ON "SMALL" COEFFICIENTS OF THE POWER SERIES OF ENTIRE FUNCTIONS

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§ 1. Introduction

The second named author has proved (see [1]) some years ago the following

Theorem A. Let \( f(z) \) be a transcendental entire function and let

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]

and

\[
f(z) = \sum_{n=0}^{\infty} b_n (z-1)^n
\]

be the power series of \( f(z) \) around the points 1. \( z=0 \) and \( z=1 \) respectively. Let \( Z_0(n) \) resp. \( Z_1(n) \) denote the number of zeros in the sequence \( a_0, a_1, \ldots, a_{n-1} \) resp. \( b_0, b_1, \ldots, b_{n-1} \). Then

\[
\liminf_{n \to \infty} \frac{Z_0(n) + Z_1(n)}{n} \leq 1.
\]

Theorem A can be formulated somewhat vaguely in the following way: if there are "many" zeros among the coefficients of the power series of a transcendental entire function about the point \( z=0 \), then there can not be "too many" zeros among the coefficients of its power series about the point \( z=1 \).

In the present paper we shall prove a generalization of Theorem A, whose statement can be expressed somewhat vaguely as follows: if "many" of the coefficients of the power series of an entire transcendental function about the point \( z=0 \) are "small", then there can not be "too many" "small" coefficients in the power series of the same function about the point \( z=1 \). Of course one has to give a precise meaning to the word "small" in this context.

Roughly speaking we shall call a coefficient of the power series of an entire function \( f(z) \) "small" if it is smaller in absolute value than the corresponding coefficient of the power series of an other entire function \( g(z) \) which is of smaller order of magnitude than \( f(z) \). To make this definition quite definite, we have to decide how to compare the orders of magnitude of the entire functions \( f(z) \) and \( g(z) \). This could be done for example by comparing the rate of increase for \( r \to \infty \) of the maximum modulus \( M_f(r) = \max_{|z|=r} |f(z)| \) of \( f(z) \) with that of \( g(z) \), i.e. with \( M_g(r) = \max_{|z|=r} |g(z)| \).

1 The theorem remains valid if we consider the power series of \( f(z) \) around two arbitrary different points \( \alpha \) and \( \beta \).
For our purposes it is however more convenient to compare the orders of magnitude of \( f(z) \) and \( g(z) \) by comparing the maximal terms of their power series. We shall prove the following

**Theorem 1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n (z-1)^n \) be a transcendental entire function and \( g(z) = \sum_{n=0}^{\infty} c_n z^n \) be an arbitrary entire function. Put

\[
\mu_f(r) = \max_n |a_n| r^n
\]

and

\[
\mu_g(r) = \max_n |c_n| r^n.
\]

Let us suppose that there exists a number \( \delta \) such that \( 0 < \delta = 1 \) and

\[
(1.1) \quad \liminf_{r \to \infty} \frac{\mu_g(r^{1/\delta})}{\mu_f(r)} < \infty.
\]

Let \( S_0(n) \) denote the number of indices \( k < n \) for which \( |a_k| \leq |c_k| \) and \( S_1(n) \) the number of indices \( k < n \) for which \( |b_k| = |c_k| \). Then we have\(^2\)

\[
(1.2) \quad \liminf_{n \to \infty} \frac{S_0(n) + S_1(n)}{n} = 1 + 2\delta.
\]

Let us make some remarks.

**Remark 1.** If we choose \( g(z) = 0 \), then (1.1) holds for any \( \delta > 0 \), further in this case \( S_0(n) = Z_0(n) \) and \( S_1(n) = Z_1(n) \), where \( Z_0(n) \) resp. \( Z_1(n) \) denote the number of zeros in the sequence \( a_0, a_1, \ldots, a_{n-1} \) resp. \( b_0, b_1, \ldots, b_{n-1} \). Thus in this special case Theorem 1 reduces to Theorem A, i.e. Theorem 1 is a generalization of Theorem A.

**Remark 2.** If \( f(z) \) is an entire function of order \( \sigma_f \) and \( g(z) \) an entire function of order \( \sigma_g \), then according to a well known theorem (see e.g. [2], Chapter IV, Problem 51)

\[
\limsup_{r \to \infty} \frac{\log \log \mu_f(r)}{\log r} = \sigma_f
\]

and similarly

\[
(1.3) \quad \limsup_{r \to \infty} \frac{\log \log \mu_g(r)}{\log r} = \sigma_g.
\]

Thus if \( \sigma_g < \sigma_f \) and we choose a sequence \( r_k \) \((k = 1, 2, \ldots)\) such that \( \lim_{k \to \infty} r_k = \infty \) and

\[
\lim_{k \to \infty} \frac{\log \log \mu_f(r_k)}{\log r_k} = \sigma_f \quad \text{then we have for any } \delta \text{ with } \delta > \frac{\sigma_g}{\sigma_f}
\]

\[
\lim_{k \to \infty} \frac{\mu_g(r_k^{1/\delta})}{\mu_f(r_k)} = 0.
\]

\(^2\) Clearly only the case \( \delta < \frac{1}{2} \) is interesting as otherwise (1.2) is trivial.
It follows that the conditions of Theorem 1 hold with any \( \delta \) for which \( \delta > \frac{\theta_g}{\theta_f} \) and thus we obtain

\[
\liminf_{n \to \infty} \frac{S_0(n) + S_1(n)}{n} = 1 + 2 \frac{\theta_g}{\theta_f}.
\]

In case \( \theta_g = 0 \) and \( \theta_f > 0 \) further also in case \( \theta_f = \infty \) and \( \theta_g < \infty \) it follows

\[
\liminf_{n \to \infty} \frac{S_0(n) + S_1(n)}{n} \leq 1.
\]

We shall show later (Theorem 4) that (1.4) can be improved, namely the factor 2 on the right hand side of (1.4) is unnecessary.

Of course, Theorem 1 can also be applied to certain pairs of entire functions which are either both of order 0 or both of order \( \infty \).

**Remark 3.** Clearly instead of considering the power series of \( f(z) \) about \( z = 0 \) and \( z = 1 \) we could consider its power series about \( z = \alpha \) and \( z = \beta \) where \( \alpha \) and \( \beta \) are arbitrary complex numbers and \( \alpha \neq \beta \). As a matter of fact in this case we have to apply Theorem 1 to the function \( f((\beta - \alpha)z + \alpha) \) instead of \( f(z) \) and \( g((\beta - \alpha)z) \) instead of \( g(z) \).

§ 2 contains estimations of certain interpolatory polynomials. In § 3 we give the proof of Theorem 1, further of the related Theorems 2, 3 and 4. The method of proof of Theorem 1 is the same as that of Theorem A given in [1] but besides the tools used there, the results of § 2 are also needed. § 4 contains the discussion of an example, which shows that our results are not far from being best possible.

**§ 2. Estimates of certain interpolatory polynomials**

Let \( Q_n(z) \) \((n = 1, 2, \ldots)\) be the unique polynomial of degree \( 2n - 1 \) which satisfies the following conditions:

\[
Q_n^{(0)}(0) = p_j \quad \text{for} \quad j = 0, 1, \ldots, n - 1
\]

and

\[
Q_n^{(1)}(1) = q_j \quad \text{for} \quad j = 0, 1, \ldots, n - 1
\]

where \( p_j \) and \( q_j \) \((j = 0, 1, \ldots, n - 1)\) are arbitrary complex numbers. An explicit formula for \( Q_n(z) \) has been given by P. Johansen [3]; this can be written as follows:

\[
Q_n(z) = (1 - z)^n \sum_{k=0}^{n-1} z^k \left[ \sum_{s=0}^{k} p_{k-s} \binom{n+s-1}{s} \right] +
\]

\[
z^n \sum_{k=0}^{n-1} (z - 1)^k \left[ \sum_{s=0}^{k} (-1)^s q_{k-s} \binom{n+s-1}{s} \right].
\]
Formula (2.3) can be transformed as follows:

\[
Q_n(z) = \sum_{m=0}^{2n-1} z^m \left\{ \sum_{k=0}^{n-1} \binom{n}{m-k}(-1)^{m-k} \left[ \sum_{s=0}^{k} p_{k-s} \binom{n+s-1}{s} \right] \right\} + \\
+ \sum_{t=0}^{n-1} z^{n+t} \left\{ \sum_{k=t}^{n-1} \binom{k}{t}(-1)^{k-t} \left[ \sum_{s=0}^{k} (-1)^s q_{k-s} \binom{n+s-1}{s} \right] \right\}.
\]

(2.4)

Thus we obtain

\[
\frac{Q^{(n)}_n(z)}{n!} = \sum_{t=0}^{n-1} z^{n+t} \binom{n+t}{n} Q_{n,t}
\]

where

\[
Q_{n,t} = (-1)^n \sum_{k=t}^{n-1} \binom{n}{n+t-k}(-1)^{k-t} \left[ \sum_{s=0}^{k} p_{k-s} \binom{n+s-1}{s} \right] + \\
+ \sum_{k=t}^{n-1} \binom{k}{t}(-1)^{k-t} \left[ \sum_{s=0}^{k} (-1)^s q_{k-s} \binom{n+s-1}{s} \right].
\]

Now evidently

\[
|Q_{n,t}| = \left( \max_{0 \leq u \leq n-1} |p_u| \right) \sum_{k=t}^{n-1} \binom{n}{n+t-k} \left[ \sum_{s=0}^{k} \binom{n+s-1}{s} \right] + \\
+ \left( \max_{0 \leq v \leq n-1} |q_v| \right) \sum_{k=t}^{n-1} \binom{k}{t} \left[ \sum_{s=0}^{k} \binom{n+s-1}{s} \right].
\]

As however

\[
\sum_{s=0}^{k} \binom{n+s-1}{s} = \binom{n+k}{k} < 2^{n+k}
\]

we obtain

\[
\sum_{k=t}^{n-1} \binom{n}{n+t-k} \left[ \sum_{s=0}^{k} \binom{n+s-1}{s} \right] < 2^{n+t} \cdot 3^n < 12^n
\]

further

\[
\sum_{k=t}^{n-1} \binom{k}{t} \left[ \sum_{s=0}^{k} \binom{n+s-1}{s} \right] < 8^n.
\]

Thus we obtain

\[
|Q_{n,t}| < 12^n \left( \max_{0 \leq u \leq n-1} |p_u| + \max_{0 \leq v \leq n-1} |q_v| \right) \quad \text{for} \quad t = 0, 1, \ldots, n-1.
\]

Thus we have proved the following

**Lemma 1.** Let \( Q_n(z) \) \((n=1, 2, \ldots)\) be the unique polynomial of degree \( 2n-1 \) determined by the conditions (2.1) and (2.2). Then we have

\[
\max_{|z|=1} \frac{|Q^{(n)}_n(z)|}{n!} < 12^n \left( \max_{0 \leq u \leq n-1} |p_u| + \max_{0 \leq v \leq n-1} |q_v| \right) \sum_{t=0}^{n-1} \binom{n+t}{n} q^t.
\]
§ 3. Proof of Theorems 1—4

We can suppose without the restriction of generality, that \( f(z) \) is real for 
real \( z \), i.e. that all the coefficients \( a_n \) (and thus all the \( b_n \)) are real. As a matter of 
fact suppose that Theorem 1 is already proved for such functions. If 
\( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is an arbitrary entire function, put 
\( f_1(z) = \sum_{n=0}^{\infty} a'_n z^n \) resp. \( f_2(z) = \sum_{n=0}^{\infty} a''_n z^n \) 
where \( a'_n \) and \( a''_n \) denotes the real and imaginary part of \( a_n \) respectively 
\( (n = 0, 1, \ldots) \). Clearly

\[
\mu_f(r) = \mu_{f_1}(r) + \mu_{f_2}(r)
\]

thus if \( f(z) \) satisfies (1.1), at least one of the functions \( f_1(z) \) and \( f_2(z) \) satisfies 
it too. Suppose e.g. that \( f_1(z) \) satisfies (1.1) i.e.

\[
\liminf_{r \to \infty} \frac{\mu_{f_1}(r^{1/2})}{\mu_{f_1}(r)} < \infty.
\]

Put \( f(z) = \sum_{n=0}^{\infty} b_n (z-1)^n \), further \( b_n = b'_n + ib''_n \); now if \( |a_n| \leq |c_n| \) then \( |a'_n| \leq |c_n| \), 
and similarly if \( |b_n| \leq |c_n| \) then \( |b'_n| \leq |c_n| \). Thus if \( S_0(n) \) resp. \( S'_1(n) \) denotes 
the number of indices \( k < n \) for which \( |a'_n| \leq |c_n| \) resp. \( |b'_n| \leq |c_n| \) then 
\( S_0(n) \leq S'_1(n) \) and \( S_1(n) \leq S'_1(n) \). As by supposition the statement of Theorem 1 
is valid for \( f_1(z) \), it follows that it is valid for \( f(z) \) too. Therefore in what follows 
we shall suppose that \( f(z) \) is real on the real axis, i.e. that the coefficients 
\( a_n \) and \( b_n \) are all real numbers.

To prove Theorem 1 we shall need the following Lemma A, which has 
been used in proving Theorem A in [1].

**Lemma A.** If \( h(x) \) is a real function which has \( n \) continuous derivatives in 
the closed interval \([0, 1]\) and if \( N(n) \) denotes the number of zeros of \( h^{(n)}(x) \) in the 
interval \([0, 1]\) further \( Z_0(n, h) \) and \( Z_1(n, h) \) denotes the number of zeros in the 
sequence \( h(0), h'(0), \ldots, h^{(n-1)}(0) \) and in the sequence \( h(1), h'(1), \ldots, h^{(n-1)}(1) \) 
respectively, then we have

\[
Z_0(n, h) + Z_1(n, h) - n = N(n).
\]

We shall apply Lemma A to the function

\[
h_n(z) = f(z) - Q_n(z) \quad (n = 1, 2, \ldots)
\]

where \( Q_n(z) \) is the unique polynomial of order \( 2n-1 \) defined by the following conditions:

\[
\frac{Q^{(0)}(j)}{j!} = \begin{cases} 
0 & \text{for } 0 \leq j < 8' n \\
a_j & \text{if } 8' n \leq j \leq n-1 \text{ and } |a_j| = |c_j| \\
0 & \text{if } 8' n \leq j \leq n-1 \text{ and } |a_j| > |c_j|
\end{cases}
\]

(3.2a)

\[
\frac{Q^{(1)}(j)}{j!} = \begin{cases} 
0 & \text{for } 0 \leq j < 8' n \\
b_j & \text{if } 8' n \leq j \leq n-1 \text{ and } |b_j| = |c_j| \\
0 & \text{if } 8' n \leq j \leq n-1 \text{ and } |b_j| > |c_j|
\end{cases}
\]

(3.2b)
Here $\delta'$ is an arbitrary number such that $\delta < \delta' < 1$.
 Clearly if $h_n(z)$ is defined by (3.1) and $Q_n(z)$ by (3.2a–b) we have

\begin{equation}
S_0(n) = Z_0(n, h_n) + \delta' n
\end{equation}

and

\begin{equation}
S_1(n) = Z_1(n, h_n) + \delta' n.
\end{equation}

Thus it follows by Lemma A that denoting by $N_n(n)$ the number of zeros of $h_n^{(n)}(z)$ in the interval $[0, 1]$, we have

\begin{equation}
S_0(n) + S_1(n) \leq N_n(n) + n(1 + 2\delta').
\end{equation}

Now we shall need Jensen's theorem which we shall use in the following form (see [4], p. 127, formula (13)):

**Lemma B.** Let $H(z)$ be an analytic function which is regular in a circle $|z| \leq R (R > 1)$ and $H(0) \neq 0$. Let $\mathcal{N}_o$ denote the number of zeros of $H(z)$ in the (closed) unit circle, and choose a number $\varrho$ with $1 < \varrho < R$. Then

\begin{equation}
\mathcal{N}_o \leq \frac{\text{Max} \log \left| \frac{H(z)}{H(0)} \right|}{\log \varrho}.
\end{equation}

It follows from (3.5) and (3.6) that if $h_n^{(n)}(0) \neq 0$, then for any $\varrho > 1$ we have

\begin{equation}
\frac{S_0(n) + S_1(n)}{n} \leq \frac{\text{Max} \log \left| \frac{h_n^{(n)}(z)}{h_n^{(n)}(0)} \right|}{n \log \varrho} + (1 + 2\delta').
\end{equation}

Now let us choose a sequence $r_s$ of positive numbers, such that $\lim_{s \to \infty} r_s = \infty$ and

\begin{equation}
\lim_{s \to \infty} \frac{\mu_H \left( r_s^\frac{1}{2} \right)}{\mu_f (r_s)} = G < \infty.
\end{equation}

Such a sequence $r_s$ exists by supposition. Let $n_s$ denote the index of the maximal term of the series $\sum_{n=0}^\infty a_n z^n$ for $|z| = r_s$, i.e. suppose

\begin{equation}
|a_{n_s}| r_s^n \approx |a_m| r_s^m \quad \text{for} \quad m = 0, 1, \ldots
\end{equation}

It follows by Lemma 1 that

\begin{equation}
\frac{Q_n^{(n)}(0)}{f^{(n)}(0)} \leq \frac{2 \text{Max} |c_1|}{|a_{n_s}|} 12^n s \leq C_1 (12 \lambda_{n_s})^n s
\end{equation}

where $C_1 > 0$ is a constant and

\begin{equation}
\lambda_{n_s} = \frac{1}{\frac{\delta' - \delta}{r_s}}.
\end{equation}
As clearly
\begin{equation}
(3.10) \quad \lim_{s \to \infty} \lambda_{n_s} = 0
\end{equation}

if follows that
\begin{equation}
(3.11) \quad \lim_{s \to \infty} \left| \frac{Q(s)}{f(s)} \right| = 0
\end{equation}

and thus also $h_{n_s}(0) \neq 0$ for sufficiently large $s$.

By Lemma 1 we have also for $|z| = \varepsilon > 1$
\[
\left| \frac{Q(s)}{f(s)} \right| \leq 2 \left( \sum_{t = 0}^{n_s} c_t \right) \frac{12^n s^{n_s-1}}{a_{n_s}} \left( \begin{array}{c} n_s + j \cr j \end{array} \right) q^{l+1}.
\]

Similarly as in (3.9) we obtain
\[
\max_{|z| = \varepsilon} \left| \frac{Q(s)}{f(s)} \right| \leq C_2 \left( \begin{array}{c} n_s + j \cr j \end{array} \right) q^{l+1}
\]

where $C_2 > 0$ is a constant.

Now if $s$ is sufficiently large, we have $12 \lambda_{n_s} q < 1$ and therefore
\[
\max_{|z| = \varepsilon} \left| \frac{Q(s)}{f(s)} \right| \leq \frac{C_2}{(1 - 12 \lambda_{n_s} q)^{n_s+1}}.
\]

It follows by (3.10) that
\begin{equation}
(3.12) \quad \lim_{s \to \infty} \frac{\max_{|z| = \varepsilon} \left| \frac{Q(s)}{f(s)} \right|}{n_s} = 0.
\end{equation}

Finally we have for $|z| = \varepsilon$ by (3.8)
\[
\left| \frac{f(s)}{f(s)} \right| \leq \frac{1}{(1 - \varepsilon r)}^{n_s+1}
\]

and thus
\begin{equation}
(3.13) \quad \lim_{s \to \infty} \frac{\max_{|z| = \varepsilon} \left| \frac{f(s)}{f(s)} \right|}{n_s} = 0.
\end{equation}

As we have evidently
\begin{equation}
(3.14) \quad \max_{|z| = \varepsilon} \left| \frac{h_{n_s}(0)}{h_{n_s}(0)} \right| \leq \frac{\max_{|z| = \varepsilon} \left| \frac{f(s)}{f(s)} \right| + \max_{|z| = \varepsilon} \left| \frac{Q(s)}{f(s)} \right|}{1 - \frac{Q(s)}{f(s)}}
\end{equation}
comparing (3.11), (3.12), (3.13) and (3.14) it follows that

\[
\lim_{s \to \infty} \text{Max} \log \frac{h_{n_s}^{(n_2)}(z)}{h_{n_s}^{(n_3)}(0)} = 0.
\]  

(3.15)

Combining (3.7) with (3.15) we obtain

\[
\limsup_{s \to \infty} \frac{S_0(n_s) + S_1(n_s)}{n_s} \leq 1 + 2\delta'
\]  

(3.16)

and as (3.16) holds for any $\delta' > \delta$, we obtain finally

\[
\liminf_{s \to \infty} \frac{S_0(n) + S_1(n)}{n} \leq 1 + 2\delta.
\]

Thus Theorem 1 is proved.

New we prove the following

**Theorem 2.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n (z-1)^n$ be a transcendental entire function and $g(z) = \sum_{n=0}^{\infty} c_n z^n$ an arbitrary entire function for which the suppositions of Theorem 1 hold. Let $Z_0(n)$ denote the number of zeros in the sequence $a_0, a_1, \ldots, a_{n-1}$ and $S_1(n)$ the number of indices $k < n$ for which $|b_k| \leq |c_k|$. Then

\[
\liminf_{n \to \infty} \frac{Z_0(n) + S_1(n)}{n} \leq 1 + \delta.
\]

The proof of Theorem 2 is almost the same as that of Theorem 1; as however we do not use now (3.3), only (3.4), we get $\delta$ instead of $2\delta$.

The following result contains each of Theorem 1 and Theorem 2 as special cases.

**Theorem 3.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n (z-1)^n$ be a transcendental entire function, further $g_1(z) = \sum_{n=0}^{\infty} c_n z^n$ and $g_2(z) = \sum_{n=0}^{\infty} d_n z^n$ be entire functions, such that putting

\[
\mu_{g_1}(r) = \text{Max}_{n} |c_n|r^n
\]

and

\[
\mu_{g_2}(r) = \text{Max}_{n} |d_n|r^n
\]

we have for some $\delta_1$ and $\delta_2 (\delta_1 > 0, \delta_2 > 0, \delta_1 + \delta_2 < 1)$

\[
\liminf_{r \to \infty} \frac{\mu_{g_1}(r^{\delta_1}) + \mu_{g_2}(r^{\delta_2})}{\mu_f(r)} < \infty.
\]
Let \( S_0(n) \) denote the number of indices \( k < n \) for which \( |a_k| \leq |c_k| \) and \( S_1(n) \) the number of indices \( k < n \) for which \( |b_k| \leq |c_k| \). Then

\[
\liminf_{n \to \infty} \frac{S_0(n) + S_1(n)}{n} = 1 + \delta_1 + \delta_2.
\]

The proof of Theorem 3 is essentially the same as that of Theorem 1; the only differences consist in that now we define the polynomial \( Q_n(z) \) of order \( 2n - 1 \) \( (n=1, 2, \ldots) \) as follows:

\[
\frac{Q_n^{(0)}}{j!} = \begin{cases} 
0 & \text{for } 0 \leq j < \delta_1 n \\
 a_j & \text{if } \delta_1 n \leq j \leq n - 1 \text{ and } |a_j| \leq |c_j| \\
 0 & \text{if } \delta_1 n \leq j \leq n - 1 \text{ and } |a_j| > |c_j| 
\end{cases}
\]

\[
\frac{Q_n^{(1)}}{j!} = \begin{cases} 
0 & \text{for } 0 \leq j < \delta_2 n \\
 b_j & \text{if } \delta_2 n \leq j \leq n - 1 \text{ and } |b_j| \leq |d_j| \\
 0 & \text{if } \delta_2 n \leq j \leq n - 1 \text{ and } |b_j| > |d_j|.
\end{cases}
\]

Here \( \delta_1 \) is an arbitrary number for which \( \delta_1 < \delta_1' < 1 \) and \( \delta_2 \) an arbitrary number for which \( \delta_2 < \delta_2' < 1 \).

Evidently if we choose \( g_1(z) = g_2(z) = g(z) \), then Theorem 3 reduces to Theorem 1, while if \( g_1(z) \equiv 0, g_2(z) = g(z) \), then Theorem 3 reduces to Theorem 2.

Now we shall prove the following Theorem 4, which gives a sharper inequality than (1.4).

**Theorem 4.** If \( f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n (z-1)^n \) is an entire function of order \( \varrho_f (0 < \varrho_f \leq \infty) \) and \( g(z) = \sum_{n=0}^{\infty} c_n z^n \) an entire function of order \( \varrho_g \) where \( 0 \leq \varrho_g < \varrho_f \), and if \( S_0(n) \) denotes the number of indices \( k < n \) for which \( |a_k| \leq |c_k| \) and \( S_1(n) \) the number of indices \( k < n \) for which \( |b_k| \leq |c_k| \) then

\[
\liminf_{n \to \infty} \frac{S_0(n) + S_1(n)}{n} \leq 1 + \frac{\varrho_g}{\varrho_f}.
\]

**Proof of Theorem 4.** Let us put

\[
\varphi(z) = \sum_{|a_n| < |c_n|} a_n z^n
\]

and

\[
f^*(z) = f(z) - \varphi(z) = \sum_{|a_n| \geq |c_n|} a_n z^n = \sum_{n=0}^{\infty} a_n^* z^n = \sum_{n=0}^{\infty} b_n^* (z-1)^n.
\]

Then clearly \( f^*(z) \) has the same order \( \varrho_f \) as \( f(z) \) further \( \varphi(z) \) is at most of order \( \varrho_g \). Let \( Z^*_0(n) \) denote the number of zeros in the sequence \( a_0^*, a_1^*, \ldots, a_n^*-1 \), then clearly

\[
(3.17) \quad Z^*_0(n) = S_0(n)
\]
We can without restricting the generality suppose that $c_n \geq 0$ as evidently
\[ \sum_{n=0}^{\infty} |c_n| z^n \] is an entire function of the same order as $g(z)$ [This follows e.g. by (1.3)]. Put
\[ g^*(z) = \sum_{n=0}^{\infty} \left( \sum_{m=n}^{\infty} \binom{m}{n} c_m \right) z^n = \sum_{n=0}^{\infty} c_n^* z^n. \]

It is easy to see that $g^*(z)$ is also an entire function of order $\rho_g$. As a matter of fact, one has
\[ g^*(z) = 2g(z + 1). \]

Now as
\[ f^*(z) = \sum_{n=0}^{\infty} \left( b_n - \sum_{m=n}^{\infty} \binom{m}{n} a_m \right)(z - 1)^n \]
we obtain
\[ |b_k^*| = |b_k| + \sum_{m=k}^{\infty} \binom{m}{k} c_m \quad (k = 0, 1, \ldots). \]

Thus if $|b_k| \leq |c_k|$ we have $|b_k^*| \leq |c_k^*|$. It follows that if $S_k^*(n)$ denotes the number of indices $k < n$ for which $|b_k^*| \leq |c_k^*|$, then
\[ S_k(n) \leq S_k^*(n). \]

Now we can apply Theorem 2 to the function $f^*(z)$ of order $\rho_f$ and the function $g^*(z)$ of order $\rho_g$ with any $\delta > \rho_g \rho_f$ and thus we obtain, taking (3.17) and (3.18) into account
\[ \liminf_{n \to \infty} \frac{S_0(n) + S_1(n)}{n} \leq \liminf_{n \to \infty} \frac{Z_0(n) + S_1^*(n)}{n} = 1 + \frac{\rho_g}{\rho_f} \]
Thus Theorem 4 is proved.

§ 4. An example

We shall discuss now an example, which shows that Theorem 4 is in a certain sense best possible.
Let us consider the function
\[ f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\sigma_n} \]
where \( a > 1 \) is an integer. Clearly \( f(z) \) is an entire transcendental function of order 1. Let us choose for \( g(z) \) the function

\[
g(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/\vartheta}}
\]

where \( \frac{1}{a} < \vartheta < 1 \). It is easy to see that \( g(z) \) is an entire function of order \( \vartheta \).

We consider now the power series of \( f(z) \) around the point \( z = 1 \); we have

\[
f(z) = \sum_{n=0}^{\infty} \frac{[1+(z-1)]a^n}{(a^n)!} = \sum_{m=0}^{\infty} b_m (z-1)^m
\]

where

\[
b_m = \frac{1}{m!} \sum_{a^n > m} \frac{1}{(a^n-m)!}.
\]

Let us compare now \( b_m \) with

\[
c_m = \frac{1}{(m!)^{1/\vartheta}}
\]

for a value of \( m \) which lies in the interval

\[
a^{n-1} \leq m < \frac{a^n \vartheta}{1+\varepsilon} \quad (n = 1, 2, \ldots)
\]

where \( \varepsilon \) is an arbitrary positive number.

We have clearly for any \( m \) satisfying (4.1)

\[
b_m = \frac{2}{m! \left[ \left( \frac{1+\varepsilon}{\vartheta} - 1 \right) m \right]!}
\]

It follows by Stirling's formula that if \( n \geq n_0 \) where \( n_0 \) is a suitably chosen large number and if \( m \) satisfies (4.1) we have

\[b_m \leq c_m.\]

It follows that

\[
\liminf_{n \to \infty} \frac{S_0(n)}{n} \approx 1 + \frac{\vartheta - 1}{a} \quad \text{and} \quad \liminf_{n \to \infty} \frac{S_1(n)}{n} \approx 1 + \frac{1+\varepsilon}{a}.\]
As for $\varepsilon$ we can choose an arbitrary small positive number and $a$ can be chosen arbitrarily large, it follows that the lower bound in (4.2) can be arbitrarily near to $1 + \delta$. As however by Theorem 4

\[(4.3)\]

$$\lim_{n \to \infty} \inf \frac{S_0(n) + S_1(n)}{n} \leq 1 + \delta$$

it follows that Theorem 4 (and thus also Theorem 2) is best possible in the sense that $1 + \delta$ can not be replaced in general by any smaller number in (4.3).

**Literature**