} |Q_p(z)| \leq \max_{1 \leq h \leq N_p} \left| \sum_{f=1}^{e_p} z^h m_{ph} + r_{ph} \right| + \frac{d_p \cdot 2\pi}{s_p \cdot \delta_p^2}
\]

and therefore by (14)

\[
P \left\{ \max_{z \in \mathcal{L}_p} |Q_p(z)| \geq \frac{7d_p}{s_p^2} \right\} \leq 4N_p e^{-32s_p^2}
\]

and thus with respect to (10a)—(11e) that for \( p \geq 64 \)

\[
P \left\{ \max_{z \in \mathcal{L}_p} |Q_p(z)| \geq \frac{7d_p}{p^2} \right\} \leq 8p^2 e^{-p^2}.
\]

Thus it follows that

\[
\sum_{p=p_0}^{\infty} P \left\{ \max_{z \in \mathcal{L}_p} |Q_p(z)| \geq \frac{7d_p}{p^2} \right\}
\]

converges, and therefore, with probability 1, only a finite number of the inequalities

\[
\max_{z \in \mathcal{L}_p} |Q_p(z)| \geq \frac{7d_p}{p^2}
\]

is satisfied.

This implies that the values of \( v_p \) can be chosen in such a way that

\[
\max_{z \in \mathcal{L}_p} |Q_p(z)| < \frac{7d_p}{p^2}
\]

for all \( p \geq p_0 \).

Let us put now

\[
f(z) = \sum_{p=p_0}^{\infty} \frac{1}{d_p} z^{n_{kp}} Q_p(z)
\]

where the polynomials \( Q_p(z) \) are chosen in such a way that (21) is satisfied for all \( p \geq p_0 \). Clearly \( f(z) \) is regular in \( |z| < 1 \), and also unbounded, as all its coefficients are nonnegative and \( Q_p(1) = d_p \). On the other hand, for any \( \varphi \neq 0 \mod 2\pi \) and any \( \epsilon > 0 \) with \( 0 < \varphi - \epsilon < \varphi + \epsilon < 2\pi \) we have for all values of \( p \), for which \( 2\pi/p < \varphi - \epsilon \) and \( 2\pi (1 - 1/p) > \varphi + \epsilon \), for \( \varphi - \epsilon \leq \arg z \leq \varphi + \epsilon \), \( |z| < 1 \) (by the maximum principle)

\[
\frac{1}{d_p} |Q_p(z)| \leq \frac{7}{p^2}
\]

for \( p \geq p_0 \). But this implies, that \( f(z) \) is bounded in the sector \( |z| < 1, \varphi - \epsilon \leq \leq \arg z \leq \varphi + \epsilon \), or, by other words, \( R_0 \) is the only singular radius of \( f(z) \).

Taking into account that

\[
v_{p,\varphi} \leq s_p = p^4 \leq \omega \psi_2
\]
evidently \( f(z) \) satisfies all requirements of Theorem 1., which is therewith proved.

It can be shown that the condition \( n'_k - n_k = O(\omega_k) \) with \( \omega_k \) tending arbitrarily slow to \( + \infty \) can not be replaced in Theorem 1. by \( n'_k - n_k = O(1) \). We prove namely the following result:

**Theorem 2.** Let \( n_k \) be an increasing sequence of natural numbers, such that \( n_k \) is divisible by \( 2^m \) for all \( k \geq k_m \) \((m = 1, 2, \ldots)\). Let

\[
f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}
\]

be regular and unbounded in the unit circle, where the sequence \( b_k \) of integers is bounded. Then every radius \( R_\phi \) is singular with respect to \( f(z) \).

**Proof of Theorem 2.** It suffices to show that \( f(z) \) can not be bounded in a sector \( |\tau| < 1, \alpha < \arg z < \beta \). This will be shown by proving that if \( f(z) \) would be bounded in such a sector, it would be bounded in the whole unit circle. As a matter of fact, let us suppose that \( f(z) \) is given by (23) and that \( |b_k| \leq B \) \((k = 1, 2, \ldots)\) and put

\[
f_j(z) = \sum_{b_k=j}^{B} c_k z^{n_k} \quad (|j| \leq B)
\]

Then we may write

\[
f(z) = \sum_{j=-B}^{B} z^j f_j(z)
\]

Let us consider the values \( z_l = e^{2\pi i l/2^m} \), where \( m \) is a fixed natural number, such that

\[
2^m > \frac{4\pi(B+1)}{\beta - \alpha}
\]

and \( l \) takes on the values 0, 1, \ldots, \( 2^m - 1 \). Putting

\[
F_{j+B}(r, \theta) = \left( \sum_{k \leq k_m \atop b_k=j} c_k r^{n_k} e^{i(n_k \theta)} (re^{i\theta})^j \right) \quad (-B \leq j \leq +B)
\]

we have for \( 0 \leq r < 1, 0 \leq \theta < 2\pi \) and \( l = 0, 1, \ldots, 2^m - 1 \)

\[
f(re^{i\theta} z_l) = z_l^{-B} \sum_{l=0}^{2B} F_{j}(r, \theta) z_l^j + \Lambda
\]

where \( \Lambda \) denotes a term which is bounded in the unit circle, the bound depending only on \( m \).

As a matter of fact we have

\[
|\Lambda| \leq \sum_{k < k_m} |c_k| = A
\]

\[\]
Now by (25) there are at least \(2B + 1 \) terms of the sequence \( z_l \) \((l = 0, 1, \ldots, 2^m - 1)\) lying on the arc \( \alpha - \theta < \arg z < \beta - \theta, \ |z| = 1. \) Let us denote these numbers by \( z_l, z_{l+1}, \ldots, z_{l+2B}, \) let us fix the value of \( \theta \) and put

\[
Q_\theta(r, z_l) = \sum_{j=0}^{2B} F_j(r, \theta) \xi^j.
\]

We have by the interpolation formula of Lagrange

\[
Q_\theta(r, z_l) = \sum_{j=0}^{2B} Q_\theta(r, z_{l+j}) \frac{\Omega(z)}{\Omega'(z_{l+j}) (z - z_{l+j})}
\]

where

\[
\Omega(z) = \prod_{j=0}^{2B} (z - z_{l+j}).
\]

As by supposition there exists a constant \( K \) such that \(|f(z)| \leq K\) for \(|z| < 1, \ \alpha < \arg z < \beta\) we have by (23c), (27) and (28a)

\[
|Q_\theta(r, z_{l+j})| \leq K + A \quad (j = 0, 1, \ldots, 2B).
\]

Thus it follows, that for \(|\zeta| = 1\) we have

\[
|Q_\theta(r, \xi)| \leq \frac{(K + A) (2B + 1)}{\left(\sin \frac{\pi}{2^m}\right)^{2B}}.
\]

It follows from (23c) for \(l = 0\) that

\[
|f(\rho e^{i\theta})| \leq \frac{(K + A) (2B + 1)}{\left(\sin \frac{\pi}{2^m}\right)^{2B}} + A \quad \text{for} \quad 0 \leq \rho < 1 \quad \text{and} \quad 0 \leq \theta < 2\pi.
\]

As the bound on the right hand side of (32) does not depend on \(r\) or \(\theta\), it follows that \(f(z)\) is bounded in the whole unit circle, which contradicts our hypothesis. Thus Theorem 2. is proved.

It remains an open question, whether condition (2d) is best possible. In other words, the following problem is still unsolved:

Let

\[
f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}
\]

be regular and unbounded in \(|z| < 1\). Suppose that

\[
\lim \inf_{(k-j) \to +\infty} (n_k - n_j)^{\frac{1}{k-j}} = q > 1
\]

Is it true that all radii \(R_\varphi\) \((0 \leq \varphi < 2\pi)\) are singular for \(f(z)\)?

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HATVÁNSOROK SZINGULÁRIS SUGARAIRÓL

ERDŐS P. és RÉNYI A.

Kivonat

Legyen f(z) az egységtörőben reguláris és nem korlátos függvény. A \( z = re^{i\theta} \) (0 ≤ r < 1) sugarral, melyet a rövidség kedvéért \( R_\theta \)-vel jelölünk, D. GAIER és W. MEYER—KÖNIG nyomán (lásd [1], [2]) szingulárisnak nevezzük, ha f(z) nem korlátos a \( |z| < 1 \), \( q - \varepsilon < \arg z < q + \varepsilon \) köreikben, akármiennyin kis pozitív szám is \( \varepsilon \). A nem-sziguláris sugarakat reguláris sugarnak nevezzük.

A jelen dolgozatban a következő tételeket bizonyítjuk be:

1. tétel. Legyen \( n_k \) természetes számok egy növekvő sorozata, amelyre

\[
\lim inf \frac{1}{(n_k - n_j)^{k-j}} = 1.
\]

Legyen \( \omega_k \) egy tetszőlegesen lassan végสlőenhez tartó számsorozat. Akkor létezik olyan

\[
f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}
\]

alakú hatványsorral bíró, az egységkörben reguláris és nem korlátos f(z) függvény, amelynek csak egyetlen szinguláris sugara van, és amelynek \( n_k \) kitevői eleget tesznek a

\[
0 \leq n'_k - n_k \leq \omega_k
\]
feltételnél.

Az 1. tétele a dolgozatban valószínűségszámítási módszerrel van bizonyítva.

2. tétele. Legyen \( A_m \) (m = 1, 2, ...) egy természetes számokból álló tetszőleges növekvő sorozat és \( n_k \) egy olyan természetes számokból álló sorozat, amely azzal a tulajdonsággal bír, hogy az \( n_k \) sorozat tagjai véges sok kivétellel oszthatók \( A_m \)-mel (m = 1, 2, ...). Legyen \( b_k \) tetszőleges egész számokból álló korlátos sorozat. Tegyük fel, hogy

\[
f(z) = \sum_{n=1}^{\infty} c_k z^{n_k + b_k}
\]

az egységtörőben reguláris és nem korlátos függvény. Akkor f(z)-re vonatkozólag az egységtörő minden sugara szinguláris.
О СИНГУЛЯРНЫХ РАДИУСАХ СТЕПЕННЫХ РЯДОВ

Резюме

Пусть функция $f(z)$ регулярна и неограничена в единичном кружке. Радиус $z = re^{i\varphi}$ ($0 \leq r < 1$), обозначаемый для краткости через $R_\varphi$, следуя D. Gaier-y и W. Meyer—Konig-y (см. [1], [2]), называется сингулярным, если $f(z)$ неограничена в круговом секторе $|z| < 1$, $\varphi - \varepsilon < \arg z < \varphi + \varepsilon$ при любом положительном $\varepsilon$. Несингулярные радиусы называются регулярными. В настоящей работе доказываются следующие теоремы:

Теорема 1. Пусть $n_k$ есть возрастающая последовательность натуральных чисел, для которой

(1) \[ \lim \inf_{(k-j) \to \infty} (n_k - n_j)^{\frac{1}{k-j}} = 1. \]

Пусть $\omega_k$ есть как угодно медленно стремящаяся к бесконечности числовая последовательность. Тогда существует такая регулярная и неограниченная в единичном кружке функция $f(z)$, разлагаемая в степенной ряд вида

(2) \[ f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}, \]

которая имеет лишь единственный сингулярный радиус и для которой выполнено условие

(3) \[ 0 \leq n_k - n_k \leq \omega_k. \]

Теорема 1 доказывается в работе теоретико-вероятностным методом.

Теорема 2. Пусть $A_m$ ($m = 1, 2, \ldots$) любая возрастающая последовательность натуральных чисел, а $n_k$ последовательность натуральных чисел, за исключением конечного числа делящихся на $A_m$ ($m = 1, 2, \ldots$). Пусть $b_k$ любая ограниченная последовательность целых чисел. Предположим, что функция

(4) \[ f(z) = \sum_{k=1}^{\infty} c_k z^{n_k + b_k}, \]

регулярна и неограничена в единичном кружке. Тогда относительно $f(z)$ всякий радиус единичного кружка сингулярен.