ON THE EXCURSIONS OF TWO-DIMENSIONAL RANDOM WALK AND WIENER PROCESS

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Abstract

Consider a simple symmetric random walk on the plane. Its portion between two consecutive returns to zero are called excursions. We study the sum of the excursions when the two largest ones are eliminated from the sum. Similar investigations are carried out for two-dimensional Wiener process.

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1. Introduction and main results

Let X_1, X_2,... be a sequence of independent identically distributed random vectors with

$$\mathbf{P}\{X_1=(0,1)\}=\mathbf{P}\{X_1=(0,-1)\}=\mathbf{P}\{X_1=(1,0)\}=\mathbf{P}\{X_1=(-1,0)\}=\frac{1}{4}$$

and let $S_0=\mathbf{0}$, $S_n=X_1+X_2+...+X_n$ $(n=1,2,...)$ be a random walk on $\mathbf{Z^2}$ $(\mathbf{0}=(0,0))$. Its local time is defined by

$$\xi(\mathbf{a},n)=\#\{k;\ 0<k\le n,\ S_k=\mathbf{a}\},$$

where $\mathbf{a}=(a_1,a_2)$ is a lattice point on the plane. Put $\xi(n)=\xi(\mathbf{0},n)$. Let $\log_j$ denote the j-th iterated logarithm.

Erdős and Taylor (1960) proved the following results:

Theorem A1 (Erdős and Taylor (1960)):

$$\lim_{n\to\infty}\mathbf{P}\{\xi(n)<x\log n\} = 1-e^{-\pi x},\quad x>0. \tag{1.1}$$

$$\limsup_{n\to\infty}\frac{\pi\xi(n)}{\log n\log_3 n} = 1 \quad \text{a.s.} \tag{1.2}$$

$$\lim_{n\to\infty}\frac{\xi(n)(\log_2 n)^{1+\varepsilon}}{\log n} = \infty \quad \text{a.s.},\quad \varepsilon>0. \tag{1.3}$$

$$\liminf_{n\to\infty}\frac{\xi(n)\log_2 n}{\log n} = 0 \quad \text{a.s.} \tag{1.4}$$

Introduce

$$\begin{aligned}&\rho_0=0,\\ &\rho_k=\inf\{n;\ n>\rho_{k-1},\ S_n=\mathbf{0}\},\quad k=1,2,...\end{aligned} \tag{1.5}$$

the consecutive return times of the planar random walk to the origin. Put $\tau_k=\rho_k-\rho_{k-1}$, $k=1,2,\ldots$. The portion of the random walk between ρ_{k-1} and ρ_k is called the k-th excursion.

Theorem A1 can be rewritten as

Theorem A2 (Erdős and Taylor (1960)):

$$\lim_{N\to\infty} \mathbf{P}\{\log \rho_N < xN\} = e^{-\pi/x}, \quad x > 0. \tag{1.6}$$

$$\liminf_{N\to\infty} \frac{\log_2 N}{N} \log \rho_N = \pi \qquad \text{a.s.} \tag{1.7}$$

$$\lim_{N\to\infty} \frac{\log \rho_N}{N(\log N)^{1+\varepsilon}} = 0 \qquad \text{a.s.}, \quad \varepsilon > 0. \tag{1.8}$$

$$\limsup_{N\to\infty} \frac{\log \rho_N}{N \log N} = \infty \qquad \text{a.s.} \tag{1.9}$$

We need also the following result.

Theorem B (Dvoretzky and Erdős (1951), Erdős and Taylor (1960)):

$$\mathbf{P}\{\rho_1 > n\} = \mathbf{P}\{\xi(n) = 0\} = \frac{\pi}{\log n} + O((\log n)^{-2}), \tag{1.10}$$

as $n \to \infty$.

Now let $\kappa(n)$ be the last return to the origin before time n, i.e.

$$\kappa(n) = \max\{j;\ j \le n,\ S_j = \mathbf{0}\} = \rho_{\xi(n)}. \tag{1.11}$$

Denote by

$$M_n^{(1)} \ge M_n^{(2)} \ge \ldots \ge M_n^{(\xi(n)+1)}$$

the order statistics of the sequence $\tau_1, \tau_2, \ldots, \tau_{\xi(n)}, n - \kappa(n)$. It was shown in Csáki et al. (1998) that

$$\lim_{n\to\infty} \frac{M_n^{(1)} + M_n^{(2)}}{n} = 1 \qquad \text{a.s.} \tag{1.12}$$

Our aim here is to investigate the upper and lower functions of

$$R(n) = n - M_n^{(1)} - M_n^{(2)} = \sum_{k=3}^{\xi(n)+1} M_n^{(k)}. \tag{1.13}$$

Recall the following definitions (Révész (1990)):

Definition 1. The function $a_1(t)$ belongs to the upper-upper class of $\{Z(t)\}$ ($a_1 \in \mathrm{UUC}(Z(t))$) if for almost all $\omega \in \Omega$ there exists a t_0 such that $Z(t) < a_1(t)$ if $t > t_0$.

Definition 2. The function $a_2(t)$ belongs to the upper-lower class of $\{Z(t)\}$ ($a_2 \in \mathrm{ULC}(Z(t))$) if for almost all $\omega \in \Omega$ there exists a sequence $t_1 < t_2 < \ldots$ such that $\lim_{k\to\infty} t_k = \infty$ and $Z(t_k) \geq a_2(t_k)$, $k = 1, 2, \ldots$

Definition 3. The function $a_3(t)$ belongs to the lower-upper class of $\{Z(t)\}$ ($a_3 \in \mathrm{LUC}(Z(t))$) if for almost all $\omega \in \Omega$ there exists a sequence $t_1 < t_2 < \ldots$ such that $\lim_{k\to\infty} t_k = \infty$ and $Z(t_k) \leq a_3(t_k)$, $k = 1, 2, \ldots$

Definition 4. The function $a_4(t)$ belongs to the lower-lower class of $\{Z(t)\}$ ($a_4 \in \mathrm{LLC}(Z(t))$) if for almost all $\omega \in \Omega$ there exists a t_0 such that $Z(t) > a_4(t)$ if $t > t_0$.

Concerning $\kappa(n)$, the following result is proved in Erdős and Taylor (1960).

Theorem C (Erdős and Taylor (1960)): *Let $f(n)$ be a non-increasing function. Then*

$$\sum_{k=1}^{\infty} f(2^{2^k}) < \infty \Longrightarrow n^{f(n)} \in \mathrm{LLC}(\kappa(n)) \tag{1.14}$$

and

$$\sum_{k=1}^{\infty} f(2^{2^k}) = \infty \Longrightarrow n^{f(n)} \in \mathrm{LUC}(\kappa(n)). \tag{1.15}$$

We prove the following results:

Theorem 1.1. *For any $\varepsilon > 0$, $c > 0$*

$$\frac{n}{\exp\left(\frac{\log n}{(\log_2 n)^{1+\varepsilon}}\right)} \in \mathrm{UUC}(R(n)), \tag{1.16}$$

$$\frac{n}{\exp\left(\frac{c\log n}{\log_2 n}\right)} \in \mathrm{ULC}(R(n)), \tag{1.17}$$

$$\exp\left(\frac{c\log n}{\log_2 n}\right) \in \mathrm{LUC}(R(n)), \tag{1.18}$$

$$\exp\left(\frac{\log n}{(\log_2 n)^{1+\varepsilon}}\right) \in \mathrm{LLC}(R(n)). \tag{1.19}$$

Similar investigations can be carried out also for two-dimensional Wiener process with slight modifications. This is necessary, since every point is polar for the Wiener process, and, on the other hand, local time (as occupation density) does not exist.

So there is no meaning to speak about excursions away from the point $\mathbf{0}$. Instead, we consider excursions away from the unit circle.

Let $W(t) = (W_1(t), W_2(t))$, $t \geq 0$ be a two-dimensional Wiener process, where $W_1(t)$ and $W_2(t)$ are two independent one-dimensional standard Wiener processes with $W_1(0) = W_2(0) = 0$. Put $||W(t)|| = \sqrt{W_1^2(t) + W_2^2(t)}$. It is well known that $\{||W(t)||, t \geq 0\}$ is a two-dimensional Bessel process, admitting countably many excursions away from the point 1. Let

$$V_t^{(1)} \geq V_t^{(2)} \geq \dots \tag{1.20}$$

be the ordered lengths of these excursions up to time t, including the interval from the origin to the first hitting of the point 1 and the possibly incomplete last excursion (the interval from the last hitting of the point 1 to t). Then

$$\sum_{i=1}^{\infty} V_t^{(i)} = t$$

and we may consider

$$Q(t) = t - V_t^{(1)} - V_t^{(2)} = \sum_{i=3}^{\infty} V_t^{(i)}. \tag{1.21}$$

The upper and lower classes for $Q(\cdot)$ are in complete analogue to those for $R(\cdot)$.

Theorem 1.2. *For any $\varepsilon > 0$, $c > 0$*

$$\frac{t}{\exp\left(\frac{\log t}{(\log_2 t)^{1+\varepsilon}}\right)} \in \mathrm{UUC}(Q(t)), \tag{1.22}$$

$$\frac{t}{\exp\left(\frac{c \log t}{\log_2 t}\right)} \in \mathrm{ULC}(Q(t)), \tag{1.23}$$

$$\exp\left(\frac{c \log t}{\log_2 t}\right) \in \mathrm{LUC}(Q(t)), \tag{1.24}$$

$$\exp\left(\frac{\log t}{(\log_2 t)^{1+\varepsilon}}\right) \in \mathrm{LLC}(Q(t)). \tag{1.25}$$

In Section 2 we prove Theorem 1.1, while Section 3 contains the proof of Theorem 1.2. Since the proof of Theorem 1.2 is similar to that of Theorem 1.1, except that the results of Erdős and Taylor (Theorems A,B,C) are not available for Wiener case, we give first the analogues of these results and then sketch the proof of Theorem 1.2.

2. Proof of Theorem 1.1.

First we prove (1.16). Let

$$f_1(n) = \exp\left(\frac{\log n}{(\log_2 n)^{1+\varepsilon}}\right) \tag{2.1}$$

with some $\varepsilon > 0$. It follows from Theorem A that we have almost surely $\xi(n) \leq \log n \log_3 n$ for large enough n. Put

$$\nu_n = \sum_{1\leq i\leq \log n \log_3 n} I\left\{\frac{n}{f_1(n)} \leq \tau_i \leq n\right\}, \tag{2.2}$$

where $I\{\}$ denotes the indicator of the event in the brackets. Let $\alpha = 1/(2+\varepsilon)$, $n_k = \left[\exp\left(e^{k^\alpha}\right)\right]$, $N_k = [\log n_k \log_3 n_k] = [\alpha(\log k)e^{k^\alpha}] + O(1)$ and

$$\overline{\nu}_k = \sum_{i=1}^{N_{k+1}} I\left\{\frac{n_k}{f_1(n_k)} \leq \tau_i \leq n_{k+1}\right\}. \tag{2.3}$$

Obviously, $\overline{\nu}_k$ has binomial distribution with parameters (N_{k+1}, p_k), where

$$p_k = \mathbf{P}\left\{\frac{n_k}{f_1(n_k)} \leq \tau_1 \leq n_{k+1}\right\}.$$

We obtain from Theorem B that as $k \to \infty$

$$p_k = \frac{\pi}{\log \frac{n_k}{f_1(n_k)}} - \frac{\pi}{\log n_{k+1}} + O((\log n_k)^{-2}) = \frac{\pi}{e^{k^\alpha}\left(1 - \frac{1}{k^{(1+\varepsilon)\alpha}}\right)} - \frac{\pi}{e^{(k+1)^\alpha}} + O(e^{-2k^\alpha})$$

$$= \left(\frac{\pi}{e^{k^\alpha}\left(1 - \frac{1}{k^{(1+\varepsilon)\alpha}}\right)} - \frac{\pi}{e^{k^\alpha}}\right) + \left(\frac{\pi}{e^{k^\alpha}} - \frac{\pi}{e^{(k+1)^\alpha}}\right) + O(e^{-2k^\alpha})$$

$$= \left(\frac{\pi}{k^{(1+\varepsilon)\alpha}e^{k^\alpha}} + \frac{\alpha\pi}{k^{1-\alpha}e^{k^\alpha}}\right)(1+o(1)) = \frac{(1+\alpha)\pi}{k^{(1+\varepsilon)/(2+\varepsilon)}e^{k^\alpha}}(1+o(1)).$$

An easy calculation shows that

$$N_{k+1}^2 p_k^2 = O\left(\frac{\log^2 k}{k^{(2+2\varepsilon)/(2+\varepsilon)}}\right). \tag{2.4}$$

For large k we have

$$\mathbf{P}\{\overline{\nu}_k \geq 2\} = \sum_{j=2}^{N_{k+1}} \binom{N_{k+1}}{j} p_k^j (1-p_k)^{N_{k+1}-j} \leq \tag{2.5}$$

$$\leq \sum_{j=2}^{N_{k+1}} \frac{(N_{k+1}p_k)^j}{2} \leq (N_{k+1}p_k)^2 = O\left(\frac{\log^2 k}{k^{(2+2\varepsilon)/(2+\varepsilon)}}\right),$$

hence by Borel-Cantelli lemma

$$\mathbf{P}\{\overline{\nu}_k \geq 2 \text{ i.o.}\} = 0, \tag{2.6}$$

i.e. with probability 1 for large enough k we have $\overline{\nu}_k \leq 1$ and a fortiori for large enough n, $\nu_n \leq 1$. Hence

$$R(n) \leq \xi(n)\frac{n}{f_1(n)} \leq \log n \log_3 n \frac{n}{f_1(n)} \tag{2.7}$$

eventually. Since $\varepsilon > 0$ is arbitrary, this shows (1.16).

Now we turn to the proof of (1.17). Define the events

$$B_N = \bigcup_{1\leq i\neq j\leq N} B_{N;i,j} \tag{2.8}$$

with

$$\begin{aligned} B_{N;i,j} &= \Bigg\{ \left(1-\frac{c}{\log N}\right)\log\tau_j < \log\tau_i < \log\tau_j, \\ & 3N/2 < \log\tau_j < 2N, \max_{1\leq r\leq N; r\neq i,j} \log\tau_r \leq N \Bigg\}. \end{aligned} \tag{2.9}$$

We show that $\mathbf{P}\{B_N \text{ i.o.}\} = 1$. Clearly, for N large enough,

$$\mathbf{P}\{B_N\} = N(N-1)q_N \left(\mathbf{P}\{\log\tau_1 < N\}\right)^{N-2},$$

where

$$q_N = \mathbf{P}\left\{\left(1-\frac{c}{\log N}\right)\log\tau_1 < \log\tau_2 < \log\tau_1,\ 3N/2 < \log\tau_1 < 2N\right\}.$$

To estimate q_N, one can condition on $\log \tau_1 = x \in (3N/2, 2N)$ and use Theorem B to see

$$\mathbf{P}\left\{\left(1 - \frac{c}{\log N}\right) x < \log \tau_2 < x\right\} \sim \frac{c\pi}{x \log N},$$

which implies for large N

$$\frac{c\pi}{4N^2 \log N} \le q_N \le \frac{2c\pi}{N^2 \log N}.$$

Using Theorem B again, it can be seen that

$$\lim_{N\to\infty} \left(\mathbf{P}\{\log \tau_1 < N\}\right)^{N-2}$$

is finite and positive, so we arrive at

$$\frac{C_1}{\log N} \le \mathbf{P}\{B_N\} \le \frac{C_2}{\log N} \tag{2.10}$$

with some positive constants C_1 and C_2.

Put $N_k = [e^k]$, then $\mathbf{P}\{B_{N_k}\} > C_1/k$, therefore $\sum_k \mathbf{P}\{B_{N_k}\} = \infty$. Now we estimate $\mathbf{P}\{B_{N_k} B_{N_l}\}$.

For large enough $k + 1 \le l$ we have

$$2N_k < \frac{3N_l}{2}\left(1 - \frac{c}{\log N_l}\right),$$

and it is easy to see that

$$\mathbf{P}\{B_{N_k} B_{N_l}\} \le \mathbf{P}\{B_{N_k}\}\mathbf{P}\{\tilde{B}_{N_k,N_l}\}, \tag{2.11}$$

where

$$\tilde{B}_{N_k,N_l} = \bigcup_{N_k < i \ne j \le N_l} B_{N_k,N_l;i,j} \tag{2.12}$$

with

$$\begin{aligned} B_{N_k,N_l;i,j} \quad = \quad & \left\{\left(1 - \frac{c}{\log N_l}\right) \log \tau_j < \log \tau_i < \log \tau_j, \right. \\ & \left. 3N_l/2 < \log \tau_j < 2N_l, \max_{N_k < r \le N_l; r \ne i,j} \log \tau_r \le N_l \right\}. \end{aligned} \tag{2.13}$$

We obtain similarly to (2.10) that with some positive constants C_3, C_4, C_5

$$\mathbf{P}\{\tilde{B}_{N_k,N_l}\} \le (N_l - N_k)^2 \frac{C_3}{N_l^2 \log N_l} \le \frac{C_4}{l} \le C_5 \mathbf{P}\{B_{N_l}\},$$

i.e.

$$\mathbf{P}\{B_{N_k} B_{N_l}\} \le C_5 \mathbf{P}\{B_{N_k}\}\mathbf{P}\{B_{N_l}\},$$

hence by Borel-Cantelli lemma, $\mathbf{P}\{B_N \text{ i.o.}\} > 0$, consequently $\mathbf{P}\{B_N \text{ i.o.}\} = 1$ by $0-1$ law.

Now let $n = 2\rho_N$. Then B_N implies $n = 2\rho_N \le e^{3N}$ for large N and since there must be a large excursion between ρ_N and $2\rho_N$ by (1.12), also

$$\log R(n) = \log R(2\rho_N) \ge \log \tau_{2,N} \ge \left(1 - \frac{c}{\log N}\right) \log \tau_{1,N},$$

where $\tau_{1,N} \ge \tau_{2,N}$ are the two largest excursions among $\tau_1, \ldots, \tau_N$. Thus B_N implies

$$\log R(n) \ge \left(1 - \frac{c}{\log N}\right) \log \tau_{1,N} \ge \left(1 - \frac{2c}{\log N}\right) \log(2\rho_N) \ge \left(1 - \frac{3c}{\log\log n}\right) \log n,$$

consequently we have this inequality infinitely often with probability 1, proving (1.17).

For the proof of (1.18) we note that $R(n) \le \kappa(n)$ and hence it follows from Theorem C.

It remains to prove (1.19).

Lemma 2.1. *For $N \ge 1$, let $\tau_{1,N} \ge \ldots \ge \tau_{N,N}$ be the order statistics of $\{\tau_i\}_{1 \le i \le N}$. Then for each fixed $k \ge 1$,*

$$\liminf_{N\to\infty} \frac{\log_2 N}{N} \log \tau_{k,N} = \pi \qquad \text{a.s.}$$

Proof. According to Theorem A2

$$\liminf_{N\to\infty} \frac{\log_2 N}{N} \log \rho_N = \pi \qquad \text{a.s.}$$

So we only have to prove the lower bound.

Fix $\varepsilon > 0$, and consider

$$p_N = \mathbf{P}\left\{\log \tau_1 \ge (1-\varepsilon)\frac{\pi N}{\log_2 N}\right\}.$$

By Theorem B, for $0 < \varepsilon_1 < \varepsilon$ and large N,

$$\frac{\log_2 N}{(1-\varepsilon_1)N} \le p_N \le \frac{2\log_2 N}{N}.$$

Therefore

$$\begin{aligned}
&\mathbf{P}\left\{\log \tau_{k,N} < (1-\varepsilon)\frac{\pi N}{\log_2 N}\right\} \\
&= \sum_{i=0}^{k-1} \binom{N}{i} p_N^i (1-p_N)^{N-i} \\
&\le 2\sum_{i=0}^{k-1} (Np_N)^i (1-p_N)^N \\
&\le 2k(2\log_2 N)^k (1-p_N)^N \\
&\le 2k(2\log_2 N)^k \exp(-(1-\varepsilon_1)^{-1}\log_2 N).
\end{aligned}$$

Taking a geometric subsequence, using Borel-Cantelli lemma and monotonicity concludes the lemma. □

Now let $\rho_N \le n < \rho_{N+1}$. Since $R(n)$ is non-decreasing, we have

$$\log R(n) \ge \log R(\rho_N) \ge \log \tau_{3,N} \ge (1-\varepsilon)\frac{\pi N}{\log_2 N}.$$

By (1.7),

$$\lim_{N\to\infty} \frac{\log \rho_N}{N(\log N)^{1+\varepsilon}} = 0 \qquad \text{a.s.}$$

Thus

$$\log R(n) \ge \frac{\log \rho_{N+1}}{(\log_2 \rho_{N+1})^{1+2\varepsilon}} \ge \frac{\log n}{(\log_2 n)^{1+2\varepsilon}}.$$

This proves (1.19) completing the proof of Theorem 1.1. □

3. Proof of Theorem 1.2.

Define two increasing sequences of stopping times (σ_n) and (θ_n) by:

(3.1) $$\sigma_0 = \inf\{t > 0; ||W(t)|| = 1\},$$

(3.2) $$\theta_n = \inf\{t > \sigma_{n-1}; ||W(t)|| = 2\}, \quad n \ge 1,$$

(3.3) $$\sigma_n = \inf\{t > \theta_n; ||W(t)|| = 1\}, \quad n \ge 1.$$

Then $\{\sigma_n - \theta_n\}_{n=1}^{\infty}$ is a sequence of i.i.d. random variables and so is $\{\sigma_n - \sigma_{n-1}\}_{n=1}^{\infty}$.

Lemma 3.1.

$$\mathbf{P}\{\sigma_1 - \theta_1 > x\} = \frac{2\log 2}{\log x} + O((\log x)^{-2}) \tag{3.4}$$

and

$$\mathbf{P}\{\sigma_1 - \sigma_0 > x\} = \frac{2\log 2}{\log x} + O((\log x)^{-2}) \tag{3.5}$$

as $x \to \infty$.

Proof. According to Kent (1978), for any $\lambda > 0$

$$\mathbf{E}\{\exp(-\lambda(\sigma_1 - \theta_1))\} = \frac{K_0(\sqrt{8\lambda})}{K_0(\sqrt{2\lambda})} \tag{3.6}$$

and

$$\mathbf{E}\{\exp(-\lambda(\theta_1 - \sigma_0))\} = \frac{I_0(\sqrt{2\lambda})}{I_0(\sqrt{8\lambda})}, \tag{3.7}$$

where I_0 and K_0 are the modified Bessel functions. We have (cf. Gradshteyn and Ryzhik, p. 961, Formula 8.447)

$$K_0(z) = \log(1/z) + \log 2 - \mathbf{C} + O(z), \qquad z \to 0^+,$$

where $\mathbf{C}$ is Euler's constant and

$$I_0(z) = 1 + O(z^2), \qquad z \to 0^+.$$

Hence

$$\frac{K_0(\sqrt{8\lambda})}{K_0(\sqrt{2\lambda})} = \frac{\log(1/\lambda) - \log 2 - 2\mathbf{C} + O(\sqrt{\lambda})}{\log(1/\lambda) + \log 2 - 2\mathbf{C} + O(\sqrt{\lambda})}$$

$$= 1 - \frac{2\log 2}{\log(1/\lambda)} + O((\log(1/\lambda))^{-2}) \tag{3.8}$$

as $\lambda \to 0^+$. From this we would obtain the main term in (3.4) by applying a Tauberian theorem (cf. Doetsch (1950), p. 511). Following its proof, we can also obtain the remainder as follows.

Start from the elementary inequalities

$$\frac{ev^2 - v}{e-1} \le 1 \le (1+e)v - ev^2 \qquad \text{for} \quad 1/e \le v \le 1,$$

$$\frac{ev^2 - v}{e-1} \le 0 \le (1+e)v - ev^2 \qquad \text{for} \quad 0 \le v \le 1/e.$$

Putting $v = e^{-u\lambda}$, denoting by $H(u)$ the distribution function of $\sigma_1 - \theta_1$, with $x = 1/\lambda$ we get

$$\int_0^x dH(u) \le \int_0^\infty ((1+e)e^{-\lambda u} - ee^{-2\lambda u})dH(u) = (1+e)\frac{K_0(\sqrt{8\lambda})}{K_0(\sqrt{2\lambda})} - e\frac{K_0(\sqrt{16\lambda})}{K_0(\sqrt{4\lambda})}$$

and similarly,

$$\int_0^x dH(u) \ge \int_0^\infty \frac{ee^{-2\lambda u} - e^{-\lambda u}}{e-1} dH(u) = \frac{e}{e-1}\frac{K_0(\sqrt{16\lambda})}{K_0(\sqrt{4\lambda})} - \frac{1}{e-1}\frac{K_0(\sqrt{8\lambda})}{K_0(\sqrt{2\lambda})}.$$

Both the upper and the lower bounds are of the form

$$A\frac{K_0(\sqrt{8\lambda})}{K_0(\sqrt{2\lambda})} + B\frac{K_0(\sqrt{16\lambda})}{K_0(\sqrt{4\lambda})}$$

with $A + B = 1$. We obtain from (3.8)

$$1 - A\frac{K_0(\sqrt{8\lambda})}{K_0(\sqrt{2\lambda})} - B\frac{K_0(\sqrt{16\lambda})}{K_0(\sqrt{4\lambda})} = \frac{2\log 2}{\log(1/\lambda)} + O((\log(1/\lambda))^{-2}).$$

Since $\mathbf{P}\{\sigma_1 - \theta_1 > x\} = 1 - \int_0^x dH(u)$, this proves (3.4).

The proof of (3.5) is similar, since independence of $\sigma_1 - \theta_1$ and $\theta_1 - \sigma_0$ gives

$$\mathbf{E}\left\{\exp\left(-\lambda(\sigma_1 - \sigma_0)\right)\right\} = \frac{I_0(\sqrt{2\lambda})K_0(\sqrt{8\lambda})}{I_0(\sqrt{8\lambda})K_0(\sqrt{2\lambda})}. \qquad \square$$

Now put $\tilde{\tau}_i = \sigma_i - \sigma_{i-1}$, $\eta(n) = \max\{N;\, \sigma_N - \sigma_0 \le n\}$. (Here n can be considered as a continuous variable, taking real numbers.) Then one can define the ordered "excursions" $\tilde{M}_n^{(1)} \ge \tilde{M}_n^{(2)} \ge \ldots \ge \tilde{M}_n^{\eta(n)+1}$ in terms of $\tilde{\tau}$ exactly in the same way as M_n were defined in terms of τ. Accordingly, we define $\tilde{R}(n) = n - \tilde{M}_n^{(1)} - \tilde{M}_n^{(2)}$. Following Erdős and Taylor (1960) one can easily see from Lemma 3.1 that all the results in Theorem A1 and Theorem A2 remain true if $\pi\xi(n)$ is replaced by $(2\log 2)\eta(n)$ and $(\log \rho_N)/\pi$ is replaced by $(\log(\sigma_N - \sigma_0))/(2\log 2)$. For $\tilde{R}$ we have

Proposition 3.1. *Theorem 1.1 remains true if $R(n)$ is replaced by $\tilde{R}(n)$.*

Proof. In fact, (1.16), (1.17) and (1.19) with R replaced by $\tilde{R}$ can be proved exactly the same way as in Section 2. We have only to show (1.18) for $\tilde{R}$.

Fix any constant $c > 0$, and let

$$B_N = \bigcup_{j=N+1}^{2N} \Big\{ 3c^{-1} N \log N < \log \tilde{\tau}_j < 4c^{-1} N \log N, \max_{1\le i\le 2N,\, i\ne j} \log \tilde{\tau}_i < N \Big\}.$$

Since $(\tilde{\tau}_i)$ are iid,

$$\mathbf{P}\{B_N\} = N \left(\mathbf{P}\{\log \tilde{\tau}_1 < N\}\right)^{2N-1} \mathbf{P}\{3c^{-1} N \log N < \log \tilde{\tau}_1 < 4c^{-1} N \log N\},$$

which, in view of Lemma 3.1, gives

$$\mathbf{P}\{B_N\} \sim \frac{c_1}{\log N}.$$

Put $N_k = 2^k$ to see that $\sum_k \mathbf{P}\{B_{N_k}\} = \infty$. Let $k < l$, then $2N_k < N_l + 1$. Accordingly, with some positive constants c_2 and c_3

$$\begin{aligned} \mathbf{P}\{B_{N_k} B_{N_l}\} &= N_k \left(\mathbf{P}\{\log \tilde{\tau}_1 < N_k\}\right)^{2N_k-1} \times \\ &\times \mathbf{P}\{3c^{-1} N_k \log N_k < \log \tilde{\tau}_1 < 4c^{-1} N_k \log N_k\} \times \\ &\times N_l \left(\mathbf{P}\{\log \tilde{\tau}_1 < N_l\}\right)^{2(N_l-N_k)-1} \times \\ &\times \mathbf{P}\{ 3c^{-1} N_l \log N_l < \log \tilde{\tau}_1 < 4c^{-1} N_l \log N_l\} \\ &\le \frac{c_2}{(\log N_k)\,(\log N_l)} \le c_3\, \mathbf{P}\{B_{N_k}\}\, \mathbf{P}\{B_{N_l}\}. \end{aligned}$$

It follows from the Borel–Cantelli lemma that $\mathbf{P}\{B_N \text{ i.o.}\} > 0$. Since the event $\{B_N \text{ i.o.}\}$ is invariant under finite permutations of $(\tilde{\tau}_i)$, the Hewitt–Savage 0–1 law confirms that its probability equals 1.

Almost surely, there are infinitely many N such that, simultaneously,

$$\sigma_{2N} - \sigma_0 - \max_{1\le i\le 2N} (\sigma_i - \sigma_{i-1}) \le (2N-1) e^N \le e^{2N},$$

$$\log(\sigma_{2N} - \sigma_0) \ge 3c^{-1} N \log N.$$

Therefore, infinitely often,

$$\tilde{R}(\sigma_{2N}) \le \sigma_{2N} - \sigma_0 - \max_{1\le i\le 2N} (\sigma_i - \sigma_{i-1}) \le e^{2N} \le \exp\Big(\frac{c \log \sigma_{2N}}{\log\log \sigma_{2N}}\Big),$$

which yields (1.18) for $\tilde{R}$. □

For the proof of Theorem 1.2 note the following inequalities valid for large enough n:

$$V_n^{(1)} + V_n^{(2)} \le \tilde{M}_n^{(1)} + \tilde{M}_n^{(2)} \le V_n^{(1)} + V_n^{(2)} + \sigma_0 + \sum_{i \le \eta(n)} (\theta_i - \sigma_{i-1}),$$

from which

$$\tilde{R}(n) \le Q(n) \le \tilde{R}(n) + \sigma_0 + \sum_{i \le \eta(n)} (\theta_i - \sigma_{i-1}).$$

It follows from (3.7) that $\mathbf{E}\{\theta_1 - \sigma_0\} < \infty$, hence by the law of large numbers,

$$\sum_{i \le \eta(n)} (\theta_i - \sigma_{i-1}) = O(\eta(n)) = O(\log n \log_3 n) \qquad \text{a.s.}$$

as $n \to \infty$.

This completes the proof of Theorem 1.2.

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