Fractional Brownian motions as "higher-order" fractional derivatives of Brownian local times

by

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Dedicated to Pál Révész on the occasion of his 65th birthday

Summary. Fractional derivatives $\mathcal{D}^\gamma$ of Brownian local times are well defined for all $\gamma < 3/2$. We show that, in the weak convergence sense, these fractional derivatives admit themselves derivatives which feature all fractional Brownian motions. Strong approximation results are also developed as counterparts of limit theorems for Brownian additive functionals which feature the fractional derivatives of Brownian local times.

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1 Introduction

(1.a) Brownian motion is perhaps the most important object of study in probability theory; besides its own interest, Brownian motion plays a cornerstone role in the construction of many important stochastic processes, among which we single out, in this paper, the stable Lévy processes on the one hand, and the fractional Brownian motions on the other hand.

We first recall some of these constructions.

i) Throughout the paper, $\{B_t,\, t \geq 0\}$ will denote a real-valued Brownian motion, with a jointly continuous family of local times $\{\ell_t^x,\ x \in \mathbb{R},\, t \geq 0\}$, which may be defined as continuous versions of the densities of the Brownian occupation measure with respect to the Lebesgue measure $\mathrm{d}x$: precisely, the local times satisfy, for any Borel function $f:\ \mathbb{R} \mapsto \mathbb{R}_+$,

$$\int_0^t f(B_s)\,\mathrm{d}s = \int_{\mathbb{R}} f(x)\ell_t^x\,\mathrm{d}x.$$

It is well known (see McKean [12], Trotter [16]) that $x \mapsto \ell_t^x$ may be chosen to be Hölder continuous of order $\frac{1}{2} - \eta$ (for $0 < \eta < \frac{1}{2}$), uniformly in t varying in a compact interval. This regularity property allows to define the fractional derivatives:

$$H_t^{(\alpha)}(x) = \int_0^t \frac{\mathrm{d}s}{(B_s - x)^\alpha}, \qquad \text{where } y^\alpha := |y|^\alpha \operatorname{sgn}(y),$$

for any $\alpha < 3/2$. Indeed, $H_t^{(\alpha)}(\cdot)$ may be defined as the absolutely convergent integral

$$\int_0^\infty \frac{\ell_t^{x+y} - \ell_t^{x-y}}{y^\alpha}\,\mathrm{d}y.$$

These (singular) integrals of Brownian local time have been considered by a number of authors (Ezawa et al. [7], Yamada [17]–[19], Biane and Yor [4], Yor [20], Fitzsimmons and Getoor [8], Bertoin [2], etc.) with various motivations; see in particular Bertoin [3] in relation with the study of Bessel processes of dimension $d \in (0,1)$, see also a presentation of a number of results in Yor [23]. As a consequence of the scaling property of Brownian motion and of the fact that $\{H_t^{(\alpha)}(0),\, t \geq 0\}$ is a Brownian additive functional, if

$$\tau_\ell = \inf\{t > 0:\ \ell_t^0 > \ell\}, \qquad \ell \geq 0,$$

then $\{H_{\tau_\ell}^{(\alpha)}(0),\, \ell \geq 0\}$ is a symmetric stable Lévy process of index $\overline{\alpha} := 1/(2-\alpha)$, which varies between 0 and 2, i.e.,

$$\mathbb{E}\left[\exp\left(i\lambda H_{\tau_\ell}^{(\alpha)}(0)\right)\right] = \exp\left(-\ell\, c_\alpha\, |\lambda|^{\overline{\alpha}}\right).$$

In particular, $\{\frac{1}{\pi}H^{(1)}_{\tau_\ell}(0),\ \ell \ge 0\}$ is a standard Cauchy process. For these results, and related discussions, see Biane and Yor [4].

ii) We now sketch one of the best known constructions of fractional Brownian motions from Brownian motion. Here, it is most convenient to start with $\{\beta_y,\ y \in \mathbb{R}\}$ a Brownian motion indexed by $\mathbb{R}$. Then it is known (see, e.g., Kahane [11], Revuz and Yor [15], Exercise I.3.9]) that for $\alpha \in (1/2, 3/2)$, $\alpha \neq 1$, the process

$$\beta^{(\alpha)}_x = \int_{\mathbb{R}} (|y|^{1-\alpha} - |y-x|^{1-\alpha})\, \mathrm{d}\beta_y, \qquad x \in \mathbb{R}, \tag{1.1}$$

is well defined, and satisfies

$$\mathbb{E}\left[\left(\beta^{(\alpha)}_x - \beta^{(\alpha)}_y\right)^2\right] = k_\alpha\, |x-y|^{\widetilde{\alpha}},$$

where $\widetilde{\alpha} := 3 - 2\alpha \in (0,2)$. Thus, using Kolmogorov's criterion, $\{\beta^{(\alpha)}_x,\ x \in \mathbb{R}\}$ may be chosen to be a continuous process, which is a (multiple of) fractional Brownian motion of order $\widetilde{\alpha}$. As explained in Section 2, for $\alpha = 1$, one ought to replace formula (1.1) by (2.3), which involves logarithms, and can be obtained from (1.1) by a limiting procedure as $\alpha \to 1$.

Remark. In their definition of the fractional Brownian motion, Mandelbrot and Van Ness [13] use the positive part function instead of absolute value (as in (1.1)); indeed there are a number of different constructions of fractional Brownian motions. See Yor [21] for such constructions involving matrices.

(1.b) The aim of the second part of our work is to obtain strong approximation for general additive functionals of Brownian motion. Let g be a Borel function on $\mathbb{R}$, and consider $A_t := \int_0^t g(B_s)\, \mathrm{d}s$. The first and second order limit theorems of Papanicolaou et al. [14] state that, as $\lambda \to \infty$,

$$\frac{A_{\lambda t}}{\lambda^{1/2}} \xrightarrow{\text{law}} \bar{g}\, \ell^0_t, \qquad \bar{g} := \int_{\mathbb{R}} g(x)\, \mathrm{d}x,$$
$$\frac{A_{\lambda t}}{\lambda^{1/4}} \xrightarrow{\text{law}} \sqrt{\langle g, g\rangle}\, \beta(\ell^0_t), \qquad \text{if } \bar{g} = 0,$$

where $\langle g, g\rangle = 2\int_{\mathbb{R}} (\int_{-\infty}^x g(u)\, \mathrm{d}u)^2\, \mathrm{d}x > 0$, and β is a Brownian motion independent of ℓ^0_t. (These results are recalled in more details in Paragraph (2.c) of Section 2). A strong approximation version of this result is given in Csáki et al. [5].

In case g does not belong to $L^1(\mathbb{R})$, or $\langle g, g\rangle = \infty$, other weak limits are available in the literature, where the limiting processes turn out to be $H^{(1)}_t(0)$ and $H^{(\alpha)}_t(0)$. In Subsection 3.2, we present strong approximation results for such limit theorems.

(1.c) We now give some details on the organization of the paper.

— Section 2 contains certain preliminaries which are necessary to proceed; in particular, we present a "duality" construction of certain Gaussian processes from Brownian motion, which will be helpful in the sequel; we also recall some well known limit theorems for additive functionals of Brownian motion.

— In Section 3, we state our main results, some of which link asymptotically $\{H_t^{(\alpha)}(x),\ x \in \mathbb{R}\}$ to fractional Brownian motions, while the others present strong approximation results.

— The remaining sections consist in the details of the proofs of our results.

2 Preliminaries

(2.a) In this paragraph, we consider a one-dimensional Brownian motion $\{\beta_y,\ y \in \mathbb{R}\}$, and a linear operator $K:\ \mathcal{LI} \to L^2(\mathbb{R})$ where $\mathcal{LI}$ denotes the set of linear combinations of indicators of intervals $\{j_x(\cdot) := \mathbb{1}_{[0,x]}(\cdot),\ x \in \mathbb{R}\}$. (For $x < 0$, $j_x = -\mathbb{1}_{[x,0]}$). Then, we define

$$\beta_x^K := \int_{\mathbb{R}} K(j_x)(y)\,\mathrm{d}\beta_y, \qquad x \in \mathbb{R}. \tag{2.1}$$

Under some mild conditions on K, we may choose this process to be continuous in x, and the identity

$$\int_{\mathbb{R}} f(z)\,\mathrm{d}\beta_z^K = \int_{\mathbb{R}} K(f)(y)\,\mathrm{d}\beta_y, \tag{2.2}$$

holds for every $f \in \mathcal{LI}$, as follows directly from (2.1) by linear combination.

(2.b) We shall be particularly interested in the construction of β^K in the cases where $K = \mathcal{H}$, the classical Hilbert transform, or $K = \mathcal{H}^\alpha := \mathcal{D}^{\alpha-1}$, the fractional derivative of order $(\alpha-1)$; we recall the definitions of these operators:

$$\begin{aligned} \mathcal{H}(f)(x) &= \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,\mathrm{d}y, \\ \mathcal{H}^\alpha(f)(x) := \mathcal{D}^{\alpha-1}(f)(x) &= \frac{1}{\Gamma(1-\alpha)}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{(x-y)^\alpha}\,\mathrm{d}y. \end{aligned}$$

We now recall some well known formulae:

$$\begin{aligned}\mathcal{H}(j_x)(y) &= \frac{1}{\pi}\log\frac{|y|}{|y-x|},\\ \mathcal{H}^\alpha(j_x)(y) &= \frac{|y|^{1-\alpha}-|y-x|^{1-\alpha}}{\Gamma(2-\alpha)}.\end{aligned}$$

As a consequence, we now obtain the following precisions as to our constructions in Paragraph (2.a):

$$\beta_x^{\mathcal{H}} = \frac{1}{\pi}\int_{\mathbb{R}} \log\frac{|y|}{|y-x|}\,\mathrm{d}\beta_y, \tag{2.3}$$

whereas

$$\beta_x^{(\alpha)} := \beta_x^{\mathcal{H}^\alpha} = \int_{\mathbb{R}} \mathcal{H}^\alpha(j_x)(y)\,\mathrm{d}\beta_y,$$

which agrees (up to a multiplicative constant) with our definition in the Introduction (for fractional Brownian motion of order $\widetilde{\alpha}$).

(2.c) We now recall first and second order limit theorems about additive functionals of $\{B_t,\, t\geq 0\}$ with local time $\{\ell_t^x,\, t\geq 0,\, x\in\mathbb{R}\}$:

(i) if $f\in L^1(\mathbb{R})$ and $\overline{f} := \int_{\mathbb{R}} f(x)\,\mathrm{d}x$, then as n goes to infinity,

$$n\int_0^t f(nB_s)\,\mathrm{d}s \longrightarrow \overline{f}\,\ell_t^0, \qquad \text{a.s.};$$

(ii) if $f\in L^2(\mathbb{R})\cap L^1(\mathbb{R})$, then as n goes to infinity,

$$\left\{B_t,\ n^{1/2}\int_0^t f(nB_s)\,\mathrm{d}B_s,\ t\geq 0\right\} \xrightarrow{\text{law}} \left\{B_t,\ \int_{\mathbb{R}} f(y)\,\mathrm{d}_y\mathbb{B}(y;\ell_t^0),\ t\geq 0\right\}, \tag{2.4}$$

where $\{\mathbb{B}(y;\ell),\ y\in\mathbb{R},\ \ell\geq 0\}$ denotes a Brownian sheet indexed by $\mathbb{R}\times\mathbb{R}_+$, independent of $\{B_t,\, t\geq 0\}$.

Remark. In fact, we need a weaker condition than $f\in L^1(\mathbb{R})$ for (2.4) to hold:

(iia) if $f\in L^2(\mathbb{R})$ and $\lim_{n\to\infty} n^{-1/2}\int_{-cn}^{cn}|f(u)|\,\mathrm{d}u = 0$ for all $c>0$, then (2.4) holds as $n\to\infty$.

To see this we have only to show that as n goes to infinity

$$\langle n^{1/2}\int_0^\cdot f(nB_s)\,\mathrm{d}B_s, B\rangle(t) = n^{1/2}\int_0^t f(nB_s)\,\mathrm{d}s \to 0, \qquad \text{a.s.}$$

Indeed, under the condition in (iia) we have

$$n^{1/2}\left|\int_0^t f(nB_s)\,\mathrm{d}s\right| = n^{1/2}\left|\int_{\mathbb{R}} f(nz)\ell_t^z\,\mathrm{d}z\right| \le \frac{\sup_{z\in\mathbb{R}}\ell_t^z}{n^{1/2}}\int_{\underline{b}n}^{\overline{b}n}|f(u)|\,\mathrm{d}u = o(1), \qquad \text{a.s.,}$$

where $\overline{b} = \sup_{s\in[0,t]} B_s$, $\underline{b} = \inf_{s\in[0,t]} B_s$. We are grateful to the referee for this remark.

Part (ii) and (iia) should essentially be thought of as a convenient way of expressing the Papanicolaou–Stroock–Varadhan [14] limit theorem for

$$n^{3/2}\int_0^t g(nB_s)\,\mathrm{d}s, \qquad n\to\infty,$$

when $\overline{g} = 0$. For more detailed presentations of these variants of the Papanicolaou–Stroock–Varadhan theorems, we refer the reader to Chapter XIII of Revuz and Yor [15] (cf, also Hu and Yor [10]). Nonetheless, in order to have a precise reference to $\mathbb{B}$ in our next discussions, we present the following variant of (ii):

$$\left\{B_t,\ \frac{\ell_t^{\varepsilon x}-\ell_t^0}{2\sqrt{\varepsilon}},\ x\in\mathbb{R},\ t\ge 0\right\} \xrightarrow{\text{law}} \left\{B_t,\ \mathbb{B}(x;\ell_t^0),\ x\in\mathbb{R},\ t\ge 0\right\}, \tag{2.5}$$

as ε goes to 0.

(2.d) As an illustration of our approach, consider the asymptotics of

$$M_t^{(\varepsilon)} := \int_0^t \frac{\mathrm{d}\,B_s}{|B_s|^\alpha}\mathbf{1}_{\{|B_s|\ge\varepsilon\}}, \qquad \text{as} \quad \varepsilon\to 0,$$

depending on α.

(a) $\alpha < 1/2$. Then

$$M_t^{(\varepsilon)} \to \int_0^t \frac{\mathrm{d}\,B_s}{|B_s|^\alpha} \qquad \varepsilon\to 0, \tag{2.6}$$

since the stochastic integral on the right hand side is well defined. More precisely, we can show the existence of a jointly continuous version of $\{M_t^{(\varepsilon)},\ \varepsilon>0,\ t\ge 0\}$ thanks to Kolmogorov's criterion, and the following estimate:

$$\mathbb{E}\left(\sup_{0\le s\le t}|M_s^{(\varepsilon)} - M_s^{(\varepsilon')}|^k\right) \le C_{k,t}\,|\varepsilon-\varepsilon'|^{(1-2\alpha)k/2},$$

from which one deduces that $M_t^{(\varepsilon)}$ can be chosen Hölder continuous in ε, of order $\frac{1}{2}-\alpha-\eta$ (for $0<\eta<\frac{1}{2}-\alpha$), uniformly in t on compact sets of $\mathbb{R}_+$. Once this choice is made, the convergence in (2.6) holds almost surely and in L^p.

(b) $\alpha = 1/2$. Then

$$(\log(1/\varepsilon))^{-1/2} M_t^{(\varepsilon)} \xrightarrow{\text{law}} 2^{1/2} \beta(\ell_t^0),$$

where $\{\beta(u),\, u \geq 0\}$ is a Brownian motion, independent of the local time ℓ_t^0. This can be seen similarly (in fact simpler) to the proof of Theorem 3.3 below.

(c) $\alpha > 1/2$. Then

$$\varepsilon^{\alpha-1/2} M_t^{(\varepsilon)} \xrightarrow{\text{law}} \left(\frac{2}{2\alpha-1}\right)^{1/2} \beta(\ell_t^0),$$

where, as before, $\{\beta(u),\, u \geq 0\}$ is a Brownian motion, independent of the local time ℓ_t^0. This can be seen by applying (2.4) (cf. (iia) in the Remark) with $n = 1/\varepsilon$ and $f(u) = |u|^{-\alpha} \mathbf{1}_{\{|u| \geq 1\}}$. In fact, we have

$$\left(\frac{2}{2\alpha-1}\right)^{1/2} \beta(u) = \int_{\mathbb{R}} \frac{\mathbf{1}_{\{|y|\geq 1\}}}{|y|^{\alpha}} \, \mathrm{d}_y \mathbb{B}(y; u), \qquad u \geq 0,$$

where $\mathbb{B}(\cdot,\cdot)$ is the Brownian sheet of (2.4).

(2.e) We now need to go back to our definitions in paragraph (2.b), and to extend them when $\{\beta_y,\, y \in \mathbb{R}\}$ is replaced by $\{\mathbb{B}(y;\ell),\, y \in \mathbb{R},\, \ell \geq 0\}$. Thus, we define

$$\begin{aligned} \mathbb{B}^{\mathcal{H}}(x;\ell) &= \int_{\mathbb{R}} \mathcal{H}(j_x)(y) \, \mathrm{d}_y \mathbb{B}(y;\ell), \\ \mathbb{B}^{(\alpha)}(x;\ell) &= \int_{\mathbb{R}} \mathcal{H}^{\alpha}(j_x)(y) \, \mathrm{d}_y \mathbb{B}(y;\ell). \end{aligned}$$

Both processes may be chosen to be jointly continuous; the first process is a Brownian sheet indexed by $\mathbb{R}\times\mathbb{R}_+$, while (with obvious meaning) the second process is a "fractional Brownian motion of order $\widetilde{\alpha}$ in the first variable and a Brownian motion in the second variable".

3 Statements of results

3.1 Weak convergence

In this subsection, we state a number of convergence in law results (except for Theorem 3.4 which is a strong limit theorem); these convergences in law hold for continuous processes indexed by $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, and it is understood that the convergence in law corresponds to the topology of uniform convergence on compacta which is endowed on $\mathcal{C}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^k)$ (for suitable k; mostly $k = 1$ or $k = 2$). Moreover, we use constantly the notation defined in the previous section and all these convergences in law hold jointly, although we shall not repeat this below. In particular, ℓ_t^0 stands for the local time at 0 of B_t.

Theorem 3.1 *The following convergence in law holds as ε goes to 0:*

$$\left\{ B_t,\ \frac{H_t^{(1)}(\varepsilon x) - H_t^{(1)}(0)}{\pi \varepsilon^{1/2}} \right\} \xrightarrow{\text{law}} \left\{ B_t,\ \mathbb{B}^{\mathcal{H}}(x; \ell_t^0) \right\}, \tag{3.1}$$

where $\mathbb{B}^{\mathcal{H}}(\cdot, \cdot)$ is a standard Brownian sheet independent of $\{B_t,\ t \geq 0\}$, defined in paragraphs (2.c) and (2.e).

Theorem 3.2 *Let $1/2 < \alpha < 3/2$ and $\alpha \neq 1$. As ε goes to 0,*

$$\left\{ B_t,\ \frac{H_t^{(\alpha)}(\varepsilon x) - H_t^{(\alpha)}(0)}{\Gamma(2-\alpha)\, \varepsilon^{3/2-\alpha}} \right\} \xrightarrow{\text{law}} \left\{ B_t,\ \mathbb{B}^{(\alpha)}(x; \ell_t^0) \right\}. \tag{3.2}$$

The case $\alpha = 1/2$ is critical, while the case $\alpha < 1/2$ is easier than the other cases, in that the mode of convergence is the almost sure convergence, and the limit process is yet a fractional derivative of local times.

Theorem 3.3 *We have,*

$$\left\{ B_t,\ \frac{H_t^{(1/2)}(\varepsilon x) - H_t^{(1/2)}(0)}{\varepsilon \sqrt{\log(1/\varepsilon)}} \right\} \xrightarrow{\text{law}} \left\{ B_t,\ x\, \beta(\ell_t^0) \right\}, \tag{3.3}$$

where β is a standard Brownian motion independent of $(B, \mathbb{B})$.

Theorem 3.4 *Let $0 < \alpha < 1/2$. Then, as $\varepsilon \to 0$,*

$$\begin{aligned} \frac{H_t^{(\alpha)}(\varepsilon x) - H_t^{(\alpha)}(0)}{\varepsilon} \quad \longrightarrow \quad & (\alpha x)\, \text{p.v.} \int_0^t \frac{\mathrm{d}s}{|B_s|^{\alpha+1}} \\ & := (\alpha x) \int_{-\infty}^{\infty} \frac{\ell_t^z - \ell_t^0}{|z|^{\alpha+1}}\, \mathrm{d}z, \qquad \text{a.s.} \end{aligned} \tag{3.4}$$

3.2 Strong approximation

Recall the following weak convergence results (see Yamada [18] and Fitzsimmons and Getoor [8]): Let $f \in L^2(\mathbb{R})$, $\bar{f} = \int_{\mathbb{R}} f(x)\,\mathrm{d}x$, then as $\lambda \to \infty$,

$$\frac{1}{\sqrt{\lambda}} \int_0^{\lambda t} (\mathcal{H}f)(B_s)\,\mathrm{d}s \quad \xrightarrow{\text{law}} \quad \frac{\bar{f}}{\pi} H_t^{(1)}(0),$$
$$\frac{1}{\lambda^{1-\alpha/2}} \int_0^{\lambda t} (\mathcal{H}^\alpha f)(B_s)\,\mathrm{d}s \quad \xrightarrow{\text{law}} \quad \frac{\bar{f}}{\Gamma(1-\alpha)} H_t^{(\alpha)}(0).$$

We get the strong approximation analogues of these limit theorems.

Theorem 3.5 *Let f be a Borel function on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} x^\kappa |f(x)|\,\mathrm{d}x < \infty, \tag{3.5}$$

for some $\kappa > 0$. Then for all sufficiently small $\varepsilon > 0$, when $t \to \infty$,

$$\int_0^t (\mathcal{H}f)(B_s)\,\mathrm{d}s = \frac{\bar{f}}{\pi} H_t^{(1)}(0) + o(t^{1/2-\varepsilon}), \qquad \text{a.s.} \tag{3.6}$$

Theorem 3.6 *Let f be a Borel function satisfying* (3.5), *and* $0 < \alpha < 3/2$ (*with* $\alpha \neq 1$). *For all sufficiently small $\varepsilon > 0$, when $t \to \infty$,*

$$\int_0^t (\mathcal{H}^\alpha f)(B_s)\,\mathrm{d}s = \frac{\bar{f}}{\Gamma(1-\alpha)} H_t^{(\alpha)}(0) + o(t^{1-\alpha/2-\varepsilon}), \qquad \text{a.s.} \tag{3.7}$$

Remark. (i) It is possible to get more precision for (3.6) and (3.7). In fact, our proofs in Section 5 reveal that, for any $\varepsilon > 0$,

$$\int_0^t (\mathcal{H}f)(B_s)\,\mathrm{d}s \;=\; \frac{\bar{f}}{\pi} H_t^{(1)}(0) + o(t^{\gamma_1+\varepsilon}), \qquad \text{a.s.},$$
$$\int_0^t (\mathcal{H}^\alpha f)(B_s)\,\mathrm{d}s \;=\; \frac{\bar{f}}{\Gamma(1-\alpha)} H_t^{(\alpha)}(0) + o(t^{\gamma_2+\varepsilon}), \qquad \text{a.s.},$$

where

$$\gamma_1 \;:=\; \frac{1+\kappa}{2(1+2\kappa)},$$

$$\gamma_2 := \frac{1}{4} + \frac{(1-\alpha)_+}{2} + \frac{3/2 - \alpha - (1-\alpha)_+}{2(1+2\kappa)},$$

and $(1-\alpha)_+ = \max(1-\alpha, 0)$ denotes the positive part of $(1-\alpha)$. Since $\gamma_1 < 1/2$ and $\gamma_2 < 1 - \alpha/2$, we immediately get (3.6) and (3.7).

(ii) As a consequence of Theorem 3.5, the law of the iterated logarithm (LIL) of Hu and Shi [9] proved for $H_t^{(1)}(0)$ remains true for the additive functional in (3.6), i.e., we have

$$\limsup_{t\to\infty} \frac{\int_0^t (\mathcal{H}f)(B_s)\,\mathrm{d}s}{(2t\log\log t)^{1/2}} = \frac{2\bar{f}}{\pi}, \qquad \text{a.s.}$$

Similarly, Theorem 3.6 and the LIL of Csáki et al. [6] for $H_t^{(\alpha)}(0)$ together imply that there exists a constant $c(\alpha) \in (0,\infty)$, depending only on α, such that

$$\limsup_{t\to\infty} \frac{\int_0^t (\mathcal{H}^\alpha f)(B_s)\,\mathrm{d}s}{t^{1-\alpha/2}(\log\log t)^{\alpha/2}} = c(\alpha), \qquad \text{a.s.}$$

The explicit value of $c(\alpha)$ remains unknown.

4 Proofs of Theorems 3.1–3.4

4.1 Proofs of Theorems 3.1 and 3.2

The arguments in the proofs of Theorems 3.1 and 3.2 are quite classical, and rely upon the use of Itô's formula, applied as follows: for a sufficiently regular function ϕ, we consider $\Phi(\xi) := \int_0^\xi \phi(x)\,\mathrm{d}x$ and we write

$$\text{(4.1)} \qquad n^{-1/2}\Phi(nB_t) = n^{1/2}\int_0^t \phi(nB_s)\,\mathrm{d}B_s + \frac{n^{3/2}}{2}\int_0^t \phi'(nB_s)\,\mathrm{d}s.$$

In our applications, we shall take

$$\phi(\xi) = f(\xi - x) - f(\xi),$$

for some particular f's, precisely,

$$f_1(x) = \log|x|, \qquad \text{and} \qquad f_\alpha(x) = |x|^{1-\alpha} \quad \left(\frac{1}{2} < \alpha < \frac{3}{2},\ \alpha \neq 1\right).$$

Step 1. For the functions ϕ_α associated with the f_α's, the asymptotic result (2.4) holds, for any given $x \in \mathbb{R}$ (cf. also the Remark following (2.4)). Thus, in order to conclude the convergence in law (as continuous processes in (x,t)), we still need to prove the tightness of the laws of the processes

$$M_n^f(x,t) := n^{1/2} \int_0^t \left(f(nB_s - x) - f(nB_s)\right) \, \mathrm{d}B_s.$$

For this purpose, we use the following inequalities

$$\begin{aligned} & \mathbb{E}\left[\sup_{0\le s\le t} |M_n^f(x,s) - M_n^f(y,s)|^p\right] \\ \le\ & c_p\, n^{p/2} \mathbb{E}\left[\left(\int_0^t (f(nB_s - x) - f(nB_s - y))^2 \, \mathrm{d}s\right)^{p/2}\right] \\ \le\ & c_{p,t} \left(\int_{\mathbb{R}} (f(\xi - x) - f(\xi - y))^2 \, \mathrm{d}\xi\right)^{p/2}, \end{aligned}$$

where, for the first inequality, we have used the Burkholder–Davis–Gundy inequalities (see for example Theorem IV.4.1 of Revuz and Yor [15]), and for the second, we have used the fact that

$$\mathbb{E}\left[\sup_{\xi\in\mathbb{R}} (\ell_t^\xi)^k\right] < \infty,$$

for any k and t (see e.g. Barlow and Yor [1]). Thus, for any f such that there exists $\varepsilon > 0$ satisfying

$$\int_{\mathbb{R}} (f(\xi - x) - f(\xi - y))^2 \, \mathrm{d}\xi \le C\,|x-y|^\varepsilon, \tag{4.2}$$

we can deduce the tightness of the laws of $\{M_n^f(x,t),\ x \in \mathbb{R},\ t \ge 0\}$.

In particular, for f_1, we get $\varepsilon = 1$ and for f_α, $\varepsilon = 3 - 2\alpha$.

Step 2. To finish the proofs of Theorems 3.1 and 3.2, we need to show that for the functions Φ_α associated with the functions f_α, as indicated in the beginning of this subsection, the left hand side of (4.1) is negligible as $n \to \infty$. More precisely, we shall show that, as n goes to infinity,

$$\sup_{s\in[0,t],\, x\in[-A,A]} n^{-1/2} \left| \int_0^{nB_s} (f_\alpha(\xi - x) - f_\alpha(\xi)) \, \mathrm{d}\xi \right| \to 0, \qquad \text{a.s.}$$

Indeed, we have,

$$\sup_{y\in[-C,C],\, x\in[-A,A]} n^{-1/2} \left| \int_0^{ny} (f_\alpha(\xi - x) - f_\alpha(\xi)) \, \mathrm{d}\xi \right| \to 0, \tag{4.3}$$

In the particular case of f_1, this boils down to

$$n^{-1/2}\int_0^{nC} \log\left(1+\frac{A}{\xi}\right)\, \mathrm{d}\xi \to 0,$$

which is obviously satisfied since the integral (without the $n^{-1/2}$ term) is $\mathcal{O}(\log n)$. In the general case of f_α (with $1/2 < \alpha < 3/2$ and $\alpha \neq 1$),

$$\begin{aligned}\int_0^{nC} |f_\alpha(\xi - x) - f_\alpha(\xi)|\, \mathrm{d}\xi \;&=\; \int_0^{A} |f_\alpha(\xi - x) - f_\alpha(\xi)|\, \mathrm{d}\xi \\ &\quad + \int_A^{nC} \xi^{1-\alpha}\left|(1-\frac{x}{\xi})^{1-\alpha} - 1\right|\, \mathrm{d}\xi.\end{aligned}$$

The first integral on the right hand side is bounded, whereas the second is $\mathcal{O}(\int_A^{nC} \xi^{-\alpha}\, \mathrm{d}\xi) = \mathcal{O}(n^{1-\alpha}+1)$, uniformly for $x \in [-A, A]$. This yields (4.3).

4.2 Proof of Theorem 3.3

Similar discussions as in Subsection 4.1 in view of (4.1) reveal that we only need to prove

$$\frac{n^{1/2}}{(\log n)^{1/2}}\int_0^t \left((nB_s - x)^{1/2} - (nB_s)^{1/2}\right)\, \mathrm{d}B_s \xrightarrow{\text{law}} \frac{x}{2}\,\beta(\ell_t^0). \tag{4.4}$$

To check (4.4), consider the sequence of continuous local martingales $\{M_n^x\}_{n\geq 2}$ defined by

$$M_n^x(t) = \frac{n^{1/2}}{(\log n)^{1/2}}\int_0^t \left((nB_s - x)^{1/2} - (nB_s)^{1/2}\right)\, \mathrm{d}B_s, \qquad t \geq 0.$$

By the Dambis–Dubins–Schwarz theorem (see Theorem V.1.6 of Revuz and Yor [15]), there exists a sequence of standard Brownian motions $\{\beta_n^x\}_{n\geq 2}$, such that

$$M_n^x(t) = \beta_n^x(\langle M_n^x\rangle(t)),$$

where

$$\langle M_n^x\rangle(t) = \frac{n}{\log n}\int_0^t \left((nB_s - x)^{1/2} - (nB_s)^{1/2}\right)^2\, \mathrm{d}s, \tag{4.5}$$

is the associated increasing process of M_n^x. Let us also look at $\langle M_n^x, B\rangle$, the so-called bracket process of M_n^x and B:

$$\langle M_n^x, B\rangle(t) = \frac{n^{1/2}}{(\log n)^{1/2}}\int_0^t \left((nB_s - x)^{1/2} - (nB_s)^{1/2}\right)\, \mathrm{d}s. \tag{4.6}$$

We shall show that, as n goes to infinity, (uniformly for (t,x) in all compacta of $\mathbb{R}_+ \times \mathbb{R}$),

$$\langle M_n^x \rangle(t) \to \frac{x^2}{4}\ell_t^0, \qquad \text{a.s.} \tag{4.7}$$

$$\langle M_n^x, B \rangle(t) \to 0, \qquad \text{a.s.} \tag{4.8}$$

$$\langle M_n^x - xM_n^1 \rangle(t) \to 0, \qquad \text{a.s.} \tag{4.9}$$

Then, an asymptotic version of Knight's theorem (see Theorem XIII.2.3 of Revuz and Yor [15]) implies the asymptotic independence of β_n^x and $\langle M_n^x \rangle(t)$, so that we have

$$M_n^x(t) \xrightarrow{\text{law}} x\,\beta\left(\frac{\ell_t^0}{4}\right),$$

where β is a standard Brownian motion independent of ℓ_t^0. This yields (4.4) by scaling.

The rest of this subsection is devoted to the proof of (4.7)–(4.9).

Let us first check (4.7). By (4.5) and the occupation time formula,

$$\begin{aligned}\langle M_n^x \rangle(t) &= \frac{n}{\log n}\int_{\mathbb{R}} \left((ny-x)^{1/2} - (ny)^{1/2}\right)^2 \ell_t^y \,\mathrm{d}y \\ &= \frac{1}{\log n}\int_{\mathbb{R}} \left((z-x)^{1/2} - z^{1/2}\right)^2 \ell_t^{z/n} \,\mathrm{d}z,\end{aligned}$$

by a change of variables $z = ny$. Write $\overline{b} = \overline{b}(t) = \sup_{s\in[0,t]} B_s$ and $\underline{b} = \underline{b}(t) = \inf_{s\in[0,t]} B_s$. Since $\ell_t^a = 0$ for any $a \notin [\,\underline{b},\ \overline{b}\,]$, we arrive at:

$$\begin{aligned}\langle M_n^x \rangle(t) &= \frac{1}{\log n}\int_{z/n\in[\,\underline{b},\overline{b}\,]} \left((z-x)^{1/2} - z^{1/2}\right)^2 \ell_t^{z/n} \,\mathrm{d}z \\ &= \mathrm{I} + \mathrm{II},\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}\mathrm{I} &= \frac{1}{\log n}\int_{z/n\in[\,\underline{b},\overline{b}\,]} \left((z-x)^{1/2} - z^{1/2}\right)^2 (\ell_t^{z/n} - \ell_t^0) \,\mathrm{d}z, \\ \mathrm{II} &= \frac{\ell_t^0}{\log n}\int_{z/n\in[\,\underline{b},\overline{b}\,]} \left((z-x)^{1/2} - z^{1/2}\right)^2 \mathrm{d}z.\end{aligned}$$

It is known (see McKean [12]) that $x \mapsto \ell_t^x$ is Hölder continuous of any order $\nu < 1/2$, uniformly for t in all compacta. Therefore, as n goes to infinity,

$$\begin{aligned}|\mathrm{I}| &\le \frac{c_1}{\log n}\int_{z/n\in[\,\underline{b},\overline{b}\,]} \left((z-x)^{1/2} - z^{1/2}\right)^2 (z/n)^{1/3} \,\mathrm{d}z \\ &= \mathcal{O}\left(\frac{1}{\log n}\right) = o(1).\end{aligned} \tag{4.11}$$

To estimate II, observe that a deterministic argument gives

$$\int_{z/n\in[\underline{b},\overline{b}]} \left((z-x)^{1/2}-z^{1/2}\right)^2 \mathrm{d}z \sim \frac{x^2}{4}\log n, \qquad n\to\infty,$$

which yields

$$\mathrm{II} \to \frac{x^2}{4}\ell_t^0, \qquad \text{a.s.}$$

This, jointly considered with (4.11) and (4.10), yields (4.7).

The proofs of (4.8) and (4.9) are very similar, and a lot easier. Indeed, by (4.6) and the occupation time formula,

$$\begin{aligned}
\langle M_n^x, B\rangle(t) &= \frac{n^{1/2}}{(\log n)^{1/2}}\int_{\mathbb{R}} \left((ny-x)^{1/2}-(ny)^{1/2}\right)\ell_t^y\,\mathrm{d}y \\
&= \frac{1}{(n\log n)^{1/2}}\int_{\mathbb{R}} \left((z-x)^{1/2}-z^{1/2}\right)\ell_t^{z/n}\,\mathrm{d}z \\
&= \frac{1}{(n\log n)^{1/2}}\int_{z/n\in[\underline{b},\overline{b}]} \left((z-x)^{1/2}-z^{1/2}\right)\ell_t^{z/n}\,\mathrm{d}z.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
|\langle M_n^x, B\rangle(t)| &\le \frac{\sup_{a\in\mathbb{R}}\ell_t^a}{(n\log n)^{1/2}}\int_{z/n\in[\underline{b},\overline{b}]} \left|(z-x)^{1/2}-z^{1/2}\right|\,\mathrm{d}z \\
&= \mathcal{O}\left(\frac{1}{(\log n)^{1/2}}\right),
\end{aligned}$$

which implies (4.8).

Finally, to check (4.9), we observe that

$$\begin{aligned}
\langle M_n^x - xM_n^1\rangle(t) &= \frac{n}{\log n}\int_0^t \left((nB_s-x)^{1/2}-x(nB_s-1)^{1/2}+(x-1)(nB_s)^{1/2}\right)^2\,\mathrm{d}s \\
&= \frac{1}{\log n}\int_{\mathbb{R}} \left((z-x)^{1/2}-x(z-1)^{1/2}+(x-1)z^{1/2}\right)^2\ell_t^{z/n}\,\mathrm{d}z \\
&\le \frac{\sup_{a\in\mathbb{R}}\ell_t^a}{\log n}\int_{\mathbb{R}} \left((z-x)^{1/2}-x(z-1)^{1/2}+(x-1)z^{1/2}\right)^2\,\mathrm{d}z \\
&= \mathcal{O}\left(\frac{1}{\log n}\right).
\end{aligned}$$

This yields (4.9), and completes the proof of Theorem 3.3. □

4.3 Proof of Theorem 3.4

Observe that

$$\begin{aligned}\frac{H_t^{(\alpha)}(\varepsilon x) - H_t^{(\alpha)}(0)}{\varepsilon} &= \frac{1}{\varepsilon}\int_0^\infty \frac{(\ell_t^{\varepsilon x+y} - \ell_t^{\varepsilon x-y}) - (\ell_t^y - \ell_t^{-y})}{y^\alpha}\,\mathrm{d}y \\ &= \frac{1}{\varepsilon}\int_0^\infty \frac{\ell_t^{\varepsilon x+y} - \ell_t^y}{y^\alpha}\,\mathrm{d}y - \frac{1}{\varepsilon}\int_0^\infty \frac{\ell_t^{\varepsilon x-y} - \ell_t^{-y}}{y^\alpha}\,\mathrm{d}y.\end{aligned}$$

We only have to check that

$$\frac{1}{\varepsilon}\int_0^\infty \frac{\ell_t^{\varepsilon+y} - \ell_t^y}{y^\alpha}\,\mathrm{d}y \to \alpha\int_0^\infty \frac{\ell_t^z - \ell_t^0}{z^{\alpha+1}}\,\mathrm{d}z, \qquad \text{a.s.} \tag{4.12}$$

To this end, let us write $M = M(t) := 1 + \sup_{s\in[0,t]} B_s$ to see that for small ε,

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^\infty \frac{\ell_t^{\varepsilon+y} - \ell_t^y}{y^\alpha}\,\mathrm{d}y &= \frac{1}{\varepsilon}\int_0^M \frac{\ell_t^{\varepsilon+y} - \ell_t^0}{y^\alpha}\,\mathrm{d}y - \frac{1}{\varepsilon}\int_0^M \frac{\ell_t^y - \ell_t^0}{y^\alpha}\,\mathrm{d}y \\ &= \frac{1}{\varepsilon}\int_\varepsilon^{M+\varepsilon} \frac{\ell_t^z - \ell_t^0}{(z-\varepsilon)^\alpha}\,\mathrm{d}z - \frac{1}{\varepsilon}\int_0^M \frac{\ell_t^z - \ell_t^0}{z^\alpha}\,\mathrm{d}z \\ &= \frac{1}{\varepsilon}\int_\varepsilon^M (\ell_t^z - \ell_t^0)\left(\frac{1}{(z-\varepsilon)^\alpha} - \frac{1}{z^\alpha}\right)\mathrm{d}z - \frac{1}{\varepsilon}\int_0^\varepsilon \frac{\ell_t^z - \ell_t^0}{z^\alpha}\,\mathrm{d}z \\ &\quad + \frac{1}{\varepsilon}\int_M^{M+\varepsilon} \frac{\ell_t^z - \ell_t^0}{(z-\varepsilon)^\alpha}\,\mathrm{d}z.\end{aligned}$$

The first term on the right hand side converges almost surely to

$$\alpha\int_0^M \frac{\ell_t^z - \ell_t^0}{z^{\alpha+1}}\,\mathrm{d}z,$$

the second term to 0, and the third term, which is $\frac{1}{\varepsilon}\int_M^{M+\varepsilon} \frac{-\ell_t^0}{(z-\varepsilon)^\alpha}\,\mathrm{d}z$, converges almost surely to

$$-\frac{\ell_t^0}{M^\alpha} = \alpha\int_M^\infty \frac{\ell_t^z - \ell_t^0}{z^{\alpha+1}}\,\mathrm{d}z.$$

This implies (4.12). Theorem (3.4) is proved. □

5 Proofs of Theorems 3.5 and 3.6

Our aim is to estimate

$$\int_0^t (\mathcal{H}^\alpha f)(B_s)\,\mathrm{d}s - \frac{\overline{f}}{\Gamma(1-\alpha)}\,H_t^{(\alpha)}(0),$$

where $\mathcal{H}^\alpha f = \mathcal{D}^{\alpha-1} f$ denotes the fractional derivative of f, and $\overline{f} = \int_{\mathbb{R}} f(x)\,\mathrm{d}x$ (which is finite thanks to (3.5)).

By Fubini's theorem,

$$\begin{aligned}
&\int_0^t (\mathcal{H}^\alpha f)(B_s)\,\mathrm{d}s - \frac{\overline{f}}{\Gamma(1-\alpha)}\,H_t^{(\alpha)}(0) \\
&= \frac{1}{\Gamma(1-\alpha)}\int_{\mathbb{R}}\Big(\int_0^t \frac{\mathrm{d}s}{(B_s-x)^\alpha} - \int_0^t \frac{\mathrm{d}s}{B_s^\alpha}\Big) f(x)\,\mathrm{d}x \\
&= \frac{1}{\Gamma(1-\alpha)}(\mathrm{I}+\mathrm{II}). \qquad (5.1)
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I} &= \int_{|x|\geq t^\beta}\Big(\int_0^t \frac{\mathrm{d}s}{(B_s-x)^\alpha} - \int_0^t \frac{\mathrm{d}s}{B_s^\alpha}\Big) f(x)\,\mathrm{d}x, \\
\mathrm{II} &= \int_{|x|< t^\beta}\Big(\int_0^t \frac{\mathrm{d}s}{(B_s-x)^\alpha} - \int_0^t \frac{\mathrm{d}s}{B_s^\alpha}\Big) f(x)\,\mathrm{d}x,
\end{aligned}$$

for some $\beta \in (0, 1/2]$ whose value is to be determined later. Observe that the same identity holds for $\int_0^t (\mathcal{H}f)(B_s)\,\mathrm{d}s - \frac{\overline{f}}{\pi}H_t^{(1)}(0)$ when taking $\alpha = 1$, except that instead of $1/\Gamma(1-\alpha)$ we get $1/\pi$ for the normalizing constant.

Assume for the moment that we could prove the following: for $\varepsilon > 0$ and $0 < \beta \leq 1/2$, almost surely, when t goes to infinity,

$$\sup_{z\in\mathbb{R}}\Big|\int_0^t \frac{\mathrm{d}s}{(B_s-z)^\alpha}\Big| = o(t^{1-\alpha/2+\varepsilon}), \qquad (5.2)$$

$$\sup_{|z|\leq t^\beta}\Big|\int_1^\infty \frac{\ell_t^{y+z}-\ell_t^{y}}{y^\alpha}\,\mathrm{d}y\Big| = o(t^{\gamma_3+\beta/2+\varepsilon}), \qquad (5.3)$$

$$\sup_{z\in\mathbb{R}}\Big|\int_0^1 \frac{\ell_t^{y+z}-\ell_t^{-y+z}}{y^\alpha}\,\mathrm{d}y\Big| = o(t^{1/4+\varepsilon}), \qquad (5.4)$$

where

$$\gamma_3 = \frac{1}{4} + \frac{(1-\alpha)_+}{2} = \begin{cases} 3/4-\alpha/2, & \text{if } 0<\alpha<1, \\ 1/4, & \text{if } 1\leq\alpha<3/2. \end{cases}$$

By (5.2) and (3.5),

$$\begin{aligned}
|\mathrm{I}| &\leq o(t^{1-\alpha/2+\varepsilon})\int_{|x|\geq t^\beta}|f(x)|\,\mathrm{d}x \\
&\leq o(t^{1-\alpha/2+\varepsilon-\kappa\beta})\int_{|x|\geq t^\beta}|x|^\kappa\,|f(x)|\,\mathrm{d}x \\
&= o(t^{1-\alpha/2+\varepsilon-\kappa\beta}). \qquad (5.5)
\end{aligned}$$

Now let us look at II. By the occupation time formula,

$$\int_0^t \frac{\mathrm{d}s}{(B_s - x)^\alpha} - \int_0^t \frac{\mathrm{d}s}{B_s^\alpha} = \int_0^\infty \frac{\ell_t^{y+x} - \ell_t^{-y+x} - \ell_t^y + \ell_t^{-y}}{y^\alpha}\, \mathrm{d}y.$$

Since f is integrable, we have

$$|\mathrm{II}| \le \mathrm{const} \times \sup_{|x|<t^\beta} \left| \int_0^\infty \frac{\ell_t^{y+x} - \ell_t^{-y+x} - \ell_t^y + \ell_t^{-y}}{y^\alpha}\, \mathrm{d}y \right|,$$

which, in view of (5.3) and (5.4), implies

$$\mathrm{II} = o(t^{\gamma_3+\beta/2+\varepsilon}) + o(t^{1/4+\varepsilon}) = o(t^{\gamma_3+\beta/2+\varepsilon}), \qquad \text{a.s.}$$

This, combined with (5.5) and (5.1), yields that, almost surely,

$$\int_0^t (\mathcal{H}^\alpha f)(B_s)\, \mathrm{d}s - \frac{\overline{f}}{\Gamma(1-\alpha)} H_t^{(\alpha)}(0) = o(t^{1-\alpha/2+\varepsilon-\kappa\beta}) + o(t^{\gamma_3+\beta/2+\varepsilon}).$$

(The expression on the left hand side should be $\int_0^t (\mathcal{H} f)(B_s)\, \mathrm{d}s - \frac{\overline{f}}{\pi} H_t^{(1)}(0)$ if $\alpha = 1$). Choosing

$$\beta = \frac{1-\alpha/2-\gamma_3}{\kappa+1/2},$$

(which lies in $(0,\, 1/2)$) completes the proof of Theorems 3.5 and 3.6.

The rest of the section is devoted to verification of (5.2)–(5.4). We first recall the following useful inequality due to Barlow and Yor [1]: for any $t>0$, $\varepsilon \in (0, 1/2]$ and $\gamma \ge 1$,

$$\mathbb{E}\left[\left(\sup_{0\le s\le t} \sup_{x\ne y} \frac{|\ell_s^x - \ell_s^y|}{|x-y|^{1/2-\varepsilon}}\right)^\gamma\right] \le c_1\, t^{(1+2\varepsilon)\gamma/4}, \tag{5.6}$$

where $c_1 = c_1(\gamma, \varepsilon)$.

Inequality (5.6) allows us to control the almost sure asymptotics (when t is large) of expressions like $\sup_{x\ne y} |\ell_t^x - \ell_t^y|/|x-y|^{1/2-\varepsilon}$. Indeed, let $\nu \in [0, 1/2)$ and let $\varepsilon > 0$. By Chebyshev's inequality and (5.6), for any $\gamma \ge 1$ and $n \ge 1$,

$$\mathbb{P}\left(\sup_{0\le s\le n} \sup_{x\ne y} \frac{|\ell_s^x - \ell_s^y|}{|x-y|^\nu} > n^{(1-\nu)/2+\varepsilon}\right) \le c_2(\gamma, \nu, \varepsilon)\, n^{-\gamma\varepsilon}.$$

Take $\gamma = 2/\varepsilon$ and use the Borel–Cantelli lemma to see that

$$\sup_{0\le s\le n} \sup_{x\ne y} \frac{|\ell_s^x - \ell_s^y|}{|x-y|^\nu} = \mathcal{O}(n^{(1-\nu)/2+\varepsilon}), \qquad \text{a.s.}$$

Since $t \mapsto \sup_{0\le s\le t}\sup_{x\neq y}|\ell_s^x-\ell_s^y|/|x-y|^\nu$ is non-decreasing, and since $\varepsilon>0$ can be arbitrarily small, we have proved that, for any $\nu\in[0,1/2)$ and $\varepsilon>0$, when $t\to\infty$,

$$\sup_{x\neq y}\frac{|\ell_t^x-\ell_t^y|}{|x-y|^\nu}=o(t^{(1-\nu)/2+\varepsilon}),\qquad \text{a.s.} \tag{5.7}$$

Now we prove (5.2)–(5.4) separately.

Proof of (5.2). By the triangular inequality and the usual LIL, when t goes to infinity,

$$\sup_{|z|\ge t^{(1+\varepsilon)/2}}\left|\int_0^t\frac{\mathrm{d}s}{(B_s-z)^\alpha}\right|\le \sup_{|z|\ge t^{(1+\varepsilon)/2}}\int_0^t\frac{\mathrm{d}s}{(z/2)^\alpha}=o(t^{1-\alpha/2}),\qquad \text{a.s.} \tag{5.8}$$

To treat the case $|z|<t^{(1+\varepsilon)/2}$, note that by the usual LIL, for large t,

$$\begin{aligned}\int_0^t\frac{\mathrm{d}s}{(B_s-z)^\alpha} &= \int_0^\infty\frac{\ell_t^{y+z}-\ell_t^{-y+z}}{y^\alpha}\,\mathrm{d}y\\ &= \int_0^{2t^{(1+\varepsilon)/2}}\frac{\ell_t^{y+z}-\ell_t^{-y+z}}{y^\alpha}\,\mathrm{d}y. \end{aligned}\tag{5.9}$$

Let $\nu\in[0,1/2)$ satisfying $\alpha-\nu<1$. Then by (5.7),

$$\begin{aligned}\sup_{z\in\mathbb{R}}\left|\int_0^{2t^{(1+\varepsilon)/2}}\frac{\ell_t^{y+z}-\ell_t^{-y+z}}{y^\alpha}\,\mathrm{d}y\right| &\le \sup_{y\in\mathbb{R},\,z\in\mathbb{R}}\frac{|\ell_t^{y+z}-\ell_t^{-y+z}|}{y^\nu}\int_0^{2t^{(1+\varepsilon)/2}}\frac{\mathrm{d}x}{x^{\alpha-\nu}}\\ &= o(t^{(1-\nu)/2+\varepsilon})\,\mathcal{O}\left(t^{(1-\alpha+\nu)(1+\varepsilon)/2}\right)\\ &= o\left(t^{1-\alpha/2+2\varepsilon}\right),\end{aligned}$$

which, jointly considered with (5.8) and (5.9), yields

$$\sup_{z\in\mathbb{R}}\left|\int_0^t\frac{\mathrm{d}s}{(B_s-z)^\alpha}\right|=o(t^{1-\alpha/2+2\varepsilon}),\qquad \text{a.s.,}$$

as desired. □

Proof of (5.3). Let $0<\beta\le 1/2$. By the usual LIL, for $|z|\le t^\beta$,

$$\int_1^\infty\frac{\ell_t^{y+z}-\ell_t^y}{y^\alpha}\,\mathrm{d}y=\int_1^{t^{1/2+\varepsilon}}\frac{\ell_t^{y+z}-\ell_t^y}{y^\alpha}\,\mathrm{d}y.$$

Using (5.7), we have

$$\sup_{|z|\le t^\beta}\sup_{y\in\mathbb{R}}|\ell_t^{y+z}-\ell_t^y|=o(t^{1/4+\beta/2+\varepsilon}),\qquad \text{a.s.}$$

As a consequence, with probability one,

$$\sup_{|z|\le t^\beta}\left|\int_1^\infty \frac{\ell_t^{y+z}-\ell_t^y}{y^\alpha}\,\mathrm{d}y\right| = o(t^{1/4+\beta/2+\varepsilon})\int_1^{t^{1/2+\varepsilon}}\frac{\mathrm{d}y}{y^\alpha}.$$

This implies (5.3). □

Proof of (5.4). Let $\nu\in[0,1/2)$ be such that $\alpha-\nu<1$. By (5.7), almost surely,

$$\sup_{z\in\mathbb{R}}\left|\int_0^1 \frac{\ell_t^{y+z}-\ell_t^{-y+z}}{y^\alpha}\,\mathrm{d}y\right| \le \sup_{z\in\mathbb{R}}\sup_{0<y\le 1}\frac{|\ell_t^{y+z}-\ell_t^{-y+z}|}{y^\nu}\int_0^1\frac{\mathrm{d}x}{x^{\alpha-\nu}} = o(t^{(1-\nu)/2+\varepsilon}).$$

Since ν can be as close to $1/2$ as possible, this yields (5.4). □

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