Increment sizes of the principal value of Brownian local time

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Summary. Let W be a standard Brownian motion, and define $Y(t) = \int_0^t \mathrm{d}s/W(s)$ as Cauchy's principal value related to local time. We determine: (a) the modulus of continuity of Y in the sense of P. Lévy; (b) the large increments of Y.

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1. Introduction

Let $\{W(t);\, t \geqslant 0\}$ be a one-dimensional Brownian motion with $W(0) = 0$, and let $\{L(t,x);\, t \geqslant 0,\, x \in \mathbb{R}\}$ denote its local time process. That is, for any Borel function $f \geqslant 0$,

$$\int_0^t f(W(s))\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x)L(t,x)\,\mathrm{d}x, \qquad t \geqslant 0.$$

We are interested in the process

$$Y(t) = \int_0^t \frac{\mathrm{d}s}{W(s)}, \qquad t \geqslant 0. \tag{1.1}$$

Rigorously speaking, the integral $\int_0^t \mathrm{d}s/W(s)$ should be considered in the sense of Cauchy's principal value, i.e., $Y(t)$ is defined by

$$Y(t) = \lim_{\varepsilon \to 0+} \int_0^t \frac{\mathrm{d}s}{W(s)} \mathbb{1}_{\{|W(s)| \geqslant \varepsilon\}} = \int_0^\infty \frac{L(t,x) - L(t,-x)}{x}\,\mathrm{d}x. \tag{1.2}$$

Since $x \mapsto L(t,x)$ is Hölder continuous of order ν, for any $\nu < 1/2$, the integral on the extreme right in (1.2) is almost surely absolutely convergent.

The study of Cauchy's principal value of Brownian local time goes back at least to Itô and McKean [12], and has become very active since the late 70s, due to applications in various branches of stochastic analysis. For example, it is a natural example in Fukushima [10]'s theory for Dirichlet processes and zero-energy additive functionals (these processes cannot be treated in the frame of the usual Itô calculus techniques). Another important fact is that principal values of local times can be represented as the Hilbert transform, or more generally, fractional derivatives, of local times. The latter plays an important role in a class of limit theorems for occupation times of Brownian motion, discovered by Papanicolaou, Stroock and Varadhan [14]. Also, the principal values of Brownian local times are the key ingredient in establishing Bertoin [1]'s excursion theory for Bessel processes of small dimensions. For a detailed account of various motivations, historical facts and general properties of principal values of local times, we refer to the recent collection of research papers in Yor [18], to Chapter 10 of the lecture notes by Yor [19], and to the survey paper by Yamada [17].

The process $Y(\cdot)$ defined in (1.1)–(1.2) is continuous, having zero quadratic variation. Although it is not used in this paper, we mention an interesting property: stopped at some suitably chosen random times, the principal values give all the possible symmetric stable processes (cf. Biane and Yor [3], Fitzsimmons and Getoor [9], Bertoin [2]).

It is easily seen that $Y(\cdot)$ inherits a scaling property from Brownian motion, namely, for any fixed $a>0$, $t\mapsto a^{-1/2}Y(at)$ has the same law as $t\mapsto Y(t)$. Although the aforementioned zero quadratic variation property distinguishes $Y(\cdot)$ from Brownian motion (in particular, $Y(\cdot)$ is not a semimartingale), it is a kind of folklore that Y behaves somewhat like a Brownian motion. Let us first recall (cf. [11]) the global and local almost sure asymptotics of $Y(\cdot)$:

$$\limsup_{t\to\infty}\frac{Y(t)}{\sqrt{t\log\log t}}=\sqrt{8}\,,\qquad \text{a.s.} \tag{1.3}$$

$$\limsup_{t\to 0}\frac{Y(t)}{\sqrt{t\log\log(1/t)}}=\sqrt{8}\,,\qquad \text{a.s.} \tag{1.4}$$

Comparing (1.3)–(1.4) with the corresponding laws of the iterated logarithm (LIL's) for Brownian motion, we see that $\frac{1}{2}Y(t)$ and $W(t)$ satisfy exactly the same global and local LIL's.

The aim of this paper is to get a uniform version of (1.3)–(1.4) for the increments of $Y(t)$. Our first result characterizes its modulus of continuity in the sense of P. Lévy.

Theorem 1.1. *With probability one,*

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{|Y(t+s)-Y(t)|}{\sqrt{h\log(1/h)}}=2.$$

Remark 1.1.1. So $\frac{1}{\sqrt{2}}Y(t)$ and $W(t)$ have the same moduli of continuity (and the same remark applies to Theorem 1.2 below). We have already seen that $\frac{1}{2}Y(t)$ and $W(t)$ satisfy the same LIL's. Heuristically speaking, that a factor $\sqrt{2}$ is missing in the modulus of continuity comes from the fact that the Hausdorff dimension of the zero set of W is $\frac{1}{2}$.

Our second result concerns the large increments of $Y(\cdot)$. The length of time window, in which the increments are considered, denoted by a_T, will be supposed to satisfy the following condition:

$$
(1.5)\qquad \begin{cases} 0 < a_T \leqslant T, \\ T \mapsto a_T \text{ and } T \mapsto T/a_T \text{ are both non-decreasing,} \\ \lim_{T\to\infty} \dfrac{\log(T/a_T)}{\log\log T} = \infty. \end{cases}
$$

Here is our main result concerning the large increments of $Y(\cdot)$.

Theorem 1.2. *Under* (1.5),

$$
\lim_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \frac{|Y(t+s)-Y(t)|}{\sqrt{a_T\log(T/a_T)}} = 2, \qquad \text{a.s.}
$$

We note that in Csáki et al. [5], we have already proved the upper bounds in Theorems 1.1 and 1.2 with a different constant, and assuming only the first two conditions of (1.5). In particular, under these conditions, we established

$$
\limsup_{h\to 0} \sup_{0\leqslant t\leqslant 1-h} \sup_{0\leqslant s\leqslant h} \frac{|Y(t+s)-Y(t)|}{\sqrt{h\log(1/h)}} \leqslant 3\cdot 2^{7/6}, \qquad \text{a.s.}
$$

$$
\limsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \frac{|Y(t+s)-Y(t)|}{\sqrt{a_T\log(T/a_T)}} \leqslant 3\cdot 2^{7/6}, \qquad \text{a.s.}
$$

These bounds were proved using estimates for local times. The reason for which we are able to get lower bounds and the exact constants is that we shall be using different techniques, based on fine analysis of Bessel processes. Also, due to third condition of (1.5), presently we have as well $a_T/T \to 0$ as $T \to \infty$. Hence we could have stated Theorem 1.2 with $\sup_{0\leqslant t\leqslant T}$ instead of $\sup_{0\leqslant t\leqslant T-a_T}$ in its present form. The proof of the latter version of Theorem 1.2 would require only a few slight changes in its current proof. We prefer to keep the present form of Theorem 1.2, for it could be still true as an exact $\limsup_{T\to\infty}$ statement under only the first two conditions of (1.5).

The rest of the paper is organized as follows. The upper bounds in Theorems 1.1 and 1.2 are proved in Section 2, and the lower bounds in Section 3. The proof of the upper bounds is the harder part, requiring careful analysis on path decompositions and deep properties of three-dimensional Bessel processes. The proof of the lower bounds mainly consists in choosing some “nice” random stopping times.

Notation. Throughout the paper, the letter c with subscripts denotes some finite and positive universal constants. When the constants depend on a parameter, say p, they are denoted by $c(p)$ with subscripts.

2. Upper bounds

The upper bounds in Theorems 1.1 and 1.2 are based on some key probability estimates which are stated below as Lemmas 2.7 and 2.8. Let us first recall some known results. Recall that $W(\cdot)$ is a standard Brownian motion, and that $Y(\cdot)$ is defined in (1.1).

The first useful result concerns the distribution of $Y(s)$ for any fixed s. This was evaluated by Biane and Yor [3].

Fact 2.1. *The density function of $Y(s)$ is given by: for $s > 0$ and $x > 0$,*

$$\frac{\mathbb{P}(Y(s) \in \mathrm{d}x)}{\mathrm{d}x} = \sqrt{\frac{2}{\pi^3 s}} \sum_{k=0}^{\infty} (-1)^k \exp\left(- \frac{(2k+1)^2 x^2}{8s} \right).$$

Comment 2.1.1. From Fact 2.1 it follows that

$$\lim_{\lambda \to \infty} \frac{1}{\lambda^2} \log \mathbb{P}(Y(1) > \lambda) = -\frac{1}{8}. \tag{2.1}$$

Moreover, for $s > 0$ and $\lambda > 0$,

$$\mathbb{P}(|Y(s)| > \lambda \sqrt{s}) \leqslant c_1 \exp\left(-\frac{\lambda^2}{8} \right), \tag{2.2}$$

where c_1 is a universal constant (cf. [11]). □

The second theorem we shall make use of in the proof of the upper bounds is time-reversal for Bessel processes, cf. Exercise XI.1.23 in Revuz and Yor [16]. We recall that a three-dimensional Bessel process is the Euclidean modulus of an $\mathbb{R}^3$-valued Brownian motion.

Fact 2.2. *Let $T_0 = \inf\{t > 0 : W(t) = 0\}$, the first hitting time of W at 0. Then for any $x > 0$,*

$$\{W(t);\, 0 \leqslant t \leqslant T_0 \mid W(0) = x\} \overset{\text{law}}{=} \{R(\mathcal{L}_x - t);\, 0 \leqslant t \leqslant \mathcal{L}_x\}, \tag{2.3}$$

where $R(\cdot)$ is a three-dimensional Bessel process starting from 0, and $\mathcal{L}_x = \sup\{t > 0 : R(t) = x\}$, the last exit time from x.

Comment 2.2.1. Observe that (2.3) also guarantees the identity in law between T_0 (given $W(0) = x$) and $\mathcal{L}_x$. Thus

$$\mathbb{P}(u < \mathcal{L}_x \leqslant v) = \int_u^v \frac{x}{(2\pi y^3)^{1/2}} \exp\left(-\frac{x^2}{2y}\right) \mathrm{d}y \leqslant \frac{(v-u)x}{u^{3/2}}, \tag{2.4}$$

for any $v > u > 0$ and $x > 0$. □

The next identity in law, due to Pitman and Yor [15], relates a particular additive functional of the three-dimensional Bessel process and the range of one-dimensional Brownian motion. This will be frequently applied to our situation.

Fact 2.3. *Let $R(\cdot)$ be as before a three-dimensional Bessel process starting from 0. The following identity in law holds:*

$$\int_0^1 \frac{\mathrm{d}s}{R(s)} \stackrel{\text{law}}{=} \sup_{0\leqslant s\leqslant 1} W(s) - \inf_{0\leqslant s\leqslant 1} W(s).$$

Comment 2.3.1. It is an immediate consequence of Fact 2.3 and Feller's exact distribution function of the range of Brownian motion (cf. [8]) that

$$\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}s}{R(s)} > \lambda\right) \leqslant c_2 \exp\left(-\frac{\lambda^2}{2}\right), \qquad \lambda > 0, \tag{2.5}$$

with some absolute constant c_2. By applying the diffusion comparison theorem stated in Theorem XI.3.7 of Revuz and Yor [16] to squared Bessel processes, we deduce the intuitively clear fact that a three-dimensional Bessel process starting from $x > 0$ is stochastically greater than a three-dimensional Bessel process starting from 0. Consequently, for any $x > 0$ and $\lambda > 0$,

$$\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}s}{R(s)} > \lambda \,\Big|\, R(0) = x\right) \leqslant \mathbb{P}\left(\int_0^1 \frac{\mathrm{d}s}{R(s)} > \lambda\right).$$

By means of the Markov and scaling properties we arrive at:

$$\mathbb{P}\left(\int_u^v \frac{\mathrm{d}s}{R(s)} > \lambda\right) \leqslant c_2 \exp\left(-\frac{\lambda^2}{2(v-u)}\right), \tag{2.6}$$

for $v > u \geqslant 0$ and $\lambda > 0$. $\square$

We start the proof of the upper bounds in Theorems 1.1 and 1.2 with an elementary estimate.

Lemma 2.4. *Let M and N be independent random variables such that for all $x \geqslant 0$,*

$$\mathbb{P}(M > x) \leqslant \mu \exp(-\alpha x^2), \qquad \mathbb{P}(N > x) \leqslant \nu \exp(-\beta x^2), \tag{2.7}$$

for some positive constants μ, ν, α and β. Then for all $x \geqslant 0$,

$$\mathbb{P}(M + N > x) \leqslant c_3\,(1 + x^2) \exp\left(-\frac{\alpha\beta x^2}{\alpha + \beta}\right),$$

where $c_3 = c_3(\mu, \nu, \beta) = \mu + 2\nu + 2\beta\mu\nu$.

Remark. We note that the term in the exponential here is sharp.

Proof. We have

$$\mathbb{P}(M + N > x) \leqslant \mathbb{P}(M > x) + \mathbb{P}(N > x) + \mathbb{P}(N > x - M,\ 0 \leqslant M \leqslant x).$$

Observe that the last probability term on the right hand side is

$$\begin{aligned}
&\leqslant \mathbb{E}\left(\nu \mathrm{e}^{-\beta(x-M)^2}\, \mathbb{1}_{\{0 \leqslant M \leqslant x\}}\right) \\
&= -\nu \int_{y \in [0,x]} \mathrm{e}^{-\beta(x-y)^2}\, \mathrm{d}_y \mathbb{P}(M > y) \\
&\leqslant \nu \mathrm{e}^{-\beta x^2} + 2\beta\nu \int_{y \in [0,x]} (x - y)\mathrm{e}^{-\beta(x-y)^2} \mathbb{P}(M > y)\, \mathrm{d}y \\
&\leqslant \nu \mathrm{e}^{-\beta x^2} + 2\beta\mu\nu x \int_{y \in [0,x]} \mathrm{e}^{-\beta(x-y)^2 - \alpha y^2}\, \mathrm{d}y.
\end{aligned}$$

Since $\beta(x - y)^2 + \alpha y^2 \geqslant \alpha\beta x^2/(\alpha + \beta)$, this yields,

$$\mathbb{P}(M + N > x) \leqslant \mu \mathrm{e}^{-\alpha x^2} + 2\nu \mathrm{e}^{-\beta x^2} + 2\beta\mu\nu x^2 \exp\left(-\frac{\alpha\beta x^2}{\alpha + \beta}\right),$$

as desired. $\square$

It is intuitively clear that, if we want $Y(\cdot)$ to get extraordinarily large increments, the Brownian motion $W(\cdot)$ should be close to 0. However, due to the fact that $Y(\cdot)$ is defined only as a principal value, we have to treat it carefully. The next preliminary estimates (Lemmas 2.5 and 2.6) concern the two different situations: (i) $W(\cdot)$ is away from 0; and (ii) $W(\cdot)$ is close to 0. The tail probability of the increment of $Y(\cdot)$ in the second situation is greater, as expected.

Lemma 2.5. *For any positive δ, t, h, and any $\lambda \geqslant 1$ and $a \geqslant 1$,*

$$\begin{aligned}(2.8)\qquad &\mathbb{P}\left(|Y(t+h)-Y(t)| > \lambda h^{1/2},\ \inf_{s\in[t,t+h]}|W(s)|>0\right)\\ &\leqslant \frac{c_4(\delta)\, a^2\, h^{1/2}}{(t+h)^{1/2}} \exp\left(-\frac{\lambda^2}{6(1+\delta)}\right) + 4\exp\left(-\frac{(a-\lambda^{-1})^2}{2}\right).\end{aligned}$$

Proof. Let $t>0$, $h>0$ and $\lambda \geqslant 1$. Write $I_1 = I_1(t,h,\lambda)$ for the probability expression on the left hand side of (2.8). Then

$$\begin{aligned} I_1 &= \mathbb{P}\left(|Y(t+h)-Y(t)| > \lambda h^{1/2},\ \inf_{s\in[t,t+h]}|W(s)|>0\right)\\ &= \mathbb{P}\left(|Y(t/h+1)-Y(t/h)| > \lambda,\ \inf_{s\in[t/h,t/h+1]}|W(s)|>0\right).\end{aligned}$$

Define

$$f(\lambda,x) = \mathbb{P}\left(\left|\int_0^1 \frac{\mathrm{d}s}{W(s)}\right| > \lambda,\ T_0>1 \,\middle|\, W(0)=x\right).$$

It follows from symmetry and the Markov property that

$$(2.9)\qquad I_1 = 2\int_0^\infty \mathbb{P}(W(t/h)\in \mathrm{d}x)\, f(\lambda,x).$$

Let us now estimate $f(\lambda,x)$. According to (and in the notation of) Fact 2.2,

$$f(\lambda,x) = \mathbb{P}\left(\int_{\mathcal{L}_x-1}^{\mathcal{L}_x} \frac{\mathrm{d}s}{R(s)} > \lambda,\ \mathcal{L}_x>1\right).$$

Fix a $\theta\in(0,1]$. Then

$$\begin{aligned} f(\lambda,x) &= \sum_{k=1}^\infty \mathbb{P}\left(\int_{\mathcal{L}_x-1}^{\mathcal{L}_x}\frac{\mathrm{d}s}{R(s)} > \lambda,\ (k-1)\theta+1<\mathcal{L}_x\leqslant k\theta+1\right)\\ (2.10)\qquad &\leqslant \sum_{k=1}^\infty \mathbb{P}\left(\int_{(k-1)\theta}^{k\theta+1}\frac{\mathrm{d}s}{R(s)} > \lambda,\ (k-1)\theta+1<\mathcal{L}_x\leqslant k\theta+1\right).\end{aligned}$$

By (2.6),

$$\mathbb{P}\left(\int_{(k-1)\theta}^{k\theta+1} \frac{\mathrm{d}s}{R(s)} > \lambda\right) \leqslant c_2 \exp\left(-\frac{\lambda^2}{2(\theta+1)}\right),$$

whereas by (2.4),

$$\mathbb{P}\Big((k-1)\theta + 1 < \mathcal{L}_x \leqslant k\theta + 1 \Big) \leqslant \frac{\theta x}{(1+(k-1)\theta)^{3/2}} \leqslant \frac{x}{k^{3/2}\theta^{1/2}}.$$

Going back to (2.10) and using Hölder's inequality, we get that, for any $p > 1$ and $q > 1$ with $1/p + 1/q = 1$,

$$\begin{aligned} f(\lambda, x) &\leqslant \sum_{k=1}^{\infty} c_2^{1/p} \exp\left(-\frac{\lambda^2}{2(\theta+1)p}\right) \frac{x^{1/q}}{k^{3/(2q)}\theta^{1/(2q)}} \\ &= c_2^{1/p} \exp\left(-\frac{\lambda^2}{2(\theta+1)p}\right) \frac{x^{1/q}}{\theta^{1/(2q)}} \zeta\left(\frac{3}{2q}\right), \end{aligned}$$

where $\zeta(\cdot)$ denotes the Riemann zeta function.

Now let $a \geqslant 1$ and $\delta > 0$. We first treat the situation when $x \leqslant a$. Note that $\zeta(3/(2q)) < \infty$ for $q < 3/2$. We can choose $\theta = \theta(\delta) \in (0,1]$ and $p = p(\delta) > 3$ such that $q \in [1,\, 3/2)$ and that $(\theta+1)p = 3(1+\delta)$. Accordingly,

$$f(\lambda, x) \leqslant c_5(\delta)\, a \exp\left(-\frac{\lambda^2}{6(1+\delta)}\right), \qquad \text{for } x \leqslant a. \tag{2.11}$$

The other situation is $x > a$. By definition,

$$\begin{aligned} f(\lambda, x) &\leqslant \mathbb{P}\left(\inf_{s\in[0,1]} W(s) < \lambda^{-1} \,\Big|\, W(0) = x\right) \\ &\leqslant \mathbb{P}\left(\inf_{s\in[0,1]} (W(s) - W(0)) < -(a - \lambda^{-1})\right) \\ &\leqslant 2\exp\left(-\frac{(a-\lambda^{-1})^2}{2}\right). \end{aligned} \tag{2.12}$$

Plugging (2.11)–(2.12) into (2.9), we see that

$$\begin{aligned} I_1 &\leqslant 2c_5(\delta)\, a \exp\Big(-\frac{\lambda^2}{6(1+\delta)}\Big)\, \mathbb{P}\Big(0 < W(\frac{t}{h}) \leqslant a\Big) \\ &\quad + 4\exp\left(-\frac{(a-\lambda^{-1})^2}{2}\right) \\ &\leqslant \frac{2c_5(\delta)\, a^2\, h^{1/2}}{(t+h)^{1/2}} \exp\left(-\frac{\lambda^2}{6(1+\delta)}\right) + 4\exp\left(-\frac{(a-\lambda^{-1})^2}{2}\right), \end{aligned}$$

proving the lemma. $\square$

Lemma 2.6. *For any positive δ, t, h, λ, and any $b \geqslant 1$,*

$$\mathbb{P}\left(|Y(t+h)-Y(t)| > \lambda h^{1/2},\ \inf_{s\in[t,t+h]}|W(s)|=0\right) \leqslant c_6 \frac{bh^{1/2}}{(t+h)^{1/2}} \exp\left(-\frac{\lambda^2}{8(1+\delta)}\right) + 4\exp\left(-\frac{b^2}{2}\right). \tag{2.13}$$

Proof. Write $I_2 = I_2(t,h,\lambda)$ for the probability on the left hand side of (2.13). It is easy to estimate I_2 when $t=0$. Indeed, in this situation, we can use (2.2) to see that

$$I_2 = \mathbb{P}\left(|Y(h)| > \lambda h^{1/2},\ \inf_{s\in[0,h]}|W(s)|=0\right) \leqslant c_1 \exp\left(-\frac{\lambda^2}{8}\right), \qquad \text{when } t=0. \tag{2.14}$$

Now assume $t>0$. Note that

$$I_2 = 2\int_0^\infty \mathbb{P}(W(t/h)\in \mathrm{d}x)\, g(\lambda,x), \tag{2.15}$$

where

$$g(\lambda,x) = \mathbb{P}\left(\left|\int_0^1 \frac{\mathrm{d}s}{W(s)}\right| > \lambda,\ \inf_{s\in[0,1]}|W(s)|=0 \,\Big|\, W(0)=x\right).$$

Recall T_0 from Fact 2.2. Combining Fact 2.2 (in its notation) with the strong Markov and scaling properties, we obtain:

$$\begin{aligned} g(\lambda,x) &= \mathbb{P}\left(\left|\int_0^1 \frac{\mathrm{d}s}{W(s)}\right| > \lambda,\ T_0 \leqslant 1 \,\Big|\, W(0)=x\right) \\ &\leqslant \mathbb{P}\left(\mathcal{L}_x \leqslant 1,\ \int_0^{\mathcal{L}_x} \frac{\mathrm{d}s}{R(s)} + \sqrt{1-\mathcal{L}_x}\,|Y(1)| > \lambda\right), \end{aligned}$$

where R and $Y(1)$ are assumed to be independent. Now fix an integer $n \geqslant 1$. Clearly,

$$\begin{aligned} g(\lambda,x) &\leqslant \sum_{k=1}^n \mathbb{P}\left(\int_0^{k/n} \frac{\mathrm{d}s}{R(s)} + \sqrt{1-\frac{k-1}{n}}\,|Y(1)| > \lambda\right) \\ &= \sum_{k=1}^n \mathbb{P}\left(\sqrt{\frac{k}{n}}\int_0^1 \frac{\mathrm{d}s}{R(s)} + \sqrt{1-\frac{k-1}{n}}\,|Y(1)| > \lambda\right). \end{aligned}$$

We want to apply Lemma 2.4 to $M = \sqrt{k/n}\int_0^1 \mathrm{d}s/R(s)$, $N = \sqrt{1-(k-1)/n}\,|Y(1)|$. In view of (2.5) and (2.2), it is seen that (2.7) is satisfied with $\mu = c_2$, $\alpha = n/(2k)$, $\nu = c_1$ and $\beta = 1/8(1-(k-1)/n)$. It follows from Lemma 2.4 that

$$\begin{aligned} g(\lambda, x) &\leqslant c_7 \sum_{k=1}^{n} \frac{1+\lambda^2}{1-(k-1)/n} \exp\left(-\frac{\lambda^2}{8+(8-6k)/n}\right) \\ &\leqslant c_7\, n^2 (1+\lambda^2) \exp\left(-\frac{\lambda^2}{8(1+1/n)}\right). \end{aligned}$$

Let $\delta > 0$. We can choose $n = n(\delta)$ such that $n^{-1} \leqslant \delta/2$. Therefore,

$$\begin{aligned} g(\lambda, x) &\leqslant c_8(\delta)\,(1+\lambda^2) \exp\left(-\frac{\lambda^2}{8(1+\delta/2)}\right) \\ &\leqslant c_9(\delta)\, \exp\left(-\frac{\lambda^2}{8(1+\delta)}\right), \end{aligned} \tag{2.16}$$

the last inequality following from the fact that $(1+\lambda^2)\exp(\frac{\lambda^2}{8(1+\delta)} - \frac{\lambda^2}{8(1+\delta/2)})$ is uniformly bounded in $\lambda > 0$.

The estimate (2.16), which holds uniformly in x, is not accurate enough when x is large. Let $b \geqslant 1$. When $x \geqslant b$, we have

$$g(\lambda, x) \leqslant \mathbb{P}\left(\inf_{s\in[0,1]} (W(s) - W(0)) \leqslant -b\right) \leqslant 2\exp\left(-\frac{b^2}{2}\right). \tag{2.17}$$

Using (2.16) for $x \in [0, b]$ and (2.17) for $x \in (b, \infty)$, and in view of (2.15), we obtain that, for $t > 0$,

$$\begin{aligned} I_2 &\leqslant c_9(\delta)\, \exp\left(-\frac{\lambda^2}{8(1+\delta)}\right) \mathbb{P}(0 \leqslant W(t/h) \leqslant b) + 4\exp\left(-\frac{b^2}{2}\right) \\ &\leqslant c_9(\delta)\, \frac{h^{1/2} b}{(t+h)^{1/2}} \exp\left(-\frac{\lambda^2}{8(1+\delta)}\right) + 4\exp\left(-\frac{b^2}{2}\right). \end{aligned}$$

This, together with (2.14), completes the proof of Lemma 2.6. □

The next two probability estimates (Lemmas 2.7 and 2.8) are the main ingredient in the proof of the upper bounds in Theorems 1.1 and 1.2. More precisely, Lemma 2.7 is our key probability estimate, which will be reinforced later in Lemma 2.8 into a maximal inequality.

Lemma 2.7. *For all positive numbers δ, t and h, and all $\lambda \geqslant 1$,*

$$\begin{aligned}&\mathbb{P}\left(|Y(t+h)-Y(t)|>\lambda h^{1/2}\right)\\&\qquad\leqslant c_{10}(\delta)\left(\frac{h}{t+h}\right)^{1/2}\exp\left(-\frac{\lambda^2}{8(1+\delta)}\right)+8\,\exp\left(-\frac{\lambda^2}{2}\right).\end{aligned}$$

Proof. Take $\delta = 1/6$ and $a = \lambda + \lambda^{-1}$ in (2.8) to see that, for $\lambda \geqslant 1$,

$$\begin{aligned}&\mathbb{P}\left(|Y(t+h)-Y(t)|>\lambda h^{1/2},\ \inf_{s\in[t,t+h]}|W(s)|>0\right)\\&\qquad\leqslant \frac{c_{11}\,\lambda^2\,h^{1/2}}{(t+h)^{1/2}}\exp\left(-\frac{\lambda^2}{7}\right)+4\exp\left(-\frac{\lambda^2}{2}\right)\\&\qquad\leqslant \frac{c_{12}\,h^{1/2}}{(t+h)^{1/2}}\exp\left(-\frac{\lambda^2}{8}\right)+4\exp\left(-\frac{\lambda^2}{2}\right).\end{aligned}\tag{2.18}$$

On the other hand, replacing δ by 2δ and taking $b = \lambda$ in (2.13), we get that

$$\begin{aligned}&\mathbb{P}\left(|Y(t+h)-Y(t)|>\lambda h^{1/2},\ \inf_{s\in[t,t+h]}|W(s)|=0\right)\\&\qquad\leqslant \frac{c_6\,\lambda\,h^{1/2}}{(t+h)^{1/2}}\exp\left(-\frac{\lambda^2}{8(1+2\delta)}\right)+4\exp\left(-\frac{\lambda^2}{2}\right)\\&\qquad\leqslant \frac{c_{13}(\delta)\,h^{1/2}}{(t+h)^{1/2}}\exp\left(-\frac{\lambda^2}{8(1+\delta)}\right)+4\exp\left(-\frac{\lambda^2}{2}\right).\end{aligned}\tag{2.19}$$

Combining (2.18) with (2.19) yields the lemma, with $c_{10}(\delta) = c_{12} + c_{13}(\delta)$. □

Lemma 2.8. *For $\delta > 0$, $x > 0$ and $T > 0$, $h > 0$,*

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\ \sup_{0\leqslant s\leqslant h}|Y(t+s)-Y(t)|>x\sqrt{h}\right)\\&\qquad\leqslant c_{14}(\delta)\left(\sqrt{\frac{T+h}{h}}\exp\left(-\frac{x^2}{8(1+\delta)}\right)+\frac{T+h}{h}\exp\left(-\frac{x^2}{2(1+\delta)}\right)\right).\end{aligned}$$

Proof. Lemma 2.8 is a consequence of Lemma 2.7. The dyadic approximation argument we are using here is not new, and can be found for example in the proofs of Lemma 1.1.1 in Csörgő and Révész [7], and of Lemma 2.2 in Csáki et al. [4]. The main ideas go back to Lévy [13].

For positive real number s and integer n put $s_n = 2^{-n}\lfloor 2^n s\rfloor$. We have

$$
\begin{aligned}
|Y(t+s)-Y(t)| \leqslant |Y((t+s)_n)-Y(t_n)| \\
+\sum_{j=0}^{\infty}|Y((t+s)_{n+j+1})-Y((t+s)_{n+j})| \\
+\sum_{j=0}^{\infty}|Y(t_{n+j+1})-Y(t_{n+j})|.
\end{aligned}
\tag{2.20}
$$

Consider $t \in [0,T]$ and $s \in [0,h]$. Clearly $|(t+s)_n - t_n| \leqslant h + 2^{-n}$. Therefore, by Lemma 2.7, for any $\lambda \geqslant 1$,

$$
\begin{aligned}
&\mathbb{P}\left(|Y((t+s)_n)-Y(t_n)| > \lambda\sqrt{h+2^{-n}}\right) \\
&\leqslant c_{10}(\delta)\sqrt{\frac{h+2^{-n}}{(t+s)_n}}\exp\left(-\frac{\lambda^2}{8(1+\delta)}\right)+8\exp\left(-\frac{\lambda^2}{2}\right).
\end{aligned}
\tag{2.21}
$$

Note that there exists an i with $0 \leqslant i \leqslant (T+h)2^n$, such that $(t+s)_n = i2^{-n}$, i.e. $t+s \in [i2^{-n},(i+1)2^{-n})$. If $i=0$, then the probability on the left hand side of (2.21) is simply 0. For each i with $1 \leqslant i \leqslant (T+h)2^n$, there are at most $2^n h + 3$ different values of $(t)_n$ such that $(t+s)_n = i2^{-n}$. Therefore, by (2.21),

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\ \sup_{0\leqslant s\leqslant h}|Y((t+s)_n)-Y(t_n)| > \lambda\sqrt{h+2^{-n}}\right) \\
&\leqslant (2^n h+3)\sum_{1\leqslant i\leqslant (T+h)2^n}\left(\frac{c_{10}(\delta)\sqrt{h+2^{-n}}}{\sqrt{i2^{-n}}}\exp\left(-\frac{\lambda^2}{8(1+\delta)}\right)+8\exp\left(-\frac{\lambda^2}{2}\right)\right).
\end{aligned}
$$

Since $\sum_{1\leqslant i\leqslant a} i^{-1/2} \leqslant 2\sqrt{a}$ for any $a \geqslant 1$, we arrive at:

$$
\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\ \sup_{0\leqslant s\leqslant h}|Y((t+s)_n)-Y(t_n)| > \lambda\sqrt{h+2^{-n}}\right) \leqslant \phi_1(n),
\tag{2.22}
$$

where

$$
\begin{aligned}
\phi_1(n) = c_{15}(\delta)\sqrt{(T+h)2^n}\,(2^n h+3)^{3/2}\exp\left(-\frac{\lambda^2}{8(1+\delta)}\right) \\
+8(2^n h+3)(T+h)2^n\exp\left(-\frac{\lambda^2}{2}\right).
\end{aligned}
$$

Similarly, for any $u \in [0, T+h]$ and integer $j \geqslant 0$, since $|u_{n+j+1} - u_{n+j}| \leqslant 2^{-(n+j+1)}$, we have, by Lemma 2.7, for any $\lambda_j > 0$,

$$
\begin{aligned}
&\mathbb{P}\left(|Y(u_{n+j+1})-Y(u_{n+j})| > \lambda_j 2^{-(n+j+1)/2}\right) \\
&\leqslant c_{10}(\delta)\sqrt{\frac{2^{-(n+j+1)}}{u_{n+j+1}}}\exp\left(-\frac{\lambda_j^2}{8(1+\delta)}\right)+8\exp\left(-\frac{\lambda_j^2}{2}\right),
\end{aligned}
$$

which leads to:

$$\begin{aligned}(2.23)\qquad &\mathbb{P}\left(\sup_{0\leqslant u\leqslant T+h}|Y(u_{n+j+1})-Y(u_{n+j})|>\lambda_j 2^{-(n+j+1)/2}\right)\\ &\leqslant \sum_{1\leqslant i\leqslant (T+h)2^{n+j+1}}\left(\frac{c_{10}(\delta)}{\sqrt{i}}\exp\left(-\frac{\lambda_j^2}{8(1+\delta)}\right)+8\exp\left(-\frac{\lambda_j^2}{2}\right)\right)\\ &\leqslant \phi_2(n,j),\end{aligned}$$

with

$$\phi_2(n,j)=c_{16}(\delta)\sqrt{(T+h)2^{n+j+1}}\,\exp\left(-\frac{\lambda_j^2}{8(1+\delta)}\right)+8(T+h)2^{n+j+1}\,\exp\left(-\frac{\lambda_j^2}{2}\right).$$

Collect (2.20), (2.22) and (2.23) to see that,

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}|Y(t+s)-Y(t)|>\lambda\sqrt{h+2^{-n}}+2\sum_{j=0}^{\infty}\lambda_j 2^{-(n+j+1)/2}\right)\\ &\leqslant \phi_1(n)+2\sum_{j=0}^{\infty}\phi_2(n,j).\end{aligned}$$

Let $\mu=\mu(\delta)\in(0,1)$ be such that $\sqrt{1+2\mu}+c_{20}\sqrt{8\mu}\leqslant 1+\delta$, where c_{20} is the absolute constant defined in (2.24) below. Choose $\lambda_j=\sqrt{\lambda^2+4j}$, and let $n=n(h,\delta)$ be such that $2^{-n}\in[\,\mu h,\,2\mu h]$. Then

$$\phi_1(n)+2\sum_{j=0}^{\infty}\phi_2(n,j)\leqslant c_{17}(\delta)\left(\sqrt{\frac{T+h}{h}}\,\exp\left(-\frac{\lambda^2}{8(1+\delta)}\right)+\frac{T+h}{h}\exp\left(-\frac{\lambda^2}{2}\right)\right).$$

On the other hand, since $\lambda_j\leqslant\lambda+2j^{1/2}$ and $\lambda\geqslant 1$, we have,

$$(2.24)\qquad \sum_{j=0}^{\infty}\lambda_j 2^{-(j+1)/2}\leqslant\sum_{j=0}^{\infty}(\lambda+2j^{1/2})2^{-(j+1)/2}=c_{18}\,\lambda+c_{19}\leqslant c_{20}\,\lambda,$$

(with $c_{18}=\sum_{j=0}^{\infty}2^{-(j+1)/2}$, $c_{19}=2\sum_{j=0}^{\infty}j^{1/2}2^{-(j+1)/2}$ and $c_{20}=c_{18}+c_{19}$), which implies

$$\lambda\sqrt{h+2^{-n}}+2\sum_{j=0}^{\infty}\lambda_j 2^{-(n+j+1)/2}\leqslant\sqrt{1+2\mu}\,\lambda\sqrt{h}+c_{20}\sqrt{8\mu}\,\lambda\sqrt{h}.$$

Since $\sqrt{1+2\mu}+c_{20}\sqrt{8\mu}\leqslant 1+\delta$, we have proved that

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}|Y(t+s)-Y(t)|>(1+\delta)\lambda\sqrt{h}\right)\\ &\leqslant c_{17}(\delta)\left(\sqrt{\frac{T+h}{h}}\,\exp\left(-\frac{\lambda^2}{8(1+\delta)}\right)+\frac{T+h}{h}\exp\left(-\frac{\lambda^2}{2}\right)\right).\end{aligned}$$

This holds actually for all $\lambda > 0$ (when $\lambda \in (0,1)$, we only have to take an enlarged value of $c_{17}(\delta)$ if necessary). Taking $x = (1+\delta)\lambda$, and since $\delta > 0$ is arbitrary, this yields the lemma. $\square$

Proof of the upper bounds in Theorems 1.1 and 1.2. To check the upper bound in Theorem 1.1, we write

$$\Theta(h) = \sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h} |Y(t+s) - Y(t)|.$$

Let $\delta > 0$ and $h_k = h_k(\delta) = k^{-3/\delta}$. Applying Lemma 2.8 to $T = 1$, $h = h_k$ and $x = 2(1+\delta)\sqrt{\log(1/h_k)}$ gives that

$$\begin{aligned}
&\mathbb{P}\left(\Theta(h_k) > 2(1+\delta)\sqrt{h_k \log(1/h_k)}\right) \\
&\quad\leqslant 2c_{14}(\delta)\left(\frac{1}{\sqrt{h_k}}\exp\left(-\frac{1+\delta}{2}\log\frac{1}{h_k}\right) + \frac{1}{h_k}\exp\left(-2(1+\delta)\log\frac{1}{h_k}\right)\right) \\
&\quad= 2c_{14}(\delta)\,(h_k^{\delta/2} + h_k^{1+2\delta}),
\end{aligned}$$

which is summable in k. By the Borel–Cantelli lemma, almost surely for all large k, $\Theta(h_k) \leqslant 2(1+\delta)\sqrt{h_k\log(1/h_k)}$. Now let $h \in [h_{k+1},\, h_k]$. We have

$$\frac{\Theta(h)}{\sqrt{h\log(1/h)}} \leqslant \frac{\Theta(h_k)}{\sqrt{h_{k+1}\log(1/h_k)}} \leqslant 2(1+\delta)\frac{\sqrt{h_k}}{\sqrt{h_{k+1}}}.$$

Since $\sqrt{h_{k+1}}\,/\sqrt{h_k} \to 1$ (as $k \to \infty$), we obtain that

$$\limsup_{h\to 0}\frac{\Theta(h)}{\sqrt{h\log(1/h)}} \leqslant 2(1+\delta), \qquad \text{a.s.}$$

This yields the upper bound in Theorem 1.1, as δ can be as close to 0 as possible.

The upper bound in Theorem 1.2 is proved using exactly the same argument, considering $\widetilde{\Theta}(T) = \sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}|Y(t+s) - Y(t)|$ instead of $\Theta(h)$. $\square$

3. Lower bounds

As before, $W(\cdot)$ is a standard Brownian motion, and $Y(\cdot)$ denotes the principal value defined in (1.1)–(1.2). The proof of the lower bounds in Theorems 1.1 and 1.2 relies on the following estimate.

Lemma 3.1. *For $T \geqslant 2a > 0$, $\varepsilon \in (0,1)$, $\delta > 0$ and $\lambda > 0$,*

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant T-a}(Y(t+a)-Y(t)) \leqslant \lambda\sqrt{a}\right)$$
$$\leqslant 5\left(\frac{a}{T}\right)^{\varepsilon/2} + \exp\left(-c_{21}(\delta)\left(\frac{T}{a}\right)^{(1-\varepsilon)/2} \mathrm{e}^{-(1+\delta)\lambda^2/8}\right).$$

Proof. Let us construct an increasing sequence of stopping times $\{\eta_k = \eta_k(a)\}_{k\geqslant 0}$ by: $\eta_0 = 0$ and

$$\eta_{k+1} = \inf\{t > \eta_k + a : W(t) = 0\}, \qquad k \geqslant 0.$$

Let

$$\nu_T = \nu_T(a) = \max\{i \geqslant 0 : \eta_i \leqslant T - a\}.$$

Clearly,

$$\sup_{0\leqslant t\leqslant T-a}(Y(t+a)-Y(t)) \geqslant \max_{0\leqslant i\leqslant \nu_T}(Y(\eta_i+a)-Y(\eta_i)),$$

which yields

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant T-a}(Y(t+a)-Y(t)) \leqslant \lambda\sqrt{a}\right)$$
$$\leqslant \mathbb{P}\left(\max_{0\leqslant i\leqslant \nu_T}(Y(\eta_i+a)-Y(\eta_i)) \leqslant \lambda\sqrt{a}\right)$$
$$\leqslant \mathbb{P}\left(\nu_T < \left(\frac{T}{a}\right)^{(1-\varepsilon)/2}\right)$$
$$+\,\mathbb{P}\left(\max_{0\leqslant i\leqslant (T/a)^{(1-\varepsilon)/2}}(Y(\eta_i+a)-Y(\eta_i)) \leqslant \lambda\sqrt{a}\right). \tag{3.1}$$

It was shown in Csáki and Földes [6] that for $T \geqslant 2a$ and $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\nu_T < \left(\frac{T}{a}\right)^{(1-\varepsilon)/2}\right) \leqslant 3\sqrt{2}\left(\frac{a}{T}\right)^{\varepsilon/2}. \tag{3.2}$$

On the other hand, by the strong Markov property, $\{Y(\eta_i+a)-Y(\eta_i)\}_{i\geqslant 0}$ are iid variables, having the same law as $Y(a)$. Hence,

$$\mathbb{P}\left(\max_{0\leqslant i\leqslant (T/a)^{(1-\varepsilon)/2}}(Y(\eta_i+a)-Y(\eta_i)) \leqslant \lambda\sqrt{a}\right)$$
$$= \left(1 - \mathbb{P}(Y(a) > \lambda\sqrt{a})\right)^{\lfloor (T/a)^{(1-\varepsilon)/2}\rfloor}.$$

In view of (2.1), we have, for any $\delta > 0$ and $\lambda > 0$,

$$\mathbb{P}(Y(1) > \lambda) \geqslant c_{22}(\delta)\, \mathrm{e}^{-(1+\delta)\lambda^2/8},$$

which implies that

$$\begin{aligned} \mathbb{P}\left(\max_{0 \leqslant i \leqslant (T/a)^{(1-\varepsilon)/2}} (Y(\eta_i + a) - Y(\eta_i)) \leqslant \lambda\sqrt{a} \right) \\ &\leqslant \left(1 - c_{22}(\delta)\, \mathrm{e}^{-(1+\delta)\lambda^2/8}\right)^{\lfloor (T/a)^{(1-\varepsilon)/2} \rfloor} \\ &\leqslant \exp\left(- \lfloor (T/a)^{(1-\varepsilon)/2} \rfloor\, c_{22}(\delta)\, \mathrm{e}^{-(1+\delta)\lambda^2/8} \right), \end{aligned} \tag{3.3}$$

the last inequality following from the relation that $1 - x \leqslant \mathrm{e}^{-x}$ (for $x \geqslant 0$). Assembling (3.1)–(3.3) yields the lemma. □

Proof of the lower bounds in Theorems 1.1 and 1.2. Again, these bounds are proved using the same argument. For the upper bounds, we chose to prove Theorem 1.1 (cf. Section 2), so we give the proof of the lower bound in Theorem 1.2 here. The corresponding proof for Theorem 1.1 is similar and easier.

Let a_T be a function satisfying (1.5). Since $\frac{\log(T/a_T)}{\log\log T} \to \infty$, we have $T \geqslant 2a_T$ for all large T, say $T \geqslant n_0$. Consider

$$\Lambda(T) = \sup_{0 \leqslant t \leqslant T - a_T} (Y(t + a_T) - Y(t)), \qquad T > 0.$$

Let $\delta > 0$ and $\varepsilon \in (0, 1/2)$. Define $T_k = T_k(\delta) = (1+\delta)^k$. Apply Lemma 3.1 to $T = T_k$, $a = a_{T_k}$ and $\lambda = \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}} \sqrt{\log(T/a_{T_k})}$, to see that for $T \geqslant n_0$,

$$\begin{aligned} &\mathbb{P}\left(\Lambda(T_k) \leqslant \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}} \sqrt{a_{T_k} \log(T_k/a_{T_k})} \right) \\ &\leqslant 5 \left(\frac{a_{T_k}}{T_k} \right)^{\varepsilon/2} + \exp\left(-c_{21}(\delta) \left(\frac{T_k}{a_{T_k}} \right)^{(1-\varepsilon)/2} \exp\Big(- \frac{(1-2\varepsilon)\log(T_k/a_{T_k})}{2} \Big) \right) \\ &= 5 \left(\frac{a_{T_k}}{T_k} \right)^{\varepsilon/2} + \exp\left(-c_{21}(\delta) \left(\frac{T_k}{a_{T_k}} \right)^{\varepsilon/2} \right). \end{aligned}$$

Since $\frac{\log(T/a_T)}{\log\log T} \to \infty$, we have $T/a_T \geqslant (\log T)^{3/\varepsilon}$ for large T. Therefore, $\sum_k (a_{T_k}/T_k)^{\varepsilon/2} < \infty$ and $\sum_k \exp(-c_{21}(\delta)\,(T_k/a_{T_k})^{\varepsilon/2}) < \infty$. By the Borel–Cantelli lemma, almost surely for all large k,

$$\Lambda(T_k) > \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}} \sqrt{a_{T_k} \log(T_k/a_{T_k})}.$$

Let

$$\Lambda_0(T) = \sup_{0 \leqslant t \leqslant T - a_T} \sup_{0 \leqslant s \leqslant a_T} (Y(t+s) - Y(t)).$$

Clearly $T \mapsto \Lambda_0(T)$ is non-decreasing, such that $\Lambda_0(T) \geqslant \Lambda(T)$. Therefore, for $T \in [T_k, T_{k+1}]$,

$$\frac{\Lambda_0(T)}{\sqrt{a_T \log(T/a_T)}} \geqslant \frac{\Lambda_0(T_k)}{\sqrt{a_{T_{k+1}} \log(T_{k+1}/a_{T_k})}} > \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}} \frac{\sqrt{a_{T_k} \log(T_k/a_{T_k})}}{\sqrt{a_{T_{k+1}} \log(T_{k+1}/a_{T_k})}}.$$

Since $\log(T/a_T) \to \infty$ (cf. (1.5)),

$$\frac{\log(T_k/a_{T_k})}{\log(T_{k+1}/a_{T_k})} = \frac{\log(T_k/a_{T_k})}{\log(T_k/a_{T_k}) + \log(1+\delta)} \to 1, \qquad k \to \infty,$$

and since $T \mapsto T/a_T$ is non-decreasing,

$$\frac{a_{T_k}}{a_{T_{k+1}}} \geqslant \frac{T_k}{T_{k+1}} = \frac{1}{1+\delta},$$

we obtain:

$$\liminf_{T\to\infty} \frac{\Lambda_0(T)}{\sqrt{a_T \log(T/a_T)}} \geqslant \frac{2\sqrt{1-2\varepsilon}}{1+\delta}, \qquad \text{a.s.}$$

Sending ε and δ to 0 yields the lower bound in Theorem 1.2. □

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