

# Favourite sites, favourite values and jumping sizes for random walk and Brownian motion

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We determine: (a) the joint almost sure asymptotic behaviour of the most visited site and the maximum local time of one-dimensional simple random walk or Brownian motion; (b) the maximal jumping size of the most visited site. These solve two open problems of Erdős and Révész.

*Keywords:* Brownian motion; favourite site; local time; random walk

# 1. Introduction

Consider a one-dimensional simple symmetric random walk  $\{S_n\}_{n \geq 0}$ , starting from  $S_0 = 0$ .

Let

$$\xi(n, x) = \#\{0 \leq k \leq n : S_k = x\}. \quad (1.1)$$

Thus,  $\xi(n, x)$  records the number of times the random walk visits the site  $x$  in the first  $n$  steps. Define

$$\mathbb{U}(n) = \left\{ x \in \mathbb{Z} : \xi(n, x) = \sup_{y \in \mathbb{Z}} \xi(n, y) \right\}.$$

In words,  $\mathbb{U}(n)$  is the set of the most visited sites (or favourite sites) of the random walk at time  $n$ . Let

$$U(n) = \max_{x \in \mathbb{U}(n)} x. \quad (1.2)$$

In Erdős and Révész (1984),  $U(n)$  was called the (largest) favourite site of  $\{S_k\}_{0 \leq k \leq n}$ . The choice of  $U(n)$  is irrelevant in the sense that all the results for  $U(n)$  stated in this paper remain true if “max” is replaced for example by “min” in (1.2).

The study of asymptotic properties of  $\mathbb{U}(n)$  and  $U(n)$  was initiated by Erdős and Révész (1984) and Bass and Griffin (1985), and followed by several groups of mathematicians. Nevertheless, little is known so far, due mainly to the fact that the process  $n \mapsto U(n)$  has some very peculiar asymptotic behaviours. For example, a surprising theorem of Bass and Griffin (1985) confirms the transience of  $U(n)$ , in the sense that  $\lim_{n \rightarrow \infty} |U(n)| = \infty$  almost surely. However, it is an open problem to determine the rate of escape of  $U(n)$ . Another interesting problem is from Erdős and Révész (1984), who conjectured that almost surely for all sufficiently large  $n$ ,  $\mathbb{U}(n)$  contains at most two points.

We are interested in the limsup asymptotics of  $U(n)$ . First, we recall the law of the iterated logarithm (LIL) for  $U(n)$  which was proved by Erdős and Révész (1984) and Bass and Griffin (1985), using different methods:

$$\limsup_{n \rightarrow \infty} \frac{U(n)}{\phi(n)} = 1, \quad \text{a.s.}, \quad (1.3)$$

where

$$\phi(n) = (2n \log \log n)^{1/2} = (2n \log_2 n)^{1/2}. \quad (1.4)$$

(However, the problem of characterizing the upper class functions of  $U(n)$  via an integral test in the sense of P. Lévy remains open. For a list of ten other open problems concerning favourite sites, see pp. 130–131 of Révész 1990).

We also recall the LIL for maximum local time: with

$$\xi^*(n) = \sup_{x \in \mathbb{Z}} \xi(n, x), \quad (1.5)$$

we have

$$\limsup_{n \rightarrow \infty} \frac{\xi^*(n)}{\phi(n)} = 1, \quad \text{a.s.} \quad (1.6)$$

This was first proved by Kesten (1965), and it was extended to the form of an integral test by Csáki (1989).

In their proof of (1.3), Erdős and Révész (1984) noticed that, for any  $\varepsilon > 0$ , almost surely there exist infinitely many  $n$  such that simultaneously,  $U(n) \geq (1 - \varepsilon)\phi(n)$  and  $\xi^*(n) \geq c\phi(n)$ , for some positive constant  $c$  depending on  $\varepsilon$ . This led them to ask the following question: what can be said about the joint asymptotics of  $U(n)$  and  $\xi^*(n)$ ? When  $U(n)$  is very close to its maximal possible value, how large can  $\xi^*(n)$  be? This question also appeared later in Erdős and Révész (1987), and it was also stated as Open problem 1 on p. 130 in Révész (1990).

If  $U(n)$  and  $\xi^*(n)$  were asymptotically independent, then one would expect that the limit set of  $\{(U(n)/\phi(n), \xi^*(n)/\phi(n))\}$  should be  $\{(x, y) : x^2 + y^2 \leq 1\}$ . However, things do not quite go like this. Namely we have the following result in this regard.

**Theorem 1.1.** *With probability one, the random sequence*

$$\left\{ \left( \frac{U(n)}{\phi(n)}, \frac{\xi^*(n)}{\phi(n)} \right) \right\}_{n \geq 3}$$

*is relatively compact, whose limit set is identical to the simplex*

$$\mathcal{A} = \left\{ (x, y) : y \geq 0, |x| + y \leq 1 \right\}.$$

**Remark.** Our Theorem 1.1 shows that favourite value and favourite site are somewhat negatively dependent, i.e., if the most visited site is small then the time spent in it is large, and if it is large then the time spent in it is correspondingly smaller.

The second question we are interested in is also due to Erdős and Révész (1984, 1987) (Open problem #4 on page 130 of Révész 1990): what is the maximal jumping size of  $U(n)$ ? The jumping size of  $U(n)$  is defined as  $U(n+1) - U(n)$  if it does not vanish. Concerning its large values, they asked whether it is possible to determine the function  $f(n)$  such that  $\limsup_{n \rightarrow \infty} (U(n+1) - U(n))/f(n) = 1$  almost surely.

In view of (1.3), one might expect  $U(n+1) - U(n)$  to be a constant multiple of  $\phi(n)$  along a random subsequence. What might be somewhat surprising is that it turns out to be as large as  $(1 - \varepsilon)\phi(n)$ , for any  $\varepsilon > 0$ .

**Theorem 1.2.** *We have,*

$$\limsup_{n \rightarrow \infty} \frac{U(n+1) - U(n)}{\phi(n)} = 1, \quad \text{a.s.}$$

In the study of favourite sites, it is more frequent to encounter questions than answers, and this may help explain why there are relatively few known results compared with the huge literature on random walk. We mention the work of Tóth and Werner (1997) who were interested in favourite edges instead of favourite sites, and of Csáki and Shi (1998) who studied how close a favourite site can be to the frontier of the range of random walk.

It will be clear from the proofs in Sections 4 and 5 that both Theorems 1.1 and 1.2 have their counterparts for Brownian motion, which we state as follows. Similarly to the case of random walk, we can define the set of favourite sites of Brownian motion as:

$$\mathbb{V}(t) = \left\{ x \in \mathbb{R} : L(t, x) = \sup_{y \in \mathbb{R}} L(t, y) \right\},$$

where  $L$  denotes the local time of a Brownian motion. It is known (Eisenbaum 1989 and 1997, Leuridan 1997) that, with probability one,  $\mathbb{V}(t)$  contains at most two points for all  $t > 0$ . Without loss of generality, let us choose

$$V(t) = \max_{x \in \mathbb{V}(t)} x.$$

According to Eisenbaum (1989),  $\{V(t); t > 0\}$  is a process with limits on the left almost surely.

**Theorem 1.3.** *Let  $L^*(t) = \sup_{x \in \mathbb{R}} L(t, x)$ . The limit set of  $\{(V(t)/\phi(t), L^*(t)/\phi(t))\}_{t \geq 3}$  is almost surely  $\{(x, y) : y \geq 0, |x| + y \leq 1\}$ .*

**Theorem 1.4.** *We have,*

$$\limsup_{t \rightarrow \infty} \frac{V(t) - V(t-)}{\phi(t)} = 1, \quad \text{a.s.}$$

We say a few words about the organization of this paper. Looking at Theorem 1.1, that  $\{(U(n)/\phi(n), \xi^*(n)/\phi(n))\}_{n \geq 3}$  is almost surely relatively compact follows from (1.3) and (1.6), so we only have to show that its limit set equals  $\mathcal{A}$ . This is done in two steps. Namely, we prove that, almost surely,

- (a) any  $(x, y) \in \mathcal{A}$  is a limit point of  $\{(U(n)/\phi(n), \xi^*(n)/\phi(n))\}_{n \geq 3}$ ;
- (b) all the limit points of  $\{(U(n)/\phi(n), \xi^*(n)/\phi(n))\}_{n \geq 3}$  are contained in  $\mathcal{A}$ .

Parts (a) and (b) are proved in Sections 4 and 5 respectively; furthermore, Section 4 contains the proof of Theorem 1.2. Before that, in Section 2 we present some probability estimates for multi-dimensional Brownian motion, which may be of independent interest. These estimates are applied to Brownian local times in Section 3, and enable us to obtain the key ingredient in the proofs of the theorems.

**Notation.** Throughout the paper,  $a(x) \sim b(x)$  ( $x \rightarrow x_0$ ) denotes  $\lim_{x \rightarrow x_0} a(x)/b(x) = 1$  for positive  $a(x)$  and  $b(x)$ .

## 2. Probability estimates for Brownian motion

This section contains some probability estimates for  $d$ -dimensional Brownian motion. Although only the special case  $d = 2$  is needed in the proof of the theorems, we present the estimates for any dimension  $d \geq 1$ .

In the rest of the section,  $\{B(t); t \geq 0\}$  denotes an  $\mathbb{R}^d$ -valued Brownian motion ( $d \geq 1$ ) and

$$R(t) = \|B(t)\|, \quad t \geq 0,$$

is the Euclidean modulus of  $B$ . Note that  $R$  is a  $d$ -dimensional Bessel process. For  $x \geq 0$ , let  $\mathbb{P}^x$  denote the probability measure under which  $R$  starts from  $x$ . We assume that under  $\mathbb{P}$ ,  $R(0) = 0$ . Thus  $\mathbb{P}^0 = \mathbb{P}$ . We also write  $\mathbb{E}^x$  for the expectation with respect to  $\mathbb{P}^x$ .

The square integral of  $R$  plays an important role in our approach. We write, for brevity,

$$X(t) = \int_0^t R^2(s) \, ds, \quad t \geq 0.$$

We start by recalling two useful results. The first is an intuitively clear comparison criterion for  $R$ . It can easily be proved by applying the diffusion comparison theorem stated in Revuz and Yor (1999, Theorem IX.3.7) to squared Bessel processes. The second is a special case of de Bruijn's exponential-type Tauberian theorem (see Bingham *et al.* 1987, p. 254).

**Fact 2.1.** *Let  $0 \leq x < y$ . The Bessel process  $\{R(t); t \geq 0\}$  is stochastically smaller under  $\mathbb{P}^x$  than under  $\mathbb{P}^y$ .*

**Fact 2.2.** Let  $\Lambda$  be a random variable with  $\mathbb{P}(\Lambda \geq 0) = 1$ , and let  $a > 0$  be a constant. The following conditions are equivalent:

$$\lim_{r \rightarrow \infty} \frac{1}{\sqrt{r}} \log \mathbb{E}(e^{-r\Lambda}) = -a, \quad (\text{i})$$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(\Lambda < \varepsilon) = -\frac{a^2}{4}. \quad (\text{ii})$$

We now prove a few preliminary estimates.

**Lemma 2.3.** Fix  $\theta \in (0, 1)$ ,  $t > 0$ , and  $r > r_1 > 0$ . Then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(\theta\varepsilon \leq X(t) \leq \varepsilon, r_1 \leq R(t) \leq r) = -\frac{(r_1^2 + td)^2}{8}. \quad (2.1)$$

**Proof.** The conditional Laplace transform of  $X(t)$  given  $(R(0), R(t))$  is known (cf. Pitman and Yor 1982):

$$\begin{aligned} & \mathbb{E}^x \left( e^{-\frac{\lambda^2}{2} X(t)} \mid R(t) = y \right) \\ &= \frac{\lambda t}{\sinh(\lambda t)} \frac{I_{(d-2)/2}(\lambda xy / \sinh(\lambda t))}{I_{(d-2)/2}(xy/t)} \exp\left( -\frac{(x^2 + y^2)(\lambda t \coth(\lambda t) - 1)}{2t} \right), \end{aligned} \quad (2.2)$$

where  $I_{(d-2)/2}(\cdot)$  denotes the modified Bessel function of index  $(d-2)/2$ . For convenience, we recall here the asymptotics of the modified Bessel function, (cf. Abramowitz and Stegun 1965, pp. 375 and 377 respectively):

$$I_p(z) \sim (z/2)^p / \Gamma(p+1), \quad z \rightarrow 0, \quad (2.3)$$

$$I_p(z) \sim e^z / \sqrt{2\pi z}, \quad z \rightarrow \infty. \quad (2.4)$$

In view of (2.3), we can let  $x$  go to 0 in (2.2) to see that

$$\mathbb{E} \left( e^{-\frac{\lambda^2}{2} X(t)} \mid R(t) = y \right) = \left( \frac{\lambda t}{\sinh(\lambda t)} \right)^{d/2} \exp\left( -\frac{y^2(\lambda t \coth(\lambda t) - 1)}{2t} \right). \quad (2.5)$$

Thus,

$$\begin{aligned} & \mathbb{E} \left( e^{-\frac{\lambda^2}{2} X(t)} \mid r_1 \leq R(t) \leq r \right) \\ &= \int_{r_1}^r \frac{\mathbb{P}(R(t) \in dy)}{\mathbb{P}(r_1 \leq R(t) \leq r)} \left( \frac{\lambda t}{\sinh(\lambda t)} \right)^{d/2} \exp\left( -\frac{y^2(\lambda t \coth(\lambda t) - 1)}{2t} \right). \end{aligned}$$

When  $\lambda \rightarrow \infty$ , the expression on the right-hand side is  $\exp(-(1 + o(1))\frac{r_1^2 + td}{2}\lambda)$ . Applying de Bruijn's Tauberian theorem (Fact 2.2) gives that, as  $\varepsilon \rightarrow 0$ ,

$$\mathbb{P}(X(t) < \varepsilon \mid r_1 \leq R(t) \leq r) = \exp\left(- (1 + o(1)) \frac{(r_1^2 + td)^2}{8\varepsilon}\right).$$

Therefore,

$$\mathbb{P}(X(t) < \varepsilon, r_1 \leq R(t) \leq r) = \exp\left(- (1 + o(1)) \frac{(r_1^2 + td)^2}{8\varepsilon}\right). \quad (2.6)$$

Similarly, using (2.5) and Fact 2.2, we have that, for any  $y > 0$ , when  $\varepsilon \rightarrow 0$ ,

$$\mathbb{P}(X(t) < \varepsilon \mid R(t) = y) = \exp\left(- (1 + o(1)) \frac{(y^2 + td)^2}{8\varepsilon}\right). \quad (2.7)$$

This estimate will be of use later.

The probability expression on the left-hand side of (2.1) can be written as

$$\mathbb{P}(X(t) < \varepsilon, r_1 \leq R(t) \leq r) - \mathbb{P}(X(t) < \theta\varepsilon, r_1 \leq R(t) \leq r).$$

In view of (2.6), this yields Lemma 2.3. □

**Lemma 2.4.** *For any  $x > 0$ ,*

$$\mathbb{P}^x(X(t) < \varepsilon) = \exp\left(- (1 + o(1)) \frac{(x^2 + td)^2}{8\varepsilon}\right), \quad \varepsilon \rightarrow 0. \quad (2.8)$$

**Proof.** Recall the probability transition density of  $R$  (see Revuz and Yor 1999, Chap. XI): for  $y > 0$ ,

$$\mathbb{P}^x(R(t) \in dy) = \frac{y}{t} \left(\frac{y}{x}\right)^{(d-2)/2} I_{(d-2)/2}\left(\frac{xy}{t}\right) \exp\left(-\frac{x^2 + y^2}{2t}\right) dy. \quad (2.9)$$

Writing

$$\mathbb{E}^x\left(e^{-\frac{\lambda^2}{2}X(t)}\right) = \int_0^\infty \mathbb{P}^x(R(t) \in dy) \mathbb{E}^x\left(e^{-\frac{\lambda^2}{2}X(t)} \mid R(t) = y\right),$$

and using (2.2)–(2.4) and (2.9), we easily arrive at:

$$\log \mathbb{E}^x\left(e^{-\frac{\lambda^2}{2}X(t)}\right) = - (1 + o(1)) \frac{(x^2 + td)\lambda}{2}, \quad \lambda \rightarrow \infty.$$

Lemma 2.4 now follows from an application of Fact 2.2. □

**Lemma 2.5.** Fix  $\theta \in (0, 1)$ ,  $t > 0$ ,  $\nu > 1/2$  and  $r > r_1 > 0$ . Assuming

$$(r^2 + td)^2 \theta < (r_1^2 + td)^2, \quad (2.10)$$

we have

$$\lim_{\varepsilon \rightarrow 0} \inf_{x \in [r_1, r]} \varepsilon \log \mathbb{P}^x(\theta \varepsilon \leq X(t) \leq \varepsilon, R(t) < \varepsilon^\nu) = -\frac{(r^2 + td)^2}{8}.$$

**Proof.** Recall that  $R(t) = \|B(t)\|$ . Under  $\mathbb{P}^x$ ,  $B$  can be realized as an  $\mathbb{R}^d$ -valued Brownian motion with  $\|B(0)\| = x$ . Define  $B_0(s) = B(s) - \frac{s}{t}B(t)$  (for  $s \in [0, t]$ ), which is a Brownian bridge on  $[0, t]$ , independent of  $B(t)$ . Let  $R_0(s) = \|B_0(s)\|$ . Note that  $\{R_0(s); s \in [0, t]\}$  is a so-called Bessel bridge starting from  $x$  (under  $\mathbb{P}^x$ ), “conditioned to hit 0 at time  $t$ ”, independent of  $R(t)$ .

By the triangular inequality,  $|R(s) - R_0(s)| \leq R(t)$  for all  $s \in [0, t]$ . By Minkowski’s inequality,  $|\sqrt{X(t)} - (\int_0^t R_0^2(s) ds)^{1/2}| \leq \sqrt{t} R(t)$ . Therefore, for any  $y > 0$  and  $q > 0$ ,

$$\begin{aligned} \mathbb{P}^x(X(t) \leq q | R(t) = y) &\geq \mathbb{P}^x\left(\left(\int_0^t R_0^2(s) ds\right)^{1/2} \leq \sqrt{q} - \sqrt{t}y\right) \\ &= \mathbb{P}(\sqrt{X(t)} \leq \sqrt{q} - \sqrt{t}y | R(t) = x), \end{aligned}$$

the last equality following from the time inversion:  $\{R_0(t-s); s \in [0, t]\}$  is again a Bessel bridge, starting from 0, conditioned to hit  $x$  at time  $t$ . Hence, by Fact 2.1, when  $\varepsilon$  is so small that  $\sqrt{t}\varepsilon^\nu < \sqrt{\varepsilon}$ ,

$$\begin{aligned} &\inf_{x \in [r_1, r]} \mathbb{P}^x(X(t) \leq \varepsilon, R(t) < \varepsilon^\nu) \\ &= \mathbb{P}^r(X(t) \leq \varepsilon, R(t) < \varepsilon^\nu) \\ &\geq \mathbb{P}^r(R(t) \leq \varepsilon^\nu) \mathbb{P}(X(t) \leq (\sqrt{\varepsilon} - \sqrt{t}\varepsilon^\nu)^2 | R(t) = r). \end{aligned} \quad (2.11)$$

By (2.7), when  $\varepsilon \rightarrow 0$ , the second probability term on the right-hand side of (2.11) is  $\exp(-(1 + o(1))(r^2 + td)^2/8\varepsilon)$ , whereas  $\mathbb{P}^r(R(t) < \varepsilon^\nu)$  is of polynomial order of  $\varepsilon^\nu$  (this is an immediate consequence of (2.9) and (2.3), or can be better understood using the fact that  $R^2(t)$  under  $\mathbb{P}^r$  has a non-centered chi-square law). Consequently,

$$\liminf_{\varepsilon \rightarrow 0} \inf_{x \in [r_1, r]} \varepsilon \log \mathbb{P}^x(X(t) \leq \varepsilon, R(t) < \varepsilon^\nu) \geq -\frac{(r^2 + td)^2}{8}. \quad (2.12)$$

On the other hand, by Fact 2.1,

$$\begin{aligned} \sup_{x \in [r_1, r]} \mathbb{P}^x(X(t) < \theta \varepsilon, R(t) < \varepsilon^\nu) &\leq \mathbb{P}^{r_1}(X(t) < \theta \varepsilon) \\ &= \exp\left(- (1 + o(1)) \frac{(r_1^2 + td)^2}{8\theta \varepsilon}\right), \end{aligned} \quad (2.13)$$



the last identity following from (2.8). Combining (2.12) with (2.13), and in view of (2.10), we obtain the lower bound in the lemma.

The upper bound is easy. Indeed, just as for (2.13), we have

$$\begin{aligned} \inf_{x \in [r_1, r]} \mathbb{P}^x(\theta\varepsilon \leq X(t) \leq \varepsilon, R(t) < \varepsilon^\nu) &\leq \mathbb{P}^r(X(t) \leq \varepsilon) \\ &= \exp\left(- (1 + o(1)) \frac{(r^2 + td)^2}{8\varepsilon}\right), \end{aligned}$$

as desired.  $\square$

**Lemma 2.6.** *For  $r > 0$  and  $t > 0$ ,*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{x \in [0, r]} \varepsilon \log \mathbb{P}^x\left(X(t) < \varepsilon, \sup_{0 \leq s \leq t} R(s) \geq r\right) \leq -\frac{(r^2 + td)^2}{8}. \quad (2.14)$$

**Proof.** Let

$$H(r) = \inf\{t > 0 : R(t) > r\}. \quad (2.15)$$

For each  $x \in [0, r]$ , the probability term on the left-hand side of (2.14) is

$$\leq \mathbb{P}^x\left(H(r) \leq t, X(t) - X(H(r)) < \varepsilon\right) \leq \mathbb{P}^r(X(t) < \varepsilon),$$

the second inequality following from the strong Markov property. Now (2.14) follows from Lemma 2.4.  $\square$

**Lemma 2.7.** *Let  $0 < x_1 < x_2 < r_1^* < r_2^*$  and  $r_1 < r_2 < r_1^*$  be such that*

$$r_2^* - r_1^* > r_1^* - r_1. \quad (2.16)$$

*Then*

$$\begin{aligned} \liminf_{t \rightarrow 0} \inf_{x \in [x_1, x_2]} t \log \mathbb{P}^x\left(\sup_{0 \leq s \leq t} R(s) \in [r_1^*, r_2^*], R(t) \in [r_1, r_2]\right) \\ \geq -(r_1^* - x_1)^2 - (r_1^* - r_1)^2. \end{aligned}$$

**Proof.** We first recall the tail estimate of  $R$ , which follows from a well-known tail estimate for general Gaussian processes (see for example Marcus and Shepp 1972): for  $r > 0$ ,

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} R(s) > r\right) = \exp\left(- (1 + o(1)) \frac{r^2}{2t}\right), \quad t \rightarrow 0. \quad (2.17)$$

Now look at the probability

$$\mathbb{P}^{r_1^*}(R(t) \in [r_1, r_2]) = \int_{r_1}^{r_2} \mathbb{P}^{r_1^*}(R(t) \in dy).$$

From (2.9) and (2.4), it is easily seen that for bounded (from above and below)  $y$  and  $r$ , and  $t \in (0, 1]$ ,

$$\frac{\mathbb{P}^r(R(t) \in dy)}{dy} \geq \frac{c_1}{\sqrt{t}} \exp\left(-\frac{(r-y)^2}{2t}\right), \quad (2.18)$$

where  $c_1 > 0$  is a constant depending only on  $d$  and on the bounds for  $(y, r)$ . Thus, for  $t \in (0, 1]$ ,

$$\mathbb{P}^{r_1^*}(R(t) \in [r_1, r_2]) \geq \frac{c_1(r_2 - r_1)}{\sqrt{t}} \exp\left(-\frac{(r_1^* - r_1)^2}{2t}\right). \quad (2.19)$$

On the other hand, under  $\mathbb{P}^{r_1^*}$ ,  $R$  is the modulus of an  $\mathbb{R}^d$ -valued Brownian motion  $B$  with  $\|B(0)\| = r_1^*$ . By the triangular inequality,

$$\mathbb{P}^{r_1^*}\left(\sup_{0 \leq s \leq t} R(s) > r_2^*\right) \leq \mathbb{P}\left(\sup_{0 \leq s \leq t} R(s) > r_2^* - r_1^*\right),$$

which, according to (2.17), is  $\exp(-(1+o(1))(r_2^* - r_1^*)^2/2t)$ , when  $t$  goes to 0. This, together with (2.19) and (2.16), implies that

$$\mathbb{P}^{r_1^*}\left(\sup_{0 \leq s \leq t} R(s) \leq r_2^*, R(t) \in [r_1, r_2]\right) \geq \exp\left(- (1 + o(1)) \frac{(r_1^* - r_1)^2}{2t}\right), \quad (2.20)$$

when  $t$  goes to 0.

Recall  $H(r_1^*)$  from (2.15). For  $x \in [x_1, x_2]$ , we have, by the strong Markov property,

$$\begin{aligned} & \mathbb{P}^x\left(\sup_{0 \leq s \leq t} R(s) \in [r_1^*, r_2^*], R(t) \in [r_1, r_2]\right) \\ &= \mathbb{P}^x\left(H(r_1^*) \leq t, \sup_{0 \leq s \leq t} R(s) \leq r_2^*, R(t) \in [r_1, r_2]\right) \\ &= \int_0^t \mathbb{P}^x(H(r_1^*) \in du) \mathbb{P}^{r_1^*}\left(\sup_{0 \leq s \leq t-u} R(s) \leq r_2^*, R(t-u) \in [r_1, r_2]\right) \\ &\geq \int_0^{t/2} \mathbb{P}^x(H(r_1^*) \in du) \mathbb{P}^{r_1^*}\left(\sup_{0 \leq s \leq t-u} R(s) \leq r_2^*, R(t-u) \in [r_1, r_2]\right) \\ &\geq \int_0^{t/2} \mathbb{P}^x(H(r_1^*) \in du) \exp\left(- (1 + o(1)) \frac{(r_1^* - r_1)^2}{t}\right), \end{aligned}$$

the last equality following from (2.20), where  $o(1)$  is uniform in  $x \in [x_1, x_2]$ . Lemma 2.7 will be proved if we can verify

$$\inf_{x \in [x_1, x_2]} \mathbb{P}^x(H(r_1^*) \leq t/2) \geq \exp\left(- (1 + o(1)) \frac{(r_1^* - x_1)^2}{t}\right), \quad t \rightarrow 0. \quad (2.21)$$

To this end, observe that for  $x \in [x_1, x_2]$ ,

$$\mathbb{P}^x(H(r_1^*) \leq t/2) \geq \mathbb{P}^x(R(t/2) > r_1^*) = \int_{r_1^*}^{\infty} \mathbb{P}^x(R(t/2) \in dy).$$

By (2.18), there exists a constant  $c_1$  depending only on  $(r_1^*, r_2^*, x_1, x_2, d)$ , such that for all  $x \in [x_1, x_2]$ ,

$$\begin{aligned} \mathbb{P}^x(H(r_1^*) \leq t/2) &\geq \int_{r_1^*}^{r_2^*} \mathbb{P}^x(R(t/2) \in dy) \\ &\geq \frac{2c_1}{t} \int_{r_1^*}^{r_2^*} \exp\left(-\frac{(y-x)^2}{t}\right) dy \\ &\geq \frac{2c_1}{t} \int_{r_1^*}^{r_2^*} \exp\left(-\frac{(y-x_1)^2}{t}\right) dy \\ &= \exp\left(-(1+o(1))\frac{(r_1^*-x_1)^2}{t}\right), \end{aligned}$$

which yields (2.21), thus completes the proof of Lemma 2.7.  $\square$

**Lemma 2.8.** *Let  $a \in (0, 1)$ ,  $b_2 > b_1 > \delta > 0$ ,  $\varrho > 1$ ,  $\nu > 1/2$  and  $0 < c < (\varrho - 1)/b_2$ . Assume*

$$\sqrt{b_2} - \sqrt{b_1} > \sqrt{b_1} - \sqrt{b_1 - \delta}.$$

Let

$$E_1(\varepsilon) = \left\{ \varepsilon \leq X(1 + c\varepsilon) \leq \varrho\varepsilon, R(1 + c\varepsilon) < \varepsilon^\nu, b_1 \leq \sup_{s \in [a, a+c\varepsilon]} R^2(s) \leq b_2, \right. \\ \left. \sup_{s \in [a, a+c\varepsilon]} R^2(s) \geq \delta + \sup_{s \in [0, 1+c\varepsilon] \setminus [a, a+c\varepsilon]} R^2(s) \right\}.$$

Then

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(E_1(\varepsilon)) \geq -\frac{(2(b_1 - \delta) + d)^2}{8} - \frac{5(\sqrt{b_1} - \sqrt{b_1 - \delta})^2}{c}.$$

**Proof.** Fix  $r \in (0, \sqrt{b_1 - \delta})$  such that  $\sqrt{b_2} - \sqrt{b_1} > \sqrt{b_1} - r$ . Define

$$\beta = \frac{r^2 + (1-a)d}{2r^2 + d}. \quad (2.22)$$

Since  $\beta \in (0, 1)$  and  $c < (\varrho - 1)/b_2$ , it is possible to choose  $0 < \beta_* < \beta < \beta^* < 1$  such that

$$1 - \beta_* + b_2 c + \beta^* \leq \varrho. \quad (2.23)$$

Let  $r^* \in (r, \sqrt{b_1 - \delta})$ . We also choose  $r_* \in (0, r)$  so that

$$\sqrt{b_1} - r_* \leq 2(\sqrt{b_1} - r), \quad (2.24)$$

$$\sqrt{b_1} - r_* < \sqrt{b_2} - \sqrt{b_1}, \quad (2.25)$$

$$(r^2 + (1-a)d)^2 \frac{\beta}{\beta^*} < (r_*^2 + (1-a)d)^2. \quad (2.26)$$

Consider the measurable events

$$E_2(\varepsilon) = \left\{ (1-\beta)\varepsilon \leq X(a) \leq (1-\beta_*)\varepsilon, \sup_{0 \leq s \leq a} R(s) < \sqrt{b_1 - \delta}, r \leq R(a) \leq r^* \right\},$$

$$E_3(\varepsilon) = \left\{ \sqrt{b_1} \leq \sup_{a \leq s \leq a+c\varepsilon} R(s) \leq \sqrt{b_2}, r_* \leq R(a+c\varepsilon) \leq r \right\},$$

$$E_4(\varepsilon) = \left\{ \beta\varepsilon \leq X(1+c\varepsilon) - X(a+c\varepsilon) \leq \beta^*\varepsilon, \sup_{a+c\varepsilon \leq s \leq 1+c\varepsilon} R(s) \leq \sqrt{b_1 - \delta}, R(1+c\varepsilon) < \varepsilon^\nu \right\}.$$

Observe that on  $E_3(\varepsilon)$ , we have  $X(a+c\varepsilon) - X(a) \in [0, b_2 c\varepsilon]$ . In view of (2.23), on  $E_2(\varepsilon) \cap E_3(\varepsilon) \cap E_4(\varepsilon)$ , we have  $X(1+c\varepsilon) \in [\varepsilon, \rho\varepsilon]$ . Thus

$$E_1(\varepsilon) \supset E_2(\varepsilon) \cap E_3(\varepsilon) \cap E_4(\varepsilon),$$

By the Markov property,

$$\begin{aligned} \mathbb{P}(E_1(\varepsilon)) &\geq \mathbb{P}(E_2(\varepsilon) \cap E_3(\varepsilon) \cap E_4(\varepsilon)) \\ &\geq \mathbb{P}(E_2(\varepsilon)) \inf_{x \in [r, r^*]} p_1(x, \varepsilon) \inf_{y \in [r_*, r]} p_2(y, \varepsilon), \end{aligned} \quad (2.27)$$

where, for  $x \in [r, r^*]$  and  $y \in [r_*, r]$ ,

$$p_1(x, \varepsilon) = \mathbb{P}^x \left( \sqrt{b_1} \leq \sup_{0 \leq s \leq c\varepsilon} R(s) \leq \sqrt{b_2}, r_* \leq R(c\varepsilon) \leq r \right),$$

$$p_2(y, \varepsilon) = \mathbb{P}^y \left( \beta\varepsilon \leq X(1-a) \leq \beta^*\varepsilon, \sup_{0 \leq s \leq 1-a} R(s) \leq \sqrt{b_1 - \delta}, R(1-a) < \varepsilon^\nu \right).$$

We now estimate  $\mathbb{P}(E_2(\varepsilon))$ ,  $p_1(x, \varepsilon)$  and  $p_2(y, \varepsilon)$  respectively. First, since

$$\begin{aligned} \mathbb{P}(E_2(\varepsilon)) &\geq \mathbb{P}((1-\beta)\varepsilon \leq X(a) \leq (1-\beta_*)\varepsilon, r \leq R(a) \leq r^*) \\ &\quad - \mathbb{P}\left( X(a) \leq (1-\beta_*)\varepsilon, \sup_{0 \leq s \leq a} R(s) \geq \sqrt{b_1 - \delta} \right), \end{aligned}$$

it follows from Lemmas 2.3 and 2.6 that (noting that  $b_1 - \delta > r^2$ )

$$\mathbb{P}(E_2(\varepsilon)) \geq \exp\left( -(1+o(1)) \frac{(r^2 + ad)^2}{8(1-\beta_*)\varepsilon} \right), \quad \varepsilon \rightarrow 0. \quad (2.28)$$

It is easy to treat  $p_1(x, \varepsilon)$ . Indeed, in view of (2.25), we can apply Lemma 2.7 to see that, when  $\varepsilon \rightarrow 0$ ,

$$\inf_{x \in [r, r_*]} p_1(x, \varepsilon) \geq \exp\left(- (1 + o(1)) \frac{(\sqrt{b_1} - r)^2 + (\sqrt{b_1} - r_*)^2}{c\varepsilon}\right). \quad (2.29)$$

Finally, we observe that, for all  $y \in [r_*, r]$ ,

$$\begin{aligned} p_2(y, \varepsilon) &\geq \inf_{y \in [r_*, r]} \mathbb{P}^y(\beta\varepsilon \leq X(1-a) \leq \beta^*\varepsilon, R(1-a) < \varepsilon^\nu) \\ &\quad - \sup_{y \in [0, \sqrt{b_1 - \delta}]} \mathbb{P}^y(X(1-a) \leq \beta^*\varepsilon, \sup_{0 \leq s \leq 1-a} R(s) > \sqrt{b_1 - \delta}). \end{aligned}$$

Recall (2.26). Applying Lemmas 2.5 and 2.6 yields (recalling  $b_1 - \delta > r^2$ ) that, when  $\varepsilon \rightarrow 0$ ,

$$\inf_{y \in [r_*, r]} p_2(y, \varepsilon) \geq \exp\left(- (1 + o(1)) \frac{(r^2 + (1-a)d)^2}{8\beta^*\varepsilon}\right). \quad (2.30)$$

Assembling (2.27)–(2.30) gives

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(E_1(\varepsilon)) &\geq -\frac{(r^2 + ad)^2}{8(1 - \beta_*)} - \frac{(\sqrt{b_1} - r)^2 + (\sqrt{b_1} - r_*)^2}{c} - \frac{(r^2 + (1-a)d)^2}{8\beta^*} \\ &\geq -\frac{(r^2 + ad)^2}{8(1 - \beta)} - \frac{5(\sqrt{b_1} - r)^2}{c} - \frac{(r^2 + (1-a)d)^2}{8\beta}, \end{aligned}$$

where in the last inequality, we have used  $\beta^* > \beta > \beta_*$  and the relation (2.24). In view of (2.22), we obtain:

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(E_1(\varepsilon)) \geq -\frac{(2r^2 + d)^2}{8} - \frac{5(\sqrt{b_1} - r)^2}{c}.$$

Since  $r$  can be as close to  $\sqrt{b_1 - \delta}$  as possible, this completes the proof of Lemma 2.8.  $\square$

Here is the main estimate which will be used in the proof of Theorem 1.1.

**Lemma 2.9.** *Let  $a \in (0, 1)$ ,  $b_2 > b_1 > 0$ ,  $\varrho > 1$ ,  $\nu > 1/2$  and  $0 < c < (\varrho - 1)/b_2$ . Let  $\delta \in (0, b_1)$  be such that*

$$\sqrt{b_2} - \sqrt{b_1} > \sqrt{b_1} - \sqrt{b_1 - \delta}.$$

Then

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(E_5(\varepsilon)) \geq -\frac{(2(b_1 - \delta) + d)^2}{8} - \frac{5(\sqrt{b_1} - \sqrt{b_1 - \delta})^2}{c}, \quad (2.31)$$

where

$$E_5(\varepsilon) = \left\{ \varepsilon \leq X(1) \leq \varrho\varepsilon, R(1) < \varepsilon^\nu, b_1 \leq \sup_{s \in [0,1]} R^2(s) \leq b_2, \right. \\ \left. \sup_{s \in [a, a+c\varepsilon]} R^2(s) \geq \delta + \sup_{s \in [0,1] \setminus [a, a+c\varepsilon]} R^2(s) \right\}.$$

**Proof.** Let  $E_1(\varepsilon)$  be the measurable event defined in Lemma 2.8. Note that on  $E_1(\varepsilon)$ , we have  $\sup_{s \in [a, a+c\varepsilon]} R^2(s) = \sup_{s \in [0, 1+c\varepsilon]} R^2(s)$ . By the scaling property,

$$\mathbb{P}(E_1(\varepsilon)) = \mathbb{P}\left( \frac{\varepsilon}{(1+c\varepsilon)^2} \leq X(1) \leq \frac{\varrho\varepsilon}{(1+c\varepsilon)^2}, R(1) < \frac{\varepsilon^\nu}{(1+c\varepsilon)^{1/2}}, \right. \\ \left. \frac{b_1}{1+c\varepsilon} \leq \sup_{0 \leq s \leq 1} R^2(s) \leq \frac{b_2}{1+c\varepsilon}, \right. \\ \left. \sup_{s \in I(\varepsilon)} R^2(s) \geq \frac{\delta}{1+c\varepsilon} + \sup_{s \in [0,1] \setminus I(\varepsilon)} R^2(s) \right),$$

where  $I(\varepsilon) = [a/(1+c\varepsilon), (a+c\varepsilon)/(1+c\varepsilon)]$ . Applying Lemma 2.8 gives (2.31).  $\square$

The proof of Theorem 1.2 requires a few more probability estimates.

**Lemma 2.10.** For  $r > 0$ ,  $\nu > 1/2$  and  $t > 0$ ,

$$\liminf_{\varepsilon \rightarrow 0} \inf_{x \in [0, r]} \varepsilon \log \mathbb{P}^x \left( \sup_{0 \leq s \leq t} R(s) > r, X(t) < \varepsilon, R(t) < \varepsilon^\nu \right) \\ \geq -\frac{(3r^2 + td)^2}{8}.$$

**Proof.** Let

$$p_3(x, \varepsilon) = \mathbb{P}^x \left( \sup_{0 \leq s \leq t} R(s) > r, X(t) < \varepsilon, R(t) < \varepsilon^\nu \right).$$

Recall the hitting time  $H(\cdot)$  from (2.15). For any  $x \in [0, r]$  and any  $u \in [0, t]$ ,

$$p_3(x, \varepsilon) \geq \mathbb{P}^x (H(r) \leq u, X(t) < \varepsilon, R(t) < \varepsilon^\nu) \\ \geq \mathbb{P}^x \left( H(r) \leq u, \int_{H(r)}^t R^2(v) dv < \varepsilon - r^2u, R(t) < \varepsilon^\nu \right).$$

By the strong Markov property, this leads to:

$$p_3(x, \varepsilon) \geq \int_0^u \mathbb{P}^x(H(r) \in ds) \mathbb{P}^r(X(t-s) < \varepsilon - r^2u, R(t-s) < \varepsilon^\nu) \\ = \int_0^u \mathbb{P}^x(H(r) \in ds) \mathbb{P}^{r/\sqrt{t-s}} \left( X(1) < \frac{\varepsilon - r^2u}{(t-s)^2}, R(1) < \frac{\varepsilon^\nu}{\sqrt{t-s}} \right) \\ \geq \mathbb{P}^x(H(r) \leq u) \inf_{s \in [0, u]} \mathbb{P}^{r/\sqrt{t-s}} \left( X(1) < \frac{\varepsilon - r^2u}{(t-s)^2}, R(1) < \frac{\varepsilon^\nu}{\sqrt{t-s}} \right) \\ \geq \mathbb{P}^x(H(r) \leq u) \inf_{y \in [r/\sqrt{t}, r/\sqrt{t-u}]} \mathbb{P}^y \left( X(1) < \frac{\varepsilon - r^2u}{t^2}, R(1) < \frac{\varepsilon^\nu}{\sqrt{t}} \right). \quad (2.32)$$

This holds for all  $u \in [0, t]$ . We now choose  $u = \chi\varepsilon$ , for some constant  $\chi \in (0, r^{-2})$  whose value will be determined later. In view of Fact 2.1, we have

$$\begin{aligned} \inf_{x \in [0, r]} \mathbb{P}^x(H(r) \leq u) &= \mathbb{P}(H(r) \leq u) \\ &= \mathbb{P}\left(\sup_{0 \leq v \leq u} R(v) > r\right) \\ &= \exp\left(- (1 + o(1)) \frac{r^2}{2\chi\varepsilon}\right), \quad \varepsilon \rightarrow 0, \end{aligned} \quad (2.33)$$

the last equality following from (2.17). On the other hand, Lemma 2.5 readily yields that, for our choice of  $u$ ,

$$\begin{aligned} \inf_{y \in [r/\sqrt{t}, r/\sqrt{t-u}]} \mathbb{P}^y\left(X(1) < \frac{\varepsilon - r^2u}{t^2}, R(1) < \frac{\varepsilon^\nu}{\sqrt{t}}\right) \\ = \exp\left(- (1 + o(1)) \frac{(r^2 + td)^2}{8(1 - r^2\chi)\varepsilon}\right). \end{aligned} \quad (2.34)$$

Assembling (2.32)–(2.34) gives

$$\liminf_{\varepsilon \rightarrow 0} \inf_{x \in [0, r]} \varepsilon \log p_3(x, \varepsilon) \geq -\frac{r^2}{2\chi} - \frac{(r^2 + td)^2}{8(1 - r^2\chi)},$$

for any  $\chi \in (0, r^{-2})$ . Picking up  $\chi = 2/(3r^2 + td)$  completes the proof of Lemma 2.10.  $\square$

Now we are ready to establish the main probability estimate which will be of use in the proof of Theorem 1.2.

**Lemma 2.11.** *Fix  $a \in (0, 1)$ ,  $r > 0$ ,  $\delta \in (0, r^2/3)$  and  $\nu > 1/2$ . Let*

$$E_6(\varepsilon) = \left\{ \sup_{0 \leq v \leq 1} R(v) < r, X(1) < \varepsilon, R(1) < \varepsilon^\nu, \right. \\ \left. \sup_{s \in [a, 1]} R^2(s) > \sup_{s \in [0, a]} R^2(s) + \delta \right\}.$$

Then

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(E_6(\varepsilon)) \geq -\frac{(3\delta + d)^2}{8}.$$

**Proof.** Let  $b \in (0, r^2/3 - \delta)$  and  $c \in (0, 1)$ . Note that

$$\begin{aligned} \mathbb{P}(E_6(\varepsilon)) &\geq \mathbb{P}\left(\sup_{s \in [0, a]} R^2(s) < b, X(a) < c\varepsilon, \right. \\ &\quad \left. b + \delta < \sup_{s \in [a, 1]} R^2(s) < r^2, X(1) - X(a) < (1 - c)\varepsilon, R(1) < \varepsilon^\nu\right). \end{aligned}$$

By the Markov property, this leads to:

$$\mathbb{P}(E_6(\varepsilon)) \geq p_4(\varepsilon) \inf_{x \in [0, \sqrt{b}]} p_5(x, \varepsilon), \quad (2.35)$$

where

$$p_4(\varepsilon) = \mathbb{P}\left(\sup_{s \in [0, a]} R^2(s) < b, X(a) < c\varepsilon\right),$$

$$p_5(x, \varepsilon) = \mathbb{P}^x\left(b + \delta < \sup_{s \in [0, 1-a]} R^2(s) < r^2, X(1-a) < (1-c)\varepsilon, R(1-a) < \varepsilon^\nu\right).$$

By Lemmas 2.3 and 2.6, for any  $b_1 \in (0, b)$ ,

$$p_4(\varepsilon) \geq \mathbb{P}\left(b_1 \leq \sup_{s \in [0, a]} R^2(s) \leq b, X(a) < c\varepsilon\right) \geq \exp\left(- (1 + o(1)) \frac{(b_1 + ad)^2}{8c\varepsilon}\right),$$

as  $\varepsilon$  goes to 0. Therefore

$$p_4(\varepsilon) \geq \exp\left(- (1 + o(1)) \frac{(b + ad)^2}{8c\varepsilon}\right), \quad \varepsilon \rightarrow 0. \quad (2.36)$$

On the other hand, observe that

$$\begin{aligned} \inf_{x \in [0, \sqrt{b}]} p_5(x, \varepsilon) &\geq \inf_{x \in [0, \sqrt{b+\delta}]} \mathbb{P}^x\left(\sup_{s \in [0, 1-a]} R^2(s) > b + \delta, \right. \\ &\quad \left. X(1-a) < (1-c)\varepsilon, R(1-a) < \varepsilon^\nu\right) \\ &\quad - \sup_{x \in (0, r]} \mathbb{P}^x\left(\sup_{s \in [0, 1-a]} R^2(s) \geq r^2, X(1-a) < (1-c)\varepsilon\right). \end{aligned}$$

Applying respectively Lemmas 2.10 and 2.6 to the two probability terms on the right hand side yields

$$\begin{aligned} \inf_{x \in [0, \sqrt{b}]} p_5(x, \varepsilon) &\geq \exp\left(- (1 + o(1)) \frac{(3(b + \delta) + (1-a)d)^2}{8(1-c)\varepsilon}\right) \\ &\quad - \exp\left(- (1 + o(1)) \frac{(r^2 + (1-a)d)^2}{8(1-c)\varepsilon}\right). \end{aligned}$$

Since  $3(b + \delta) < r^2$ , this leads to:

$$\inf_{x \in [0, \sqrt{b}]} p_5(x, \varepsilon) \geq \exp\left(- (1 + o(1)) \frac{(3(b + \delta) + (1-a)d)^2}{8(1-c)\varepsilon}\right). \quad (2.37)$$

Combining (2.35)–(2.37) yields that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(E_6(\varepsilon)) \geq -\frac{(b + ad)^2}{8c} - \frac{(3(b + \delta) + (1-a)d)^2}{8(1-c)},$$



for any  $c \in (0, 1)$  and  $b \in (0, r^2/3 - \delta)$ . Take  $c = (b + ad)/(4b + 3\delta + d)$  and then send  $b$  to  $0^+$  to complete the proof of Lemma 2.11.  $\square$

### 3. Brownian local time

Let  $\{W(t); t \geq 0\}$  be a one-dimensional Brownian motion starting from 0. Let  $\{L(t, x); t \geq 0, x \in \mathbb{R}\}$  denote its jointly continuous local time process, in the sense that for any Borel function  $f \geq 0$  and  $t \geq 0$ ,

$$\int_0^t f(W(s)) ds = \int_{-\infty}^{\infty} f(x) L(t, x) dx. \quad (3.1)$$

For brevity, we write

$$L^*(t) = \sup_{x \in \mathbb{R}} L(t, x), \quad t \geq 0.$$

For each  $r > 0$ , define

$$T_r = \inf\{t > 0 : W(t) > r\}, \quad (3.2)$$

the hitting time associated with  $W$ . Here is the key probability estimate in the proof of Theorem 1.1.

**Lemma 3.1.** *Let  $a \in (0, 1)$ ,  $b_2 > b_1 > 0$ ,  $\varrho > 1$  and  $0 < c < (\varrho - 1)/b_2$ . Let  $\delta \in (0, b_1)$  be such that*

$$\sqrt{b_2} - \sqrt{b_1} > \sqrt{b_1} - \sqrt{b_1 - \delta}.$$

We have,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(F_1(\varepsilon)) \geq -\frac{(b_1 - \delta + 1)^2}{2} - \frac{5(\sqrt{b_1} - \sqrt{b_1 - \delta})^2}{c},$$

where

$$F_1(\varepsilon) = \left\{ \varepsilon \leq T_1 \leq \varrho\varepsilon, b_1 \leq L^*(T_1) \leq b_2, \right. \\ \left. \sup_{x \in [a - c\varepsilon, a]} L(T_1, x) > \delta + \sup_{x \notin [a - c\varepsilon, a]} L(T_1, x) \right\}.$$

**Proof.** The Ray–Knight theorem (Ray 1963, Knight 1963) says that,  $\{L(T_1, 1 - x); x \geq 0\}$  is a continuous inhomogeneous Markov process. More precisely, when  $x \in [0, 1]$ , it is a two-dimensional squared Bessel process, starting from 0, and when  $x \geq 1$ , it becomes a squared Bessel process of “dimension 0”.

Let  $\{R(t); t \geq 0\}$  be a two-dimensional Bessel process with  $R(0) = 0$ , and  $\{Z(t); t \geq 0\}$  a Bessel process of dimension 0, with  $Z(0) = 1$ . We assume that  $R$  and  $Z$  are independent. As before, we write

$$X(t) = \int_0^t R^2(s) ds, \quad t \geq 0.$$

Observe that  $T_1 = \int_0^\infty L(T_1, 1-x) dx$ . By the Ray–Knight theorem and the scaling property for the Bessel process (writing  $x \vee y$  for  $\max(x, y)$ ),

$$\begin{aligned} \mathbb{P}(F_1(\varepsilon)) &= \mathbb{P}\left(\varepsilon \leq X(1) + R^4(1) \int_0^\infty Z^2(s) ds \leq \varrho\varepsilon, \right. \\ &\quad b_1 \leq \sup_{0 \leq s \leq 1} R^2(s) \vee R^2(1) \sup_{s \geq 0} Z^2(s) \leq b_2, \\ &\quad \left. \sup_{s \in J(\varepsilon)} R^2(s) > \delta + \sup_{s \in [0,1] \setminus J(\varepsilon)} R^2(s) \vee R^2(1) \sup_{s \geq 0} Z^2(s)\right), \end{aligned}$$

where  $J(\varepsilon) = [1-a, 1-a+c\varepsilon]$ . Let  $\varrho_1 \in (1, \varrho)$  be such that  $c < (\varrho_1 - 1)/b_2$ . For all sufficiently small  $\varepsilon > 0$ ,

$$\begin{aligned} \mathbb{P}(F_1(\varepsilon)) &\geq \mathbb{P}\left(\int_0^\infty Z^2(s) ds \leq 1, \sup_{s \geq 0} Z^2(s) \leq 2\right) \times \\ &\quad \mathbb{P}\left(R(1) < \varepsilon^\nu, \varepsilon \leq X(1) \leq \varrho_1\varepsilon, b_1 \leq \sup_{0 \leq s \leq 1} R^2(s) \leq b_2, \right. \\ &\quad \left. \sup_{s \in J(\varepsilon)} R^2(s) > \delta + \sup_{s \in [0,1] \setminus J(\varepsilon)} R^2(s)\right). \end{aligned}$$

The first probability term on the right-hand side is a positive constant. Lemma 3.1 now follows from Lemma 2.9.  $\square$

In the proof of Theorem 1.2, we need two other probability estimates for Brownian local time. Their proofs are in the same spirit as the proof of Lemma 3.1, based on the Ray–Knight theorem and using Lemmas 2.11 and 2.10 respectively in lieu of Lemma 2.9. The details are omitted.

**Lemma 3.2.** *Let  $u \in (0, 1)$ ,  $b > 0$  and  $\delta \in (0, b/3)$ . Then*

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\left(L^*(T_1) < b, T_1 < \varepsilon, \sup_{x \in [0, u]} L(T_1, x) > \sup_{x > u} L(T_1, x) + \delta\right) \\ \geq -\frac{(3\delta + 2)^2}{8}. \end{aligned}$$

**Lemma 3.3.** For  $b > 0$ ,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left( L^*(T_1) > b, T_1 < \varepsilon, \inf_{t \in [0, T_1]} W(t) > -1 \right) \geq -\frac{(3b+2)^2}{8}.$$

## 4. Theorem 1.2 and Part (a) of Theorem 1.1

In this section, we prove Theorem 1.2 and Part (a) of Theorem 1.1. Let us first recall the following strong approximation theorem which holds simultaneously for local time and random walk.

**Fact 4.1 (Révész 1990, pp. 105–107).** *After possible redefinitions of variables and processes, there exists a coupling for the simple random walk  $\{S_k\}_{k \geq 0}$  and the Wiener process  $\{W(t); t \geq 0\}$ , such that for all  $\varepsilon > 0$ , as  $n$  goes to infinity,*

$$\sup_{x \in \mathbb{Z}} \max_{0 \leq k \leq n} |\xi(k, x) - L(k, x)| = o(n^{1/4+\varepsilon}), \quad \text{a.s.}, \quad (4.1)$$

$$\max_{0 \leq k \leq n} |S_k - W(k)| = \mathcal{O}(\log n), \quad \text{a.s.}, \quad (4.2)$$

where  $\xi(n, x)$  and  $L(n, x)$  are defined in (1.1) and (3.1) respectively.

In the proof of Theorem 1.2 and of Part (a) of Theorem 1.1, we shall be working in the probability space such that the coupling for  $\{S_n\}_{n \geq 0}$  and  $W$  in Fact 4.1 holds.

**Proof of Theorem 1.2.** Fix  $\kappa \in (0, 1)$ , and define  $r_n = r_n(\kappa) = \exp(n^{1+\kappa})$ . Let  $T$  be the first hitting time process for  $W$ , see (3.2).

Let  $u \in (0, 1/100)$  be such that

$$\frac{(u^2 + 2(1-u))^2}{4} + 49u < \frac{1+2\kappa}{1+\kappa}. \quad (4.3)$$

Define

$$W^{(n)}(t) = W(t + T_{r_n}) - r_n, \quad t \in [0, T_{r_{n+1}} - T_{r_n}].$$

By the strong Markov property, these are independent Brownian pieces. We can define  $T_r^{(n)} = \inf\{t > 0 : W^{(n)}(t) > r\}$ ,  $0 < r \leq r_{n+1} - r_n$ , the first hitting time process for  $W^{(n)}$ , and also  $\{L^{(n)}(t, x); 0 \leq t \leq T_{r_{n+1}-r_n}^{(n)}, x \in \mathbb{R}\}$ , the local time of  $W^{(n)}$ .

Consider the following measurable events:

$$\begin{aligned}
A_{1,n} &= \left\{ \sup_{x \in \mathbb{R}} L^{(n)}(T_{(1-u)r_{n+1}}^{(n)}, x) < ur_{n+1}, T_{(1-u)r_{n+1}}^{(n)} < \frac{(1+2\kappa)r_{n+1}^2}{2 \log_2 r_{n+1}}, \right. \\
&\quad \left. \sup_{x \in [0, ur_{n+1}]} L^{(n)}(T_{(1-u)r_{n+1}}^{(n)}, x) > \sup_{x > ur_{n+1}} L^{(n)}(T_{(1-u)r_{n+1}}^{(n)}, x) + \frac{u^2 r_{n+1}}{3} \right\}, \\
A_{2,n} &= \left\{ \sup_{x \in \mathbb{R}} (L^{(n)}(T_{r_{n+1}-r_n}^{(n)}, x) - L^{(n)}(T_{(1-u)r_{n+1}}^{(n)}, x)) \geq 2ur_{n+1}, \right. \\
&\quad T_{r_{n+1}-r_n}^{(n)} - T_{(1-u)r_{n+1}}^{(n)} < \frac{(1+2\kappa)ur_{n+1}^2}{2 \log_2 r_{n+1}}, \\
&\quad \left. \inf_{T_{(1-u)r_{n+1}}^{(n)} \leq t \leq T_{r_{n+1}-r_n}^{(n)}} W^{(n)}(t) > (1-2u)r_{n+1} \right\}.
\end{aligned}$$

By the strong Markov property,  $\{A_{1,n}, A_{2,n}\}_{n \geq 3}$  are mutually independent events. In particular,

$$\mathbb{P}(A_{1,n} \cap A_{2,n}) = \mathbb{P}(A_{1,n})\mathbb{P}(A_{2,n}). \quad (4.4)$$

Since  $\{W^{(n)}(t), t \in [0, T_{r_{n+1}} - T_{r_n}]\}$  is again a Brownian motion,

$$\begin{aligned}
\mathbb{P}(A_{1,n}) &= \mathbb{P}\left( \sup_{x \in \mathbb{R}} L(T_{(1-u)r_{n+1}}, x) < ur_{n+1}, T_{(1-u)r_{n+1}} < \frac{(1+2\kappa)r_{n+1}^2}{2 \log_2 r_{n+1}}, \right. \\
&\quad \left. \sup_{x \in [0, ur_{n+1}]} L(T_{(1-u)r_{n+1}}, x) > \sup_{x > ur_{n+1}} L(T_{(1-u)r_{n+1}}, x) + \frac{u^2 r_{n+1}}{3} \right) \\
&= \mathbb{P}\left( \sup_{x \in \mathbb{R}} L(T_1, x) < \frac{u}{1-u}, T_1 < \frac{1+2\kappa}{2(1-u)^2 \log_2 r_{n+1}}, \right. \\
&\quad \left. \sup_{x \in [0, u/(1-u)]} L(T_1, x) > \sup_{x > u/(1-u)} L(T_1, x) + \frac{u^2}{3(1-u)} \right),
\end{aligned}$$

the second equality following from the scaling property. Applying Lemma 3.2 gives that, when  $n$  goes to infinity,

$$\mathbb{P}(A_{1,n}) \geq \exp\left( -(1+o(1)) \frac{(u^2 + 2(1-u))^2}{4(1+2\kappa)} \log_2 r_{n+1} \right). \quad (4.5)$$

Similarly,

$$\begin{aligned}
\mathbb{P}(A_{2,n}) &= \mathbb{P}\left(\sup_{x \in \mathbb{R}} L(T_{ur_{n+1}-r_n}, x) \geq 2ur_{n+1}, T_{ur_{n+1}-r_n} < \frac{(1+2\kappa)ur_{n+1}^2}{2 \log_2 r_{n+1}}, \right. \\
&\quad \left. \inf_{0 \leq t \leq T_{ur_{n+1}-r_n}} W(t) > -ur_{n+1}\right) \\
&= \mathbb{P}\left(\sup_{x \in \mathbb{R}} L(T_1, x) \geq \frac{2ur_{n+1}}{ur_{n+1}-r_n}, T_1 < \frac{(1+2\kappa)ur_{n+1}^2}{2(ur_{n+1}-r_n)^2 \log_2 r_{n+1}}, \right. \\
&\quad \left. \inf_{0 \leq t \leq T_1} W(t) > -\frac{ur_{n+1}}{ur_{n+1}-r_n}\right) \\
&\geq \mathbb{P}\left(\sup_{x \in \mathbb{R}} L(T_1, x) \geq 4, T_1 < \frac{1+2\kappa}{2u \log_2 r_{n+1}}, \inf_{0 \leq t \leq T_1} W(t) > -1\right),
\end{aligned}$$

the last inequality following from the fact that  $ur_{n+1} - r_n \in [ur_{n+1}/2, ur_{n+1}]$  for all sufficiently large  $n$ . In light of Lemma 3.3, this implies

$$\mathbb{P}(A_{2,n}) \geq \exp\left(- (1+o(1)) \frac{49u}{1+2\kappa} \log_2 r_{n+1}\right). \quad (4.6)$$

In view of (4.3)–(4.6), we have  $\sum_n \mathbb{P}(A_{1,n} \cap A_{2,n}) = \infty$ . By the Borel–Cantelli lemma,  $\mathbb{P}(A_{1,n} \cap A_{2,n}, \text{ i.o.}) = 1$ .

Now let us see what happens on  $A_{1,n} \cap A_{2,n}$ . First, we have, on  $A_{1,n} \cap A_{2,n}$ ,

$$T_{r_{n+1}-r_n}^{(n)} < \frac{(1+2\kappa)(1+u)r_{n+1}^2}{2 \log_2 r_{n+1}}. \quad (4.7)$$

By definition, for  $r \in (0, r_{n+1} - r_n]$  and  $x \in \mathbb{R}$ ,

$$T_r^{(n)} = T_{r+r_n} - T_{r_n}, \quad (4.8)$$

$$L^{(n)}(T_r^{(n)}, x) = L(T_{r+r_n}, x+r_n) - L(T_{r_n}, x+r_n). \quad (4.9)$$

Writing

$$s_n = (1-u)r_{n+1} + r_n \in (r_n, r_{n+1}),$$

$$p_n = ur_{n+1} + r_n \in (r_n, s_n),$$

we obtain, on  $A_{1,n} \cap A_{2,n}$ ,

$$\sup_{y \in \mathbb{R}} (L(T_{s_n}, y) - L(T_{r_n}, y)) < ur_{n+1}, \quad (4.10)$$

$$\sup_{y \in [r_n, p_n]} (L(T_{s_n}, y) - L(T_{r_n}, y)) > \sup_{y > p_n} (L(T_{s_n}, y) - L(T_{r_n}, y)) + \frac{u^2 r_{n+1}}{3}, \quad (4.11)$$

$$\sup_{y \in \mathbb{R}} (L(T_{r_{n+1}}, y) - L(T_{s_n}, y)) \geq 2ur_{n+1}, \quad (4.12)$$

$$\inf_{T_{s_n} \leq t \leq T_{r_{n+1}}} W(t) > (1-2u)r_{n+1} + r_n. \quad (4.13)$$

On the other hand, by (4.7) and (4.8), we have, on  $A_{1,n} \cap A_{2,n}$ ,

$$T_{r_{n+1}} - T_{r_n} < \frac{(1 + 2\kappa)(1 + u)r_{n+1}^2}{2 \log_2 r_{n+1}}. \quad (4.14)$$

Since  $\mathbb{P}(A_{1,n} \cap A_{2,n}, \text{ i.o.}) = 1$ , we know that there are almost surely infinitely many  $n$  satisfying (4.10)–(4.14). We shall now only be considering these infinitely many  $n$ .

Since  $r \mapsto T_r$  is a subordinator of index  $(1/2)$ , we have (Fristedt 1973): for any  $\varepsilon \in (0, 1/100)$ , almost surely for large  $r$ ,

$$T_r < r^{2+\varepsilon}. \quad (4.15)$$

Let  $\xi$  be the local time of the random walk  $\{S_k\}_{k \geq 0}$  satisfying (4.1) and (4.2). In view of (4.1), almost surely for large  $n$ , (4.10) implies (recalling  $\xi^*(n)$  from (1.5))

$$\xi^*([T_{s_n}]) \leq ur_{n+1} + T_{s_n}^{1/4+\varepsilon} + \xi^*([T_{r_n}]) + T_{r_n}^{1/4+\varepsilon},$$

which, in light of (4.15) and (1.6), yields

$$\xi^*([T_{s_n}]) \leq (1 + \varepsilon)ur_{n+1},$$

whereas (4.12) implies

$$\xi^*([T_{r_{n+1}}]) \geq (2 - \varepsilon)ur_{n+1}.$$

By the definition of favourite sites, these two estimates imply that,  $U([T_{r_{n+1}}])$ , the favourite site at time  $[T_{r_{n+1}}]$ , must be visited by the random walk during  $([T_{s_n}], [T_{r_{n+1}}])$ , i.e.,

$$U([T_{r_{n+1}}]) = S_j, \quad \text{for some } j = j_n(\omega) \in ([T_{s_n}], [T_{r_{n+1}}]). \quad (4.16)$$

On the other hand, (4.13) and (4.2) together confirm that

$$S_i > (1 - 2u)r_{n+1}, \quad \text{for all } i \in ([T_{s_n}], [T_{r_{n+1}}]). \quad (4.17)$$

Now consider (4.11). Again, by means of (4.1), (4.15) and (1.6), it follows from (4.11) that

$$\max_{y \in [r_n, p_n]} \xi([T_{s_n}], y) > \max_{y > p_n} \xi([T_{s_n}], y) + \left(\frac{1}{3} - \varepsilon\right)u^2 r_{n+1},$$

which, by definition, implies  $U([T_{s_n}]) \leq p_n$ . Since  $p_n < (1 - 2u)r_{n+1}$ , comparing this with (4.16) and (4.17), we conclude that, there exists a random integer  $m = m_n(\omega) \in$

$[\lfloor T_{s_n} \rfloor, \lfloor T_{r_{n+1}} \rfloor]$  such that  $U(m) = U(\lfloor T_{s_n} \rfloor)$  and that  $U(m+1) > (1-2u)r_{n+1}$ . For this  $m$ ,

$$U(m+1) - U(m) > (1-2u)r_{n+1} - p_n \geq (1-4u)r_{n+1}, \quad (4.18)$$

and, of course,  $m \in [T_{r_n} - 1, T_{r_{n+1}}]$ .

To complete the proof of Theorem 1.2, it suffices to note that by (4.14) and (4.15),  $T_{r_{n+1}} < (1+3\kappa)(1+u)r_{n+1}^2/2\log_2 r_{n+1}$ , which implies

$$r_{n+1} > \sqrt{\frac{2T_{r_{n+1}} \log_2 T_{r_{n+1}}}{(1+4\kappa)(1+u)}} \geq \sqrt{\frac{2m \log_2 m}{(1+4\kappa)(1+u)}}.$$

Going back to (4.18), we see that

$$U(m+1) - U(m) > (1-4u) \sqrt{\frac{2m \log_2 m}{(1+4\kappa)(1+u)}}.$$

We have therefore proved that

$$\limsup_{n \rightarrow \infty} \frac{U(n+1) - U(n)}{\sqrt{2n \log_2 n}} \geq \frac{1-4u}{\sqrt{(1+4\kappa)(1+u)}}, \quad \text{a.s.}$$

Since both  $u$  and  $\kappa$  can be as close to 0 as possible, this yields the lower bound in Theorem 1.2.

The upper bound is trivial. Indeed,  $U(n+1) - U(n) \leq \max_{0 \leq k \leq n} S_k - \min_{0 \leq k \leq n} S_k$ , and the upper bound immediately follows from the usual LIL for the range of random walk (see p. 44 of Révész 1990).  $\square$

**Proof of Theorem 1.1: Part (a).** Again, let us fix  $\kappa \in (0, 1)$ , and define  $r_n = \exp(n^{1+\kappa})$ . Recall  $T$  from (3.2). Let  $a \in (0, 1)$  and  $b > 0$ .

Let  $\varrho > 1$ ,  $b^* > b$  and  $0 < c < (\varrho - 1)/b^*$ . We can choose  $\delta \in (0, b)$  such that  $\sqrt{b^*} - \sqrt{b} > \sqrt{b} - \sqrt{b - \delta}$  and that

$$\frac{(b - \delta + 1)^2}{2} + \frac{5(\sqrt{b} - \sqrt{b - \delta})^2}{c} < (1 + \kappa) \frac{(b + 1)^2}{2}. \quad (4.19)$$

Fix  $\gamma > 0$  and let

$$I_n = \left[ ar_n - \frac{c\gamma r_n}{\log_2 r_n}, ar_n \right],$$

$$A_n = \left\{ \frac{\gamma r_n^2}{\log_2 r_n} \leq T_{r_n} \leq \frac{\varrho \gamma r_n^2}{\log_2 r_n}, br_n \leq L^*(T_{r_n}) \leq b^* r_n, \right. \\ \left. \sup_{x \in I_n} L(T_{r_n}, x) > \delta r_n + \sup_{x \notin I_n} L(T_{r_n}, x) \right\}.$$

By scaling, Lemma 3.1 and (4.19), for all sufficiently large  $n$ ,

$$\mathbb{P}(A_n) \geq \exp\left(- (1 + \kappa) \frac{(b + 1)^2 \log_2 r_n}{2\gamma}\right).$$

Therefore, as long as  $\gamma \geq (1 + \kappa)^2(b + 1)^2/2$ , we have  $\sum_n \mathbb{P}(A_n) = \infty$ . Rigorously speaking, we cannot directly apply the Borel–Cantelli lemma here, since the events  $(A_n)$  are not independent. However, as in the proof of Theorem 1.2, we can consider the independent Brownian pieces  $W^{(n)}$  and construct events which are based on  $W^{(n)}$ , and use the same argument to get rid of the dependence difficulty. We choose to omit tedious discussions which are much alike to those in the proof of Theorem 1.2, and claim that  $\mathbb{P}(A_n, \text{ i.o.}) = 1$ . Therefore, almost surely there exist infinitely many  $n$  such that simultaneously,

$$\begin{aligned} \frac{\gamma r_n^2}{\log_2 r_n} \leq T_{r_n} \leq \frac{\varrho \gamma r_n^2}{\log_2 r_n}, \quad b r_n \leq L^*(T_{r_n}) \leq b^* r_n, \\ \sup_{x \in I_n} L(T_{r_n}, x) > \delta r_n + \sup_{x \notin I_n} L(T_{r_n}, x). \end{aligned}$$

Applying Fact 4.1 implies that, infinitely often,

$$\frac{\gamma r_n^2}{\log_2 r_n} \leq T_{r_n} \leq \frac{\varrho \gamma r_n^2}{\log_2 r_n}, \quad (1 - \kappa) b r_n \leq \xi^*([T_{r_n}]) \leq (1 + \kappa) b^* r_n, \quad U([T_{r_n}]) \in I_n.$$

When  $\gamma r_n^2 / \log_2 r_n \leq T_{r_n} \leq \varrho \gamma r_n^2 / \log_2 r_n$ , we have (recalling  $\phi$  from (1.4))

$$\begin{aligned} b r_n &\geq \frac{(1 + o(1))b}{\sqrt{2\varrho\gamma}} \phi(T_{r_n}), \\ b^* r_n &\leq \frac{(1 + o(1))b^*}{\sqrt{2\gamma}} \phi(T_{r_n}), \\ I_n &\subset \left[ \frac{(1 + o(1))a}{\sqrt{2\varrho\gamma}} \phi(T_{r_n}), \frac{(1 + o(1))a}{\sqrt{2\gamma}} \phi(T_{r_n}) \right]. \end{aligned}$$

Thus, for any  $\kappa \in (0, 1)$ ,  $b^* > b > 0$ ,  $\varrho > 1$  and  $\gamma \geq (1 + \kappa)^2(b + 1)^2/2$ , with probability one, the random sequence  $\{(U(n)/\phi(n), \xi^*(n)/\phi(n))\}_{n \geq 3}$  visits

$$\left[ \frac{(1 - \kappa)a}{\sqrt{2\varrho\gamma}}, \frac{(1 + \kappa)a}{\sqrt{2\gamma}} \right] \times \left[ \frac{(1 - \kappa)^2 b}{\sqrt{2\varrho\gamma}}, \frac{(1 + \kappa)^2 b^*}{\sqrt{2\gamma}} \right]$$

infinitely often. Take  $\gamma = (1 + \kappa)^2(b + 1)^2/2$ . Since  $(\kappa, b^*, \varrho)$  can be as close to  $(0, b, 1)$  as possible, we conclude that  $(a/(b + 1), b/(b + 1))$  is almost surely a limit point of  $\{(U(n)/\phi(n), \xi^*(n)/\phi(n))\}_{n \geq 3}$ . For any  $x > 0$  and  $y > 0$  with  $x + y < 1$ , we can choose  $a \in (0, 1)$  and  $b > 0$  such that  $(x, y) = (a/(b + 1), b/(b + 1))$ . Thus  $(x, y)$  is a limit point



of  $\{(U(n)/\phi(n), \xi^*(n)/\phi(n))\}_{n \geq 3}$ . Of course, the positivity of  $x$  is irrelevant, since when  $x < 0$ , we can consider  $\{-S_n\}_{n \geq 0}$  instead of  $\{S_n\}_{n \geq 0}$ .

As a consequence, any  $(x, y) \in \mathcal{A}$  is almost surely a limit point of  $\{(U(n)/\phi(n), \xi^*(n)/\phi(n))\}_{n \geq 3}$ . We can interchange the order of “any  $(x, y) \in \mathcal{A}$ ” and “almost surely” by considering the points in  $\mathcal{A}$  whose coordinates are rationals.  $\square$

## 5. Proof of Theorem 1.1: Part (b)

Part (b) of Theorem 1.1 is a consequence of the following:

**Theorem 5.1.** *For any  $a \in (0, 1)$ ,*

$$\limsup_{n \rightarrow \infty} \sup_{x \geq a\phi(n)} \frac{\xi(n, x)}{\phi(n)} = 1 - a, \quad \text{a.s.}$$

More precisely, Part (b) of Theorem 1.1 follows from the upper bound in Theorem 5.1.

In view of (4.1), it is equivalent to state Theorem 5.1 as follows.

**Theorem 5.2.** *Let  $\{L(t, x); t \geq 0, x \in \mathbb{R}\}$  be a Brownian local time process, and let  $a \in (0, 1)$ . Then,*

$$\limsup_{t \rightarrow \infty} \sup_{x \geq a\phi(t)} \frac{L(t, x)}{\phi(t)} = 1 - a, \quad \text{a.s.}$$

The main ingredient in the proof of Theorem 5.2 is the following tail estimate:

**Lemma 5.3.** *For  $a > 0$  and  $b > 0$ ,*

$$\lim_{t \rightarrow \infty} \frac{1}{\log_2 t} \log \mathbb{P} \left( \sup_{x \geq a\phi(t)} L(t, x) > b\phi(t) \right) = -(a + b)^2.$$

The proof of Theorem 5.2 relies on Lemma 5.3 and the Borel–Cantelli lemma, along the same lines as in Section 4. No new ingredient is needed. So we feel free to omit the details. The rest of the section is devoted to the proof of Lemma 5.3.

**Proof of Lemma 5.3.** Let  $T$  be the first hitting time process of the Brownian motion whose local time is  $L$ , see (3.2). By the strong Markov property, for any  $u > 0$  and  $v > 0$ ,

$$\begin{aligned} \mathbb{P}\left(\sup_{x \geq u} L(t, x) > v\right) &= \int_0^t \mathbb{P}(T_u \in ds) \mathbb{P}\left(\sup_{x \geq 0} L(t-s, x) > v\right) \\ &= \int_0^t \frac{u ds}{\sqrt{2\pi s^3}} \exp\left(-\frac{u^2}{2s}\right) \mathbb{P}\left(\sup_{x \geq 0} L(1, x) > \frac{v}{\sqrt{t-s}}\right). \end{aligned}$$

It is known (see Kesten 1965, Csáki 1989) that

$$\mathbb{P}\left(\sup_{x \geq 0} L(1, x) > \lambda\right) = \exp\left(-\left(1 + o(1)\right) \frac{\lambda^2}{2}\right), \quad \lambda \rightarrow \infty.$$

Therefore,

$$\begin{aligned} \mathbb{P}\left(\sup_{x \geq a\phi(t)} L(t, x) > b\phi(t)\right) \\ = \int_0^t \frac{a\phi(t) ds}{\sqrt{2\pi s^3}} \exp\left(-\frac{(a\phi(t))^2}{2s} - (1 + o(1)) \frac{(b\phi(t))^2}{2(t-s)}\right). \end{aligned}$$

This completes the proof of Lemma 5.3 by means of an elementary argument in the spirit of Laplace's method (cf., e.g., Widder 1946).  $\square$

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