Almost Sure Limit Theorems for Sums and Maxima from the Domain of Geometric Partial Attraction of Semistable Laws

István Berkes [1]

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences

P.O.B. 127, H–1364 Budapest, Hungary

Endre Csáki [2]

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences

P.O.B. 127, H–1364 Budapest, Hungary

Sándor Csörgő [3]

Bolyai Institute, University of Szeged

Aradi vértanúk tere 1, H-6720 Szeged, Hungary

and

Department of Statistics, University of Michigan

4062 Frieze Building, Ann Arbor, MI 48109–1285, U.S.A.

Zoltán Megyesi [4]

Bolyai Institute, University of Szeged

Aradi vértanúk tere 1, H-6720 Szeged, Hungary

Dedicated to Pál Révész for his sixty-fifth birthday

Abstract. The possible limiting distributions of sums of independent identically distributed random variables along subsequences $\{k_n\} \subset \mathbb{N}$ satisfying $k_{n+1}/k_n \to c \geq 1$ are the semistable laws and the domain of geometric partial attraction of a semistable law consists of distributions attracted to it along such a subsequence. The aim of this paper is to show that sums and maxima from the domain of geometric partial attraction of a semistable law satisfy almost sure limit theorems along the whole sequence $\{n\} = \mathbb{N}$ of natural numbers, despite the fact that ordinary convergence in distribution typically takes place in both cases only along $\{k_n\}$ and related subsequences. We describe the class of all possible almost sure asymptotic distributions both for sums and maxima.

AMS 1991 Subject Classification: Primary 60F15; Secondary 60E07.
Keywords: Semistable laws, domains of geometric partial attraction, sums and maxima, asymptotic distributions, logarithmic averages, almost sure limit theorems.

[1,2] Research supported by the Hungarian National Foundation for Scientific Research, Grants T 19346 and T 29621.

[3] Research supported in part by the NSF Grant DMS–9625732 held at the University of Michigan and by the Hungarian National Foundation for Scientific Research, Grants T 032025 and T 034121.

[4] Research partially supported by the 'Students for Science' program of the Pro Renovanda Cultura Hungariae Foundation and by the Hungarian National Foundation for Scientific Research, Grants T 032025 and T 034121.

1. Introduction and results: natural norming sequences

Let $X_1, X_2, \ldots$ be independent random variables with partial sums $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N} = \{1, 2, \ldots\}$. Introduce the symbol C_G for the continuity points of a function G. In the last decade, many authors investigated the connection between the weak limit theorem

$$\mathbb{P}\left\{ \frac{S_n - b_n}{a_n} \le x \right\} \to G(x), \qquad \text{for any } x \in C_G, \tag{A}$$

and, abbreviating 'almost surely' or 'almost sure' to 'a.s.', the corresponding pointwise result

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{S_r - b_r}{a_r} \le x \right\} \to G(x), \qquad \text{a.s. for any } x \in C_G, \tag{B}$$

where $a_n > 0$ and b_n are some numerical sequences and I denotes the indicator function; throughout, we use the convention that every asymptotic relation is meant as $n \to \infty$ unless otherwise specified.

Berkes and Dehling [6] proved the surprising result that under mild moment conditions on the X_n, relation (A) implies relation (B), despite the almost sure nature of the latter. If the random variables $X_1, X_2, \ldots$ are identically distributed, then the limiting distribution in (A) is necessarily stable with some exponent $\alpha \in (0, 2\,]$, and the theorem of Berkes and Dehling yields the well-known results of Brosamler [8], Schatte [19], Lacey and Philipp [14] and Fisher [12] for (B) when $\mathbb{E}(X_1^2) < \infty$, and even the rest of the case $\alpha = 2$, and that of Peligrad and Révész [18] in the case $0 < \alpha < 2$. For a recent survey of results in the area of "logarithmic" limit theorems of type (B), see Berkes [2].

In the converse direction, Berkes, Dehling and Móri [7] constructed examples for which (B) holds while (A) does not (and Móri [17] discovered interesting natural examples, different from sums, with the same feature). Berkes, Csáki and Csörgő [5] recently showed that this amusing phenomenon for sums of independent and identically distributed random variables obtains not just for esoteric examples, but also in a well-known classical situation: for the underlying distribution of the St. Petersburg game the almost sure statement in (B) holds with a suitable G, despite the fact that convergence in (A) does not take place along the whole $\{n\} = \mathbb{N}$, and the same is true for the maximal gains in a sequence of games. Our aim here is to extend these results and thus to show that the phenomenon holds not just for isolated examples, but in fact remains true for a whole class of distributions, namely, for those in the domain of geometric partial attraction of semistable laws. To state the results we need some notation and preliminary facts.

The concept of a 'semistable' distribution first appeared in 1937 in Paul Lévy's fundamental work [15]. We shall use recent results of Megyesi [16], where the existing theory of semistable laws is inserted into the framework of the 'probabilistic' approach of

Csörgő, Haeusler and Mason [10] and Csörgő [9]. For more results and further references see [16]. To start with standard notions and notation used in this approach, let Ψ be the class of all non-positive, non-decreasing, right-continuous functions $\psi(\cdot)$ defined on the positive half-line $(0,\infty)$ such that the integrals $\int_\varepsilon^\infty \psi^2(s)\,ds < \infty$ for all $\varepsilon > 0$. Let $E_1^{(j)}, E_2^{(j)}, \ldots$, $j = 1, 2$, be two independent sequences of independent exponentially distributed random variables with mean 1, and with respective partial sums $Y_n^{(j)} = E_1^{(j)} + \cdots + E_n^{(j)}$ as jump points, and consider the standard left-continuous independent Poisson processes $N_j(u) = \sum_{n=1}^\infty I\{Y_n^{(j)} < u\}$, $0 \le u < \infty$, $j = 1, 2$. For a function $\psi \in \Psi$, define the random variables

$$W_j(\psi) = \int_1^\infty \big[N_j(s) - s\big]\,d\psi(s) + \int_0^1 N_j(s)\,d\psi(s) - \psi(1), \quad j = 1, 2,$$

where the first integral is almost surely well-defined, by the condition that $\psi \in \Psi$, as an improper Riemann integral. Let Z be a standard normal random variable such that $N_1(\cdot)$, Z, and $N_2(\cdot)$ are independent, and for $\psi_1 \in \Psi$ and $\psi_2 \in \Psi$ and a finite constant $\sigma \ge 0$ finally introduce the random variables

$$V(\psi_1, \psi_2, \sigma) = -W_1(\psi_1) + \sigma Z + W_2(\psi_2). \tag{1.1}$$

Considering then the constant, also for $\psi \in \Psi$,

$$\Theta(\psi) = \int_0^1 \frac{\psi(s)}{1 + \psi^2(s)}\,ds - \int_1^\infty \frac{\psi^3(s)}{1 + \psi^2(s)}\,ds$$

and letting $W(\psi_1, \psi_2, \sigma) = V(\psi_1, \psi_2, \sigma) + \Theta(\psi_2) - \Theta(\psi_1)$, by Theorem 3 in [10] this random variable has characteristic function given for all $t \in \mathbb{R}$ as

$$\begin{aligned} \mathbb{E}\Big(e^{itW(\psi_1,\psi_2,\sigma)}\Big) = \exp\bigg\{ -\frac{\sigma^2}{2}t^2 &+ \int_{-\infty}^0 \Big(e^{itx} - 1 - \frac{itx}{1+x^2}\Big)\,dL(x) \\ &+ \int_0^\infty \Big(e^{itx} - 1 - \frac{itx}{1+x^2}\Big)\,dR(x)\bigg\}, \end{aligned} \tag{1.2}$$

where $L(x) = \inf\{s > 0 \colon \psi_1(s) \ge x\}$, $x < 0$, and $R(x) = -\inf\{s > 0 \colon \psi_2(s) \ge -x\}$, $x > 0$. Here $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and $R(\cdot)$ is right-continuous and non-decreasing on $(0,\infty)$ with $R(\infty) = 0$, and we have $\int_{-\varepsilon}^0 x^2 dL(x) + \int_0^\varepsilon x^2 dR(x) < \infty$ for every $\varepsilon > 0$ since $\psi_1, \psi_2 \in \Psi$. Thus $V(\psi_1, \psi_2, \sigma)$ is infinitely divisible by Lévy's formula; see [13] for example. Conversely, given the right side of (1.2) with $L(\cdot)$ and $R(\cdot)$ having the properties just listed, the variable $W(\psi_1, \psi_2, \sigma)$ has this characteristic function again if we choose $\psi_1(s) = \inf\{x < 0 \colon L(x) > s\}$ and $\psi_2(s) = \inf\{x < 0 \colon -R(-x) > s\}$, $s > 0$, for then $\psi_1, \psi_2 \in \Psi$.

Thus, modulo a location constant, the class $\mathcal{I}$ of all non-degenerate infinitely divisible distributions can be identified with the class $\{(\psi_1, \psi_2, \sigma) \neq (0,0,0) \colon \psi_1, \psi_2 \in \Psi, \sigma \geq 0\}$ of triplets. We say that F is in the domain of partial attraction of a $G = G_{\psi_1,\psi_2,\sigma} \in \mathcal{I}$ if there exists a subsequence $\{k_n\}_{n=0}^{\infty} \subset \mathbb{N}$ and centering and norming constants $B_{k_n} \in \mathbb{R}$ and $A_{k_n} > 0$ such that

$$\frac{1}{A_{k_n}} \left\{ \sum_{j=1}^{k_n} X_j - B_{k_n} \right\} \xrightarrow{\mathcal{D}} V(\psi_1, \psi_2, \sigma), \tag{1.3}$$

where $X_1, X_2, \ldots$ are independent identically distributed random variables with common distribution function $F(x) = \mathbb{P}(X_1 \leq x)$, $x \in \mathbb{R}$, the symbol $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and $G_{\psi_1,\psi_2,\sigma}$ is the distribution function of the random variable $V(\psi_1, \psi_2, \sigma)$. All subsequences of $\mathbb{N}$ appearing in this paper are assumed to be unbounded. If we demand $\{k_n\} = \{n\} = \mathbb{N}$ in (1.3) then the possible limiting distributions are the stable laws and F is said to be in the domain of attraction of the stable law in question, in which case either $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$ for some $\sigma > 0$, and F is in the domain of attraction of the normal distribution, written $F \in \mathbb{D}(2)$, or $(\psi_1, \psi_2, \sigma) = (m_1 \psi^\alpha, m_2 \psi^\alpha, 0)$ for some constants $\alpha \in (0,2)$, $m_1, m_2 \geq 0$, $m_1 + m_2 > 0$, where $\psi^\alpha(s) = -s^{-1/\alpha}$, $s > 0$, in which case F is in the domain of attraction of a stable distribution of exponent α, written $F \in \mathbb{D}(\alpha)$. (The superscript α in ψ^α here, and in ψ_1^α and ψ_2^α beginning with (1.5) below, is meant as a label, not as a power exponent.) The normal being the stable law of exponent 2, let $\mathcal{S}$ denote the class of all stable laws.

The class $\mathcal{S}_* \subset \mathcal{I}$ of semistable laws arises as the class of limiting distributions in (1.3) if we place a geometric growth condition on the subsequence $\{k_n\}$, i.e., that

$$k_{n+1}/k_n \to c \quad \text{for some } c \in [1, \infty). \tag{1.4}$$

By Theorem 1 of [16], $G_{\psi_1,\psi_2,\sigma} \in \mathcal{S}_*$ if and only if either $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$ for some $\sigma > 0$, giving the normal distribution as a semistable distribution of exponent 2, or $(\psi_1, \psi_2, \sigma) = (\psi_1^\alpha, \psi_2^\alpha, 0)$, where

$$\psi_j^\alpha(s) = M_j(s)\psi^\alpha(s) = -M_j(s)s^{-1/\alpha}, \quad s > 0, \quad j = 1, 2\,, \tag{1.5}$$

for some $\alpha \in (0,2)$, defining a semistable law of exponent $\alpha \in (0,2)$, where M_1 and M_2 are non-negative, right-continuous functions on $(0, \infty)$, either identically zero or bounded away from both zero and infinity, such that $M_1 + M_2$ is not identically zero, the functions $M_j(\cdot)\psi^\alpha(\cdot)$ are non-decreasing and, if $c > 1$ in (1.4), then $M_j(cs) = M_j(s)$ for all $s > 0$, $j = 1, 2$. This property will be referred to as multiplicative periodicity with period c. If $c = 1$ in (1.4), then the functions M_1 and M_2 are necessarily constant, giving the stable special case of semistable laws, therefore $\mathcal{S} \subset \mathcal{S}_*$. For $\alpha \in (0,2)$, Lévy's original description of property (1.5) in terms of L and R in (1.2) is that there exist non-negative bounded functions $M_L(\cdot)$ on $(-\infty,0)$ and $M_R(\cdot)$ on $(0,\infty)$, one of which has a strictly positive infimum and the other one either has a strictly positive infimum or is identically zero, such that $L(x) = M_L(x)/|x|^\alpha$, $x<0$, is left-continuous and non-decreasing on $(-\infty,0)$ and $R(x) = -M_R(x)/x^\alpha$, $x>0$, is right-continuous and non-decreasing on $(0,\infty)$, while $M_L(c^{1/\alpha}x) = M_L(x)$ for all $x<0$ and $M_R(c^{1/\alpha}x) = M_R(x)$ for all $x>0$, for the same period $c>1$. Because of the inversions given above, the two descriptions are equivalent.

Now, for $G = G_{\psi_1,\psi_2,\sigma} \in \mathcal{S}_*$, we say that F is in the *domain of geometric partial attraction of G with rank $c \geq 1$*, in short $F \in \mathbb{D}_{\rm gp}^{(c)}(G)$, if (1.3) holds along a subsequence $\{k_n\} \subset \mathbb{N}$ satisfying the growth condition (1.4) if $c>1$. Recalling that $(\psi_1,\psi_2,\sigma) \neq (0,0,0)$ for $G = G_{\psi_1,\psi_2,\sigma} \in \mathcal{S}_*$, define $\mathbf{c} = \mathbf{c}(G_{0,0,\sigma}) = 1$ for any $\sigma>0$ and $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) = \inf\{c>1: \ M_j(cs) = M_j(s),\ s>0,\ \ j=1,2\}$, the minimal common period c (among those greater than 1) of the factor functions M_1 and M_2 in ψ_1^α and ψ_2^α in (1.5) for $\alpha \in (0,2)$. It turns out for the whole domain $\mathbb{D}_{\rm gp}(G) = \bigcup_{c\geq 1} \mathbb{D}_{\rm gp}^{(c)}(G)$ of geometric partial attraction of $G \in \mathcal{S}_*$ that $\mathbb{D}_{\rm gp}(G) = \bigcap_{m\in\mathbb{N}} \mathbb{D}_{\rm gp}^{(\mathbf{c}^m)}(G) = \mathbb{D}_{\rm gp}^{(\mathbf{c})}(G)$. Also, if $\mathbf{c}(G) = 1$ for $G \in \mathcal{S}_*$, then $G \in \mathcal{S}$ and $\mathbb{D}_{\rm gp}(G) = \mathbb{D}(G)$, the domain of attraction of the stable G. In other words, if $\mathbb{D}(\mathcal{S}) = \bigcup_{G\in\mathcal{S}} \mathbb{D}(G) = \bigcup_{0<\alpha\leq 2} \mathbb{D}(\alpha)$ is the classical domain of attraction and $\mathbb{D}_{\rm gp}(\mathcal{G}) = \bigcup_{G\in\mathcal{G}} \mathbb{D}_{\rm gp}(G)$ is the domain of geometric partial attraction of a class $\mathcal{G} \subset \mathcal{S}_*$, then $\mathbb{D}_{\rm gp}(\mathcal{S}) = \mathbb{D}(\mathcal{S})$.

A characterization of an $F \in \mathbb{D}_{\rm gp}(\mathcal{S}_*)$ in terms of the quantile function

$$Q(s) = \inf\{x \in \mathbb{R}: \ F(x) \geq s\}, \quad 0<s<1,$$

is given by Theorem 3 in [16]. Consider a subsequence $\{k_n\}_{n=0}^\infty \subset \mathbb{N}$ satisfying (1.4). If $c=1$ in (1.4), then put $\gamma(s)=1$, $0<s<1$. If $c>1$, then the sequence $\{k_n\}$ is eventually strictly increasing, thus for all $s \in (0,s_0)$, with $s_0 \in (0,1]$ small enough, there exists a unique $k_{n^*(s)}$ such that $k_{n^*(s)}^{-1} \leq s < k_{n^*(s)-1}^{-1}$. We define $\gamma(s) = s k_{n^*(s)}$ for $s \in (0,s_0)$ and $\gamma(s)=1$ for $s \in [s_0,1)$, so that $1 \leq \gamma(s) < c+\varepsilon$ for any fixed $\varepsilon>0$ and all $s \in (0,1)$ for the limiting $c \geq 1$ from (1.4). Let $Q_+(\cdot)$ denote the right-continuous version of the quantile function $Q(\cdot)$. Since $\mathbb{D}_{\rm gp}(G) = \mathbb{D}(G)$ for a normal $G \in \mathcal{S}_*$, we only have to describe the domain of geometric partial attraction of non-normal semistable laws, for which the characterization is the following: If $G_{\psi_1^\alpha,\psi_2^\alpha,0} \in \mathcal{S}_*$ is semistable with exponent $\alpha \in (0,2)$, so that ψ_1^α and ψ_2^α satisfy (1.5), and $F \in \mathbb{D}_{\rm gp}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ so that (1.3) hold for $V(\psi_1^\alpha,\psi_2^\alpha,0)$ and a subsequence $\{k_n\}_{n=0}^\infty \subset \mathbb{N}$ satisfying (1.4), then for all

$s \in (0,1)$,

$$\begin{aligned} Q_+(s) &= -s^{-1/\alpha}\ell(s)\big[M_1(\gamma(s)) + h_1(s)\big] \\ \text{and} & \\ Q(1-s) &= s^{-1/\alpha}\ell(s)\big[M_2(\gamma(s)) + h_2(s)\big] \end{aligned} \tag{1.6}$$

for the same $\alpha \in (0,2)$, where $\ell(\cdot)$ is a right-continuous function, slowly varying at zero, the function $\gamma(\cdot)$ is determined by the subsequence $\{k_n\}$ along which (1.3) holds and the errors h_1 and h_2 are right-continuous functions such that if M_j is continuous, then $\lim_{s\downarrow 0} h_j(s) = 0$ for the corresponding h_j, while if M_j has discontinuities, then the corresponding $h_j(s)$ may not go to zero as $s \downarrow 0$ but $\lim_{n\to\infty} h_j(t/k_n) = 0$ for every continuity point $t > 0$ of M_j, $j = 1, 2$. Conversely, if for the quantile function pertaining to F the equations in (1.6) hold with the properties of ℓ and of h_1 and h_2 just listed, for some $\alpha \in (0,2)$ and functions M_1 and M_2 satisfying the properties described at (1.5), and for $\gamma(\cdot)$ determined by a given subsequence $\{k_n\}_{n=0}^{\infty} \subset \mathbb{N}$ satisfying (1.4), then $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for the ψ_1^α and ψ_2^α given by (1.5), and, in particular, the relation (1.3) can be specified as

$$\frac{1}{k_n^{1/\alpha}\ell(1/k_n)}\left\{\sum_{j=1}^{k_n} X_j - k_n \int_{\frac{1}{k_n}}^{1-\frac{1}{k_n}} Q(u)\,du\right\} \xrightarrow{\mathcal{D}} V(\psi_1^\alpha, \psi_2^\alpha, 0). \tag{1.7}$$

To state the main results from [11], which will form the basis of our further investigations, for $\lambda > 0$ introduce the functions

$$\psi_j^{\lambda,\alpha}(s) := \lambda^{-1/\alpha}\psi_j^\alpha(s/\lambda) = -M_j(s/\lambda)s^{-1/\alpha}, \qquad s \in (0,\infty),$$

$j = 1, 2$, and the distribution function

$$G_\lambda(x) := \mathbb{P}\big\{V(\psi_1^{\lambda,\alpha}, \psi_2^{\lambda,\alpha}, 0) \le x\big\}, \qquad x \in \mathbb{R}.$$

Clearly, $G_{\psi_1^\alpha,\psi_2^\alpha,0} = G_1 = G_{\mathbf{c}^n}$ for all $n \in \mathbb{N}$ and G_λ is semistable for all $\lambda > 0$. Let now $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ along a subsequence $\{k_n\}$ satisfying (1.4), in which there is no loss of generality to assume that $c > 1$. Introduce now

$$\gamma_n := \frac{1}{\gamma(1/n)} = \frac{n}{\min\{k_m : k_m \ge n\}}, \qquad n \in \mathbb{N}.$$

Again, it is easy to see that $\gamma_n \le 1$ and $\liminf_{n\to\infty} \gamma_n = 1/c$ for the c figuring in (1.4). Now the *Merge Theorem* is as follows:

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{\sum_{j=1}^n X_j - n\int_{\frac{1}{n}}^{1-\frac{1}{n}} Q(u)\,du}{n^{1/\alpha}\ell(1/n)} \le x\right\} - G_{\gamma_n}(x)\right| \to 0. \tag{1.8}$$

In fact, Theorem 2 in [11] states more: introducing $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$, the order statistics pertaining to $X_1, X_2, \ldots X_n$, it states that the suitably centered and normalized trimmed sums $\sum_{j=l}^{n-m} X_{j,n}$, where l and m are some fixed integers, merge together with 'trimmed variants' of G_{γ_n}. The above statement corresponds to the untrimmed special case, i.e., when $l = m = 0$. Our first theorem is now the following:

Theorem 1. *Assume that $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ along a subsequence $\{k_n\}$ satisfying (1.4) with some $c > 1$. Then*

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{\sum_{j=1}^{r} X_j - r\int_{\frac{1}{r}}^{1-\frac{1}{r}} Q(u)\, du}{r^{1/\alpha}\ell(1/r)} \le x \right\} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{G_\gamma(x)}{\gamma}\, d\gamma \tag{1.9}$$

almost surely for all $x \in \mathbb{R}$.

Turning now to maxima, for two distribution functions G and H introduce

$$\mathcal{L}(G, H) = \inf\{\varepsilon > 0 : G(x - \varepsilon) - \varepsilon \le H(x) \le G(x + \varepsilon) + \varepsilon \quad \text{for all} \quad x \in \mathbb{R}\},$$

their Lévy distance. It is well-known that this distance metrizes the weak convergence of distribution functions on the line (cf. p. 33 in [13]). Recalling our previous notation for order statistics and the ingredients of the Lévy measure of non-normal semistable laws, introduce

$$H_n(x) := \mathbb{P}\left\{ \frac{X_{n,n}}{n^{1/\alpha}\ell(1/n)} \le x \right\}, \qquad x \in \mathbb{R},$$

and for $\lambda > 0$,

$$K_\lambda(x) := \begin{cases} 0, & x \le 0, \\ e^{-M_R(\lambda^{1/\alpha} x) x^{-\alpha}}, & x > 0. \end{cases} \tag{1.10}$$

Here K_λ can be identified as the distribution function of $\left|\psi_2^{\lambda,\alpha}(Y)\right|$, where Y is an exponentially distributed random variable with mean 1. When considering maxima, we will assume that $M_2 \not\equiv 0$, and hence $M_R \not\equiv 0$. Then, retaining the conditions of Theorem 1, the *Merge Theorem for Maxima* (Theorem 3 in [11]) states that

$$\mathcal{L}(H_n, K_{\gamma_n}) \to 0. \tag{1.11}$$

The merge of distribution functions, unlike in the case of sums, does not generally take place uniformly, that is, it may not hold in the supremum distance. The main reason for this is that the distribution functions K_λ are not continuous if M_R is not. The almost sure limit theorem is nevertheless true:

Theorem 2. *Assume that* $F \in \mathbb{D}_{\rm gp}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ *along the subsequence* $\{k_n\}$ *satisfying (1.4) with some* $c > 1$, *where* $\psi_2^\alpha \not\equiv 0$. *Then*

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{X_{r,r}}{r^{1/\alpha}\ell(1/r)} \le x \right\} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{K_\gamma(x)}{\gamma}\, d\gamma \tag{1.12}$$

almost surely for all $x \in \mathbb{R}$.

We note that, even though $K_\gamma(\cdot)$ may not be continuous, the limiting distribution function in (1.12) is continuous at every $x \in \mathbb{R}$, as will be shown in the proof.

The proofs of these results are in the next section, after which Section 3 is devoted to the scaling problem: what happens if, instead of $r^{1/\alpha}\ell(1/r)$ in the two theorems above, one uses a general norming sequence?

2. Proof of Theorems 1 and 2

Lemma 1. *Let* $\{X_n\}$ *be a sequence of independent identically distributed random variables with partial sums* S_n *and assume that for some numerical sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$, *satisfying*

$$C_1 (l/k)^\lambda \le \alpha_l / \alpha_k \le C_2 (l/k)^\lambda \qquad \text{for any } l > k, \tag{2.1}$$

for some $\lambda \in [1/2, \infty)$, *where* C_1 *and* C_2 *are finite positive constants, the sequence*

$$\left\{ \frac{S_n - \beta_n}{\alpha_n} \right\}_{n=1}^{\infty} \tag{2.2}$$

is bounded in probability. Then

$$\sup_n \mathbb{E}\left(\left| \frac{S_n - \beta_n}{\alpha_n} \right|^p \right) < \infty$$

for any $0 < p < 1/\lambda$.

Proof. The proof of this lemma is implicit in the proof of Theorem 6.1 in de Acosta and Giné [1]. ∎

Proof of Theorem 1. Introduce $A_n := n^{1/\alpha}\ell(1/n)$ and $B_n := n\int_{\frac{1}{n}}^{1-\frac{1}{n}} Q(u)\, du$ for the normalizing and centering sequences in (1.8) and (1.9), respectively. First we prove that

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} \mathbb{P}\left\{ \frac{1}{A_r} \left[\sum_{j=1}^{r} X_j - B_r \right] \le x \right\} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{G_\gamma(x)}{\gamma}\, d\gamma, \tag{2.3}$$

for all $x \in \mathbb{R}$. By the Merge Theorem in (1.8) it suffices to show that

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{G_{\gamma_r}(x)}{r} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{G_\gamma(x)}{\gamma}\, d\gamma, \tag{2.4}$$

for all $x \in \mathbb{R}$. First we infer that (2.4) holds along $\{k_n\}$. Indeed,

$$\begin{aligned} \frac{1}{\log k_n} \sum_{r=1}^{k_n} \frac{G_{\gamma_r}(x)}{r} &= \frac{1}{\log k_n} \sum_{r=1}^{k_0} \frac{G_{\gamma_r}(x)}{r} + \frac{1}{\log k_n} \sum_{m=1}^{n} \sum_{r=k_{m-1}+1}^{k_m} \frac{G_{\gamma_r}(x)}{r} \\ &= o(1) + \frac{1}{\log k_n} \sum_{m=1}^{n} \sum_{r=k_{m-1}+1}^{k_m} \frac{G_{r/k_m}(x)}{r/k_m} \frac{1}{k_m} \\ &= o(1) + \frac{n \log c}{\log k_n} \frac{1}{\log c} \frac{1}{n} \sum_{m=1}^{n} \sum_{r=k_{m-1}+1}^{k_m} \frac{G_{r/k_m}(x)}{r/k_m} \frac{1}{k_m} \end{aligned}$$

Here the second equality follows by the fact that $\gamma_r = r/k_m$ for $r \in \{k_{m-1}+1, \ldots, k_m\}$. Introduce

$$\overline{G}_m(s,x) := \begin{cases} \frac{G_{j/k_m}(x)}{j/k_m}, & s \in \left(\frac{j}{k_m}, \frac{j+1}{k_m} \right], \; j = k_{m-1}+1, \ldots, k_m, \\ 0, & \text{elsewhere.} \end{cases}$$

If $\mu \uparrow \lambda \in (0, \infty)$, then, by the right-continuity of $\psi_1^\alpha(\cdot)$ and $\psi_2^\alpha(\cdot)$,

$$V(\psi_1^{\mu,\alpha}, \psi_2^{\mu,\alpha}, 0) \to V(\psi_1^{\lambda,\alpha}, \psi_2^{\lambda,\alpha}, 0),$$

even almost surely. Therefore, $G_\mu(x) \to G_\lambda(x)$ for all $x \in C_{G_\lambda}$. Since $G_\lambda(x)$ is continuous for all $\lambda > 0$ (see e. g. Lemma 2 in [11]), this means all $x \in \mathbb{R}$. Hence, for all $x \in \mathbb{R}$, $\overline{G}_n(s,x) \to G_s(x)/s$ for $s \in (1/c, 1]$ and $\overline{G}_n(s,x) \to 0$ for $s \in (0, \infty) \setminus [1/c, 1]$. Thus

$$\sum_{r=k_{m-1}+1}^{k_m} \frac{G_{r/k_m}(x)}{r/k_m} \frac{1}{k_m} = \int_0^\infty \overline{G}_m(s,x)\, ds \to \int_{\frac{1}{c}}^{1} \frac{G_\gamma(x)}{\gamma}\, d\gamma,$$

as $m \to \infty$, by Lebesgue's theorem and, since (1.4) is equivalent to $n \log c / \log k_n \to 1$, all this together yields (2.4) along $\{k_n\}$. But $\log k_{n+1} / \log k_n \to 1$ and the terms $G_{\gamma_n}(x)/\gamma_n$ are non-negative and bounded, thus (2.4) and (2.3) hold along $\{n\} = \mathbb{N}$, as well. It also follows that the limiting distribution function is continuous.

The equivalence of (2.3) and (1.9) will be shown by Theorem 1 of Berkes and Dehling [6]. Therefore, we must check the conditions of that theorem.

For the normalizing constants $A_n = n^{1/\alpha} \ell(1/n)$ in (1.9), by the slow variation of ℓ, there obviously exists a positive constant C for every $\delta \in (0, 1/\alpha)$ such that

$$A_l / A_k \geq C (l/k)^\delta, \qquad l > k,$$

fulfilling the condition (2.2) in [6]. Condition (2.1) of [6] follows by Lemma 1, since (2.2) above is obviously implied by Theorem 1(i) in [11], and (2.1) can be seen by the slow variation of ℓ again. ■

The special case when the subsequence $\{n_k\}$ is the whole sequence $\mathbb{N}$ in the following lemma is exactly the statement of Lemma 2 in [5]. While the proof of Theorem 2 requires only this special case, the stronger subsequential variant here is needed to prove Theorem 5 in the next section.

Lemma 2. *Let $\{X_n\}$ be a sequence of independent identically distributed random variables with $X_{n,n} = \max_{1\le i\le n} X_i$. Then for any subsequence $\{n_k\}\subset\mathbb{N}$ and (proper or improper) distribution function Λ and any real sequences $\{a_n\}$ and $\{b_n\}$ the relations*

$$\lim_{k\to\infty}\frac{1}{\log n_k}\sum_{r=1}^{n_k}\frac{1}{r}\,\mathbb{P}\left\{\frac{X_{r,r}-b_r}{a_r}\le x\right\}=\Lambda(x)\qquad \text{for any } x\in C_\Lambda \tag{2.5}$$

and

$$\lim_{k\to\infty}\frac{1}{\log n_k}\sum_{r=1}^{n_k}\frac{1}{r}\,I\left\{\frac{X_{r,r}-b_r}{a_r}\le x\right\}=\Lambda(x)\qquad \text{a.s. for any } x\in C_\Lambda \tag{2.6}$$

are equivalent.

Proof. Introduce $M_r^* := (X_{r,r}-b_r)/a_r$. It follows exactly the same way as in the proof of Theorem B in [4] (cf. also Lemma 2 in [5]) that for any bounded Lipschitz(1) function f on $\mathbb{R}$ we have

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}\big[f(M_r^*)-\mathbb{E}(f(M_r^*))\big]\to 0\quad\text{a.s.} \tag{2.7}$$

as $n\to\infty$.

We assume first (2.5) and show (2.6). Fix some $x\in C_\Lambda$ and choose a sequence $\varepsilon_m\to 0$, $\varepsilon_m>0$, such that $x\pm\varepsilon_m\in C_\Lambda$ for each $m\in\mathbb{N}$. Let the bounded Lipschitz(1) functions $f_m^x(t)$ and $g_m^x(t)$, $t\in\mathbb{R}$, be defined by

$$f_m^x(t) := \begin{cases}1, & \text{if } t\le x-\varepsilon_m,\\ 0, & \text{if } t\ge x,\\ \text{linear in between,} & \end{cases}$$

and

$$g_m^x(t) := \begin{cases}1, & \text{if } t\le x,\\ 0, & \text{if } t\ge x+\varepsilon_m,\\ \text{linear in between.} & \end{cases}$$

Clearly, $f_m^x(t)\le I(t\le x)\le g_m^x(t)$ for all $t\in\mathbb{R}$, and hence

$$\frac{1}{\log n_k}\sum_{r=1}^{n_k}\frac{f_m^x(M_r^*)-\mathbb{E}(g_m^x(M_r^*))}{r}\le\theta_k(x)\le\frac{1}{\log n_k}\sum_{r=1}^{n_k}\frac{g_m^x(M_r^*)-\mathbb{E}(f_m^x(M_r^*))}{r} \tag{2.8}$$

for each $k, m \in \mathbb{N}$, where

$$\theta_k(x) = \frac{1}{\log n_k} \sum_{r=1}^{n_k} \frac{I(M_r^* \le x) - \mathbb{P}\{M_r^* \le x\}}{r}.$$

Denoting by $\xi_k^m(x)$ the lower bound in (2.8), we have

$$\begin{aligned}
\limsup_{k\to\infty} \xi_k^m(x) &= \limsup_{k\to\infty} \frac{1}{\log n_k} \sum_{r=1}^{n_k} \frac{f_m^x(M_r^*) - \mathbb{E}(f_m^x(M_r^*))}{r} \\
&\quad + \limsup_{k\to\infty} \frac{1}{\log n_k} \sum_{r=1}^{n_k} \frac{\mathbb{E}(f_m^x(M_r^*)) - \mathbb{E}(g_m^x(M_r^*))}{r} \\
&\ge 0 + \limsup_{k\to\infty} \frac{1}{\log n_k} \sum_{r=1}^{n_k} \frac{\mathbb{P}\{M_r^* \le x - \varepsilon_m\} - \mathbb{P}\{M_r^* \le x + \varepsilon_m\}}{r} \\
&= \Lambda(x - \varepsilon_m) - \Lambda(x + \varepsilon_m),
\end{aligned}$$

which goes to 0 as $m \to \infty$. Here the inequality holds almost surely for each $m \in \mathbb{N}$ by virtue of (2.7), and hence it holds on a set of probability 1 for all $m \in \mathbb{N}$. A similar estimate for the right-hand side of (2.8) shows that $\theta_k(x) \to 0$ a.s. as $k \to 0$, that is, (2.6) follows from (2.5) as claimed. Assuming (2.6), the converse implication follows by taking expectation. ■

Proof of Theorem 2. First we show that

$$\vartheta_n(x) := \frac{1}{\log n} \sum_{r=1}^{n} \frac{H_r(x)}{r} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{K_\gamma(x)}{\gamma}\, d\gamma, \tag{2.9}$$

for all $x \in \mathbb{R}$. By the Merge Theorem in (1.11) and the definition of the Lévy distance, for every $x \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} K_{\gamma_r}(x - \varepsilon) - \varepsilon &\le \liminf_{n\to\infty} \vartheta_n(x) \\
&\le \limsup_{n\to\infty} \vartheta_n(x) \le \limsup_{n\to\infty} \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} K_{\gamma_r}(x + \varepsilon) + \varepsilon,
\end{aligned}$$

whence (2.9) follows if we show that the limiting distribution function on the right-hand side of (2.9) is continuous and that

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{K_{\gamma_r}(x)}{r} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{K_\gamma(x)}{\gamma}\, d\gamma$$

for all $x \in \mathbb{R}$. Although the distribution functions $K_\gamma(\cdot)$ are not continuous in general, the latter can be done along the same lines as in the proof of Theorem 1 above. Introduce $\overline{K}_m(s,x)$ similarly as $\overline{G}_m(s,x)$ in the proof of Theorem 1 but with K standing everywhere in place of G. Again, it is enough to infer that

$$\sum_{r=k_{m-1}+1}^{k_m} \frac{K_{r/k_m}(x)}{r/k_m} \frac{1}{k_m} = \int_0^\infty \overline{K}_m(s,x)\,ds \to \int_{\frac{1}{c}}^1 \frac{K_\gamma(x)}{\gamma}\,d\gamma \tag{2.10}$$

as $m \to \infty$. Using the definition of $K_\gamma(\cdot)$ in (1.10), it can be seen directly that $K_\mu(x) \to K_\lambda(x)$ for each fixed $x \in \mathbb{R}$ whenever $\mu \uparrow \lambda$ if λ is a continuity point of $K_\bullet(x)$, or, equivalently by (1.10), if $\lambda^{1/\alpha}x$ is a continuity point of M_R. Since $M_R(\cdot)$ can have only a countable number of discontinuities, $K_\mu(x) \to K_\lambda(x)$ holds for each $x \in \mathbb{R}$ as $\mu \uparrow \lambda$ for almost all $\lambda > 0$, that is, with an exceptional set of Lebesgue measure 0. Lebesgue's dominated convergence theorem yields the desired convergence in (2.10) and the continuity of the limiting distribution function in (1.10) and (2.9) may be seen similarly. Now (1.12) follows from (2.9) by an application of Lemma 2. ■

3. General norming sequences: mixtures

In this section we are concerned with determining all possible limits if in (1.9) or (1.12) of Theorem 1 or 2, respectively, the norming factor $r^{1/\alpha}\ell(1/r)$ is replaced by an arbitrary one, a_r, say. The motivation comes from [3], where it was shown that in the case when the random variables $X_1, X_2, \ldots$ have zero mean and finite variance then the class of all possible almost sure limiting distribution functions of the random functions

$$\frac{1}{\log n} \sum_{r=1}^n \frac{1}{r} I\left\{ \frac{\sum_{i=1}^r X_i}{a_r} \le x \right\}, \quad x \in \mathbb{R},$$

consists of all scale mixtures of the standard normal distribution function. Similar results were proved in [3], more generally, for the case when the underlying distribution is attracted to an arbitrary stable law.

Before stating the corresponding result for our semistable case here, we need to introduce some notation. Given $c > 1$, a non-negative function $q(\cdot)$ on the interval $[1/c, 1]$ will be called a step-function if it is piecewise constant, i.e. there exist $\gamma_0 = 1/c < \gamma_1 < \cdots < \gamma_{k-1} < \gamma_k = 1$ and non-negative numbers $q_1, \ldots, q_{k-1}, q_k$ such that $q(\gamma) = \sum_{i=1}^k q_i\, I\{\gamma \in [\gamma_{i-1}, \gamma_i)\}$ for some $k \in \mathbb{N}$. Let $\mathcal{A}$ be the set of such step-functions. For $q \in \mathcal{A}$ define

$$J(x) = J(q\,; x) = \frac{1}{\log c} \int_{\frac{1}{c}}^1 \frac{G_\gamma\left(q(\gamma)x\right)}{\gamma}\,d\gamma, \quad x \in \mathbb{R}.$$

Define the class of functions $\mathcal{J} := \{J(q\,;\cdot) : q \in \mathcal{A}\}$ and let $\mathcal{J}_*$ be the set of finite mixtures of $J \in \mathcal{J}$, i.e. the set of functions of the form

$$p_1 J_1(\cdot) + \cdots + p_N J_N(\cdot), \text{ where } p_1, \ldots, p_N \geq 0, \ \sum_{i=1}^{N} p_i = 1, \ J_1(\cdot), \ldots, J_N(\cdot) \in \mathcal{J}$$

for some $N \in \mathbb{N}$. Finally, let $\mathcal{J}^*$ be the weak closure of $\mathcal{J}_*$.

Now we state all of our scaling results, the proofs of which follow after the statement of Theorem 5.

Theorem 3. *Assume the condition of Theorem 1. Then there exists a positive numerical sequence* $\{a_n\}$ *such that*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{\sum_{j=1}^{r} X_j - r\int_{\frac{1}{r}}^{1-\frac{1}{r}} Q(u)\,du}{a_r} \leq x \right\} = J(x), \tag{3.1}$$

almost surely for all $x \in C_J$, *if and only if* $J \in \mathcal{J}^*$.

Turning to maxima again, construct the class $\mathcal{K}^*$ similarly to $\mathcal{J}^*$, starting from the limiting distributions in Theorem 2. Let $\mathcal{K}$ be the set of functions

$$K(x) = K(q\,;x) = \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{K_\gamma\left(q(\gamma)x\right)}{\gamma}\, d\gamma, \quad x \in \mathbb{R},$$

where $q \in \mathcal{A}$, let $\mathcal{K}_*$ be the set of finite convex mixtures of $K \in \mathcal{K}$ and, finally, let $\mathcal{K}^*$ be the weak closure of the class $\mathcal{K}_*$. The result for maxima is then the following.

Theorem 4. *Assume the conditions of Theorem 2. Then there exists a positive numerical sequence* $\{a_n\}$ *such that*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{X_{r,r}}{a_r} \leq x \right\} = K(x), \tag{3.2}$$

almost surely for all $x \in C_K$, *if and only if* $K \in \mathcal{K}^*$.

Theorem 3 in [3] says that the class of almost sure limiting distributions of logarithmic averages pertaining to sums of independent identically distributed random variables from the domain of attraction of a stable law cannot be enlarged by taking subsequential limits. A special case of Theorem 1 in [11] shows that independent variables from the domain of geometric partial attraction of a semistable distribution function G_1 also possess a certain closure or maximum property: the non-degenerate partial asymptotic laws of their sums along arbitrary subsequences all belong to the 'family' around G_1, i.e. they are of the form $\delta G_\lambda + d$, where $\delta, \lambda > 0$, $d \in \mathbb{R}$. Theorem 5 below, which is an analogue of Theorem 3 in [3], shows that in a sense the maximum property is preserved for almost sure distributional limits as well, and even for maxima.

Theorem 5. *Assume the conditions of Theorem 1 (or Theorem 2, respectively). If the logarithmic averages on the left-hand side of (3.1) (or of (3.2)) converge to some limit* $J(x)$ *(or* $K(x)$*) almost surely for each* $x \in C_J$ *(or* $x \in C_K$*) along some subsequence* $\{n_k\}_{k=1}^{\infty}$*, then necessarily* $J \in \mathcal{J}^*$ *(or* $K \in \mathcal{K}^*$*). Furthermore, there exists a universal positive norming sequence* $\{a_n\}_{n=1}^{\infty}$ *for which the totality of subsequential almost sure limits of the corresponding logarithmic averages is identical with the whole class* $\mathcal{J}^*$ *(or* $\mathcal{K}^*$*, respectively).*

It is natural to ask all sorts of questions concerning the limiting classes $\mathcal{J}^*$ and $\mathcal{K}^*$. Are there more compact descriptions of them? Are all the proper distribution functions in $\mathcal{J}^*$ infinitely divisible? Does it contain normal distribution functions? Does it even contain infinitely divisible distribution functions? All these questions are open. It is easy to see that $\mathcal{J}^*$ and $\mathcal{K}^*$ contain improper distribution functions, and we conjecture that $\mathcal{J}^*$ contains proper distribution functions that are not infinitely divisible and it probably does not contain normal distribution functions. However, we see that all distribution functions of the form

$$J(x) = \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{G_\gamma\left(q(\gamma)x\right)}{\gamma}\, d\gamma, \qquad x \in \mathbb{R},$$

are in $\mathcal{J}^*$ for any $q \in \mathcal{L}_1^+ = \mathcal{L}_1^+[1/c, 1]$, the class of non-negative integrable functions on $[1/c, 1]$. Hence, all distribution functions of the form

$$J(x) = \frac{1}{\log c} \int_{\mathcal{L}_1^+} \int_{\frac{1}{c}}^{1} \frac{G_\gamma\left(q(\gamma)x\right)}{\gamma}\, d\gamma\, \mu(dq), \qquad x \in \mathbb{R}, \tag{3.3}$$

are also in $\mathcal{J}^*$ for any probability measure μ on $\mathcal{L}_1^+$. Analogous statements hold for $\mathcal{K}^*$. But it is unclear whether the whole class $\mathcal{J}^*$ is contained if in (3.3) we allow arbitrary sub-probability measures μ on $\mathcal{L}_1^+$, and attach the weight $1-\mu(\mathcal{L}_1^+)$ to the point $x = 0$, or whether the class thus obtained is wider than $\mathcal{J}^*$.

Turning now to the proofs, we begin with a lemma that is needed for the proof of Theorems 3 and 5.

Lemma 3. *Let* $X_1, X_2, \ldots$ *be independent random variables,* $S_n = X_1 + \cdots + X_n$*, and let* $\{A_n\}$ *and* $\{B_n\}$ *be numerical sequences,* $\{A_n\}$ *being positive, such that*

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{S_r - B_r}{A_r} \le x \right\} \to G(x) \qquad \text{a.s. for all } x \in C_G, \tag{3.4}$$

where G *is a distribution function continuous at the origin. Assume that*

$$A_l / A_k \ge C_1 (l/k)^{\beta}, \qquad l > k, \tag{3.5}$$

and

$$\mathbb{E}\left(\left|\frac{S_n - B_n}{A_n}\right|^p\right) \le C_2, \qquad n = 1, 2, \ldots, \tag{3.6}$$

for some positive constants β, p, C_1 and C_2. Then for any positive norming sequence $\{a_n\}$ and any subsequence $\{n_k\} \subset \mathbb{N}$ and (proper or improper) distribution function H the relations

$$\frac{1}{\log n_k} \sum_{r=1}^{n_k} \frac{1}{r} I\left\{\frac{S_r - B_r}{a_r} \le x\right\} \to H(x) \qquad \text{a.s. for all } x \in C_H, \text{ as } k \to \infty, \tag{3.7}$$

and

$$\frac{1}{\log n_k} \sum_{r=1}^{n_k} \frac{1}{r} \mathbb{P}\left\{\frac{S_r - B_r}{a_r} \le x\right\} \to H(x) \qquad \text{for all } x \in C_H, \text{ as } k \to \infty, \tag{3.8}$$

are equivalent.

Note before the proof that I can of course be replaced by $\mathbb{P}$ in (3.4) by Theorem 1 of Berkes and Dehling [6] if $\{n_k\} = \mathbb{N}$, where the continuity assumption on G is not needed. The lemma shows that if G is continuous at the origin, then we can replace I by $\mathbb{P}$ even if the norming factor is changed arbitrarily, and this can be done even along subsequences. Note also that the slight continuity condition is satisfied in our applications since the limiting distribution function in (1.9), inheriting this property from the distribution functions $G_\gamma(\cdot)$, $1/c \le \gamma \le 1$, is everywhere continuous.

Proof. First we prove that if $\{a_n\}$ satisfies the additional condition

$$a_n \ge K A_n, \quad n = 1, 2, \ldots, \quad \text{for some positive constant } K \tag{3.9}$$

then for any bounded Lipschitz(1) function f on $\mathbb{R}$ we have

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} \left[f\left(\frac{S_r - B_r}{a_r}\right) - \mathbb{E}\left(f\left(\frac{S_r - B_r}{a_r}\right)\right)\right] \to 0 \quad \text{a.s. as } n \to \infty. \tag{3.10}$$

Set

$$\xi_r^{(f)} = f(\{S_r - B_r\}/a_r) - \mathbb{E}(f(\{S_r - B_r\}/a_r)).$$

By a standard argument in a.s. central limit theory, (3.10) will follow if we show that

$$\left|\mathbb{E}\left(\xi_k^{(f)}\xi_l^{(f)}\right)\right| \le C\left(\frac{k}{l}\right)^\rho, \quad k \le l,$$

for some constants $C > 0$ and $\rho > 0$. Choosing the constant K_1 so that $|f(x)| \le K_1$, $|f(x) - f(y)| \le K_1|x - y|$ hold for all x, y, we get for any $k \le l$, using (3.5), (3.6) (note that in (3.6) we can assume $p < 1$) and the independence of S_k and $S_l - S_k$,

$$\begin{aligned}\Big|\mathbb{E}\Big(\xi_k^{(f)}\xi_l^{(f)}\Big)\Big| &= \Big|\mathrm{Cov}\Big(f\Big(\frac{S_k-B_k}{a_k}\Big), f\Big(\frac{S_l-B_l}{a_l}\Big)\Big)\Big| \\ &= \Big|\mathrm{Cov}\Big(f\Big(\frac{S_k-B_k}{a_k}\Big), f\Big(\frac{S_l-B_l}{a_l}\Big) - f\Big(\frac{(S_l-S_k)-(B_l-B_k)}{a_l}\Big)\Big)\Big| \\ &\le 4K_1\,\mathbb{E}\Big(K_1\Big|\frac{S_k-B_k}{a_l}\Big| \wedge 2K_1\Big) \le 8K_1^2\,\mathbb{E}\Big(\Big|\frac{S_k-a_k}{KA_l}\Big|\wedge 1\Big) \\ &\le 8K_1^2\,\mathbb{E}\Big(\Big|\frac{S_k-B_k}{KA_l}\Big|^p\Big) \le 8K_1^2C_2K^{-p}\Big(\frac{A_k}{A_l}\Big)^p \le C\Big(\frac{k}{l}\Big)^{\beta p}.\end{aligned}$$

Dropping now condition (3.9), we first deal with the case $\{n_k\}=\mathbb{N}$; instead of n_k we simply write n. Let $\{a_n\}$ be an arbitrary positive numerical sequence and assume that (3.7) holds. Fix a continuity point $x>0$ of H (negative x's can be handled similarly). Let $\varepsilon>0$ and assume that $x\pm\varepsilon$ and $\pm\varepsilon x$ are also continuity points of H (this excludes only countably many ε's). Set

$$a_n^* := \begin{cases} a_n & \text{if} \quad a_n \ge \varepsilon A_n, \\ \varepsilon A_n & \text{if} \quad a_n < \varepsilon A_n, \end{cases}$$

and let $f_{\varepsilon,x}(t)$ denote the bounded Lipschitz(1) function which is 1 for $t\le x$, 0 for $t\ge x+\varepsilon$ and linear in $[x,x+\varepsilon]$. We consider the six expressions

$$\begin{aligned} &\frac{1}{\log n}\sum_{r=1}^n \frac{1}{r} I\Big\{\frac{S_r-B_r}{a_r}\le x\Big\}, && \frac{1}{\log n}\sum_{r=1}^n \frac{1}{r} I\Big\{\frac{S_r-B_r}{a_r^*}\le x\Big\}, \\ &\frac{1}{\log n}\sum_{r=1}^n \frac{1}{r} f_{\varepsilon,x}\Big(\frac{S_r-B_r}{a_r^*}\Big), && \frac{1}{\log n}\sum_{r=1}^n \frac{1}{r}\,\mathbb{E}\Big(f_{\varepsilon,x}\Big(\frac{S_r-B_r}{a_r^*}\Big)\Big), \\ &\frac{1}{\log n}\sum_{r=1}^n \frac{1}{r}\,\mathbb{P}\Big\{\frac{S_r-B_r}{a_r^*}\le x\Big\}, && \frac{1}{\log n}\sum_{r=1}^n \frac{1}{r}\,\mathbb{P}\Big\{\frac{S_r-B_r}{a_r}\le x\Big\}, \end{aligned} \tag{3.11}$$

and estimate the difference between the consecutive ones. Clearly, the difference

$$|I\{(S_r-B_r)/a_r\le x\} - I\{(S_r-B_r)/a_r^*\le x\}|$$

equals $I\{a_r x/A_r < (S_r-B_r)/A_r \le \varepsilon x\} \le I\{|(S_r-B_r)/A_r|\le \varepsilon x\}$ for $a_r<\varepsilon A_r$ and thus the difference of the first two expressions in (8) is at most

$$\frac{1}{\log n}\sum_{r=1}^n \frac{1}{r} I\Big\{\Big|\frac{S_r-B_r}{A_r}\Big|\le \varepsilon x\Big\}. \tag{3.12}$$

By (3.4), and since $\pm\varepsilon x$ are continuity points of G, the expression in (3.12) converges a.s. to $\psi(\varepsilon) := G(\varepsilon x)-G(-\varepsilon x)$. Since G is continuous at the origin, $\psi(\varepsilon)\to 0$ if

$\varepsilon \to 0$. Thus the limsup of the difference of the first two terms in (3.11) is at most $\psi(\varepsilon)$. Replacing I by $\mathbb{P}$, the same estimate applies for the difference of the last two expressions of (3.11). Next we observe that

$$I_{(-\infty,x]} \le f_{\varepsilon,x} \le I_{(-\infty,x+\varepsilon]} \tag{3.13}$$

and thus the difference of the second and third expressions in (3.11) is bounded by

$$\left| \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{S_r - B_r}{a_r^*} \le x \right\} - \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{S_r - B_r}{a_r^*} \le x + \varepsilon \right\} \right|. \tag{3.14}$$

Relation (3.7) and the estimate for the difference of the first two expressions in (3.11) imply that for almost every outcome of the sample space and for all sufficiently large n the first and second terms in (3.11) are within $2\psi(\varepsilon)$ of $H(x)$ and $H(x+\varepsilon)$, respectively, and thus the difference in (3.14) is $\le |H(x+\varepsilon)-H(x)|+4\psi(\varepsilon) =: \psi_1(\varepsilon)$. Since $a_n^* \ge \varepsilon A_n$, the statement proved at the beginning of the proof shows that the difference between the third and fourth term in (3.11) tends to 0 a.s. as $n \to \infty$. Finally, the difference of the fourth and fifth expression in (3.11) can be estimated similarly as the difference of the second and the third, only instead of (3.13) we use the inequality $f_{\varepsilon,x-\varepsilon} \le I_{(-\infty,x]} \le f_{\varepsilon,x}$.

Hence we proved that for sufficiently large n the difference of the first and last expressions in (3.11) is $\le \psi_2(\varepsilon)$ where $\psi_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus (3.7) implies (3.8), at least if $\{n_k\} = \mathbb{N}$. But if $\{n_k\} \ne \mathbb{N}$, then we only have to write n_k in place of n everywhere in (3.11) and apply the same estimates as above, and the implication (3.10) $\Rightarrow$ (3.11) follows once again. The converse statement can be proved similarly. ■

Proof of Theorem 3. By Lemma 3 it suffices to prove that, under the condition of Theorem 1, there exists a positive numerical sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} \mathbb{P}\left\{ \frac{\sum_{j=1}^{r} X_j - r\int_{\frac{1}{r}}^{1-\frac{1}{r}} Q(u)\,du}{a_r} \le x \right\} = J(x), \tag{3.15}$$

for all $x \in C_J$, if and only if $J \in \mathcal{J}^*$.

First assume that (3.15) holds for some $\{a_n\}$ and J. We show that $J \in \mathcal{J}^*$. As in the proof of Theorem 1, let $S_n = X_1 + \cdots + X_n$, $A_n = n^{1/\alpha}\ell(1/n)$, $B_n = n\int_{\frac{1}{n}}^{1-\frac{1}{n}} Q(u)\,du$ and put $q_n = a_n/A_n$. Let $\{k_n\}$ be a sequence satisfying (1.4). Then by the Merge

Theorem in (1.8),

$$
\begin{aligned}
\sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r}\,\mathbb{P}\Big\{\frac{S_r-B_r}{a_r}\le x\Big\} &= \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r}\,\mathbb{P}\Big\{\frac{S_r-B_r}{A_r}\le \frac{xa_r}{A_r}\Big\}\\
&= \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r}\,G_{\gamma_r}\Big(\frac{xa_r}{A_r}\Big)+o(1)\\
&= \sum_{r=k_{m-1}+1}^{k_m} \frac{G_{r/k_m}(q_r x)}{r}+o(1)\\
&= \int_{k_{m-1}+1}^{k_m} \frac{G_{\lfloor t\rfloor/k_m}(q_{\lfloor t\rfloor}x)}{\lfloor t\rfloor}\,dt+o(1)\\
&= \int_{k_{m-1}+1}^{k_m} \frac{G_{t/k_m}(q_{\lfloor t\rfloor}x)}{t}\,dt+o(1)\\
&= \int_{\frac{1}{c}}^{1} \frac{G_{\gamma}(q_m(\gamma)x)}{\gamma}\,d\gamma+o(1)\,,
\end{aligned}
\tag{3.16}
$$

where $q_m(\gamma)=q_{\lfloor k_m\gamma\rfloor}$, with $\lfloor t\rfloor$ standing for the integer part of t, and hence $q_m\in\mathcal{A}$. Now put

$$
J_m(x)=\frac{1}{\log c}\int_{\frac{1}{c}}^{1}\frac{G_\gamma(q_m(\gamma)x)}{\gamma}\,d\gamma.
$$

Then by (3.15), using also that $\log k_n\sim n\log c$, we have

$$
\begin{aligned}
J(x) &= \lim_{N\to\infty}\frac{1}{\log k_N}\sum_{m=1}^{N}\sum_{r=k_{m-1}+1}^{k_m}\frac{1}{r}\,\mathbb{P}\Big\{\frac{S_r-B_r}{a_r}\le x\Big\}\\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N} J_m(x),
\end{aligned}
$$

i.e. J is the limit of convex linear combinations of $J_m\in\mathcal{J}$, whence $J\in\mathcal{J}^*$.

We prove the converse statement in several steps.

First assume that $J=J(q\,;x)\in\mathcal{J}$ for some $q\in\mathcal{A}$. Let $\{k_m\}$ satisfy (1.4). We can find a sequence $\{q_m(\gamma)\}$ of step-functions such that the break-points of $q_m(\gamma)$ are of the form ν/k_m with integer ν, $q_m(\gamma)>0$ and $q_m(\gamma)\to q(\gamma)$ weakly (pointwise at each continuity point of q). If $(k_{m-1}+1)/k_m<c^{-1}$, then we let $q_m(\gamma)=q_m(1/c)$ for $\gamma\in((k_{m-1}+1)/k_m,1/c)$. Put

$$
a_r=A_r q_m\Big(\frac{r}{k_m}\Big),\quad k_{m-1}+1\le r<k_m\,,\quad m=1,2,\ldots\,.
$$

Then reading (3.16) backwards, we get

$$\int_{\frac{1}{c}}^{1} \frac{G_\gamma(q_m(\gamma)x)}{\gamma}\, d\gamma = \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r}\, \mathbb{P}\left\{ \frac{S_r - B_r}{a_r} \le x \right\} + o(1),$$

hence (3.15) holds along the subsequence $\{k_m\}$. This implies, as in the proof of Theorem 1, that (3.15) holds along $\{n\} = \mathbb{N}$ as well.

Now let $J(\cdot)$ be of the form

$$J(\cdot) = p_1 J_1(\cdot) + \cdots + p_N J_N(\cdot)$$

with $J_i(\cdot) \in \mathcal{J}$ and let $p_i > 0$, $i = 1, \ldots, N$, be rational numbers such that $\sum_{i=1}^N p_i = 1$. By the previous arguments, we can find $a_r^{(i)}$ such that

$$\frac{1}{\log c} \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r}\, \mathbb{P}\left\{ \frac{S_r - B_r}{a_r^{(i)}} \le x \right\} = J_i(x) + o(1), \quad x \in C_{J_i}.$$

In particular, suppose that $p_i = \beta_i/\beta$ for some $\beta_i \in \mathbb{N}$, $i = 1, \ldots, N$, with $\sum_{i=1}^N \beta_i = \beta$. Define $a_r = a_r^{(i)}$ for all r and m satisfying $k_{m-1}+1 \le r \le k_m$ and $m \equiv j \pmod{\beta}$, where $j \in \mathbb{N}$ is such that $\sum_{l=1}^{i-1} \beta_l \le j \le \sum_{l=1}^{i} \beta_l - 1$, $i = 1, \ldots, N$. Then, clearly,

$$\lim_{n\to\infty} \frac{1}{\beta n \log c} \sum_{m=1}^{\beta n} \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r}\, \mathbb{P}\left\{ \frac{S_r - B_r}{a_r} \le x \right\} = p_1 J_1(x) + \cdots + p_N J_N(x) = J(x)$$

for all $x \in C_J$, and so (3.15) holds along the subsequence $\{k_{\beta n}\}$. Thus it also holds along the whole sequence $\{n\} = \mathbb{N}$.

Finally, since the functions $J \in \mathcal{J}_*$ with rational coefficients are weakly dense in $\mathcal{J}^*$, we can find a sequence $\{a_n\}$ such that (3.15) holds. For details on this point, see [3]. This completes the proof of Theorem 3. ■

Proof of Theorem 4. We proceed similarly as above. Referring to Lemma 2, it is sufficient to show that under the stated conditions there exists a numerical sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r}\, \mathbb{P}\left\{ \frac{X_{r,r}}{a_r} \le x \right\} = K(x) \tag{3.17}$$

for all $x \in C_K$, if and only if $K \in \mathcal{K}^*$. We first assume that (3.17) holds for some K and show that $K \in \mathcal{K}^*$. Using the Merge Theorem for Maxima at (1.11) we obtain similarly as in (3.16) that for any $\varepsilon > 0$ there exists a threshold number $M = M(\varepsilon)$ and step-functions q_m such that for $m \geq M$,

$$\int_{\frac{1}{c}}^{1} \frac{K_\gamma(q_m(\gamma)(x-\varepsilon))}{\gamma}\, d\gamma - \varepsilon \leq \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r}\, \mathbb{P}\left\{ \frac{X_{r,r}}{a_r} \leq x \right\} \leq \int_{\frac{1}{c}}^{1} \frac{K_\gamma(q_m(\gamma)(x+\varepsilon))}{\gamma}\, d\gamma + \varepsilon. \tag{3.18}$$

Introduce

$$K_m(x) = \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{K_\gamma(q_m(\gamma)x)}{\gamma}\, d\gamma, \quad x \in \mathbb{R}.$$

By definition, $K_m \in \mathcal{K}$. Using (3.18),

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} K_m(x-\varepsilon) - \varepsilon \leq K(x) \leq \liminf_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} K_m(x+\varepsilon) + \varepsilon,$$

which is to say that $\mathcal{L}\Big(\frac{1}{N}\sum_{m=1}^{N} K_m\, , K\Big) \to 0$ as $N \to \infty$. Thus $K \in \mathcal{K}^*$, as stated.

The converse statement follows by a straightforward modification of the construction in the proof of the preceding theorem. ■

Proof of Theorem 5. The statements concerning the subsequential maximum property of the classes $\mathcal{J}^*$ and $\mathcal{K}^*$ are implicit in the proofs of Theorems 3 and 4, respectively: use the literally subsequential original forms of Lemmas 3 and 2, and then see (3.16) or (3.18), respectively. Now, let $\mathcal{J}_\dagger$ consist of distribution functions of the form

$$J(x) = p_1 J(q_1\, ; x) + \cdots + p_N J(q_N\, ; x), \quad x \in \mathbb{R},$$

where $N \in \mathbb{N}$, $p_i > 0$, $p_i \in \mathbb{Q}$, $i = 1, \ldots, N$, $\sum_{i=1}^{N} p_i = 1$, and

$$J(q_i\, ; x) = \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{G_\gamma(q_i(\gamma)x)}{\gamma}\, d\gamma,$$

where $q_i \in \mathcal{A}$ is a non-negative rational-valued step-function with rational jump-points, $i = 1, \ldots, N$. Thus $\mathcal{J}_\dagger$ is a "rational restriction" of $\mathcal{J}_*$. Clearly, $\mathcal{J}_\dagger$ is countable and its weak closure is $\mathcal{J}^*$. Hence it suffices to construct a norming sequence $\{a_n\}$ for which the set of subsequential limits of the averages

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r}\, \mathbb{P}\left\{ \frac{\sum_{j=1}^{r} X_j - r\int_{\frac{1}{r}}^{1-\frac{1}{r}} Q(u)\, du}{a_r} \leq x \right\}$$

contains $\mathcal{J}_\dagger$. The construction of such a sequence $\{a_n\}$ can be done by a straightforward modification of the corresponding part in the proof of Theorem 3 in [3]. This completes the proof for sample sums. The case of sample maxima is entirely similar. ■

References

[1] A. de Acosta and E. Giné: Convergence of moments and related functionals in the general central limit theorem in Banach spaces, *Z. Wahrscheinlichkeitstheorie verw. Geb.* **48** (1979), 213–231.

[2] I. Berkes: Results and problems related to the pointwise central limit theorem, In: *Asymptotic Methods in Probability and Statistics* (B. Szyszkowicz, ed.), pp. 59–96, Elsevier, Amsterdam, 1998.

[3] I. Berkes and E. Csáki: On the pointwise central limit theorem and mixtures of stable distributions, *Statist. Probab. Letters* **29** (1996), 361–368.

[4] I. Berkes and E. Csáki: A universal result in almost sure central limit theory, *Stochastic Process. Appl.*, to appear.

[5] I. Berkes, E. Csáki and S. Csörgő: Almost sure limit theorems for the St. Petersburg game, *Statist. Probab. Letters* **45** (1999), 23–30.

[6] I. Berkes and H. Dehling: Some limit theorems in log density, *Ann. Probab.* **21** (1993), 1640–1670.

[7] I. Berkes, H. Dehling and T. F. Móri: Counterexamples related to the a.s. central limit theorem, *Studia Sci. Math. Hungar.* **26** (1991), 153–164.

[8] G. Brosamler, An almost everywhere central limit theorem, *Math. Proc. Cambridge Phil. Soc.* **104** (1988), 561–574.

[9] S. Csörgő: A probabilistic approach to domains of partial attraction, *Adv. in Appl. Math.* **11** (1990), 282–327.

[10] S. Csörgő, E. Haeusler and D. M. Mason: A probabilistic approach to the asymptotic distribution of sums of independent, identically distributed random variables, *Adv. in Appl. Math.* **9** (1988), 259–333.

[11] S. Csörgő and Z. Megyesi: Merging to semistable laws, *Theory of Probability and its Applications* **46** (2001), to appear.

[12] A. Fisher: A pathwise central limit theorem for random walks, Preprint, 1989.

[13] B. V. Gnedenko and A. N. Kolmogorov: *Limit Distributions for Sums of Independent Random Variables*, Addison–Wesley, Reading, Massachusetts, 1954.

[14] M. Lacey and W. Philipp: A note on the almost everywhere central limit theorem, *Statist. Probab. Letters* **9** (1990), 201–205.

[15] P. Lévy: *Théorie de l'addition des variables aléatoires*, Gauthier–Villars, Paris, 1937.

[16] Z. Megyesi: A probabilistic approach to semistable laws and their domains of partial attraction, *Acta Sci. Math. (Szeged)* **66** (2000), 403–434.

[17] T. F. Móri: The a.s. limit distribution of the longest head run, *Canadian J. Math.* **45** (1993), 1245–1262.

[18] M. Peligrad and P. Révész: On the almost sure central limit theorem, In: *Almost Everywhere Convergence II* (A. Bellow and R. Jones, eds.), pp. 209–225, Academic Press, New York, 1991.

[19] P. Schatte: On strong versions of the central limit theorem, *Math. Nachr.* **137** (1988), 249–256.