Pointwise and uniform asymptotics of the Vervaat error process

Dedicated to the memory of Arthur Hing-Chiu Chan (1946–1999), PhD 1977, Carleton University

Endre Csáki[1]

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, P.O.B. 127, H-1364, Hungary. E-mail: csaki@renyi.hu

Miklós Csörgő[2]

School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6. E-mail: mcsorgo@math.carleton.ca

Antónia Földes[3]

City University of New York, 2800 Victory BLVD., Staten Island, New York 10314, U.S.A. E-mail: afoldes@gc.cuny.edu

Zhan Shi

Laboratoire de Probabilités UMR 7599, Université Paris VI, 4 Place Jussieu, F-75252 Paris Cedex 05, France. E-mail: zhan@proba.jussieu.fr

Ričardas Zitikis[4]

Department of Statistics, University of Manitoba, 327 Machray Hall, Winnipeg, Manitoba, Canada R3T 2N2. E-mail: Ricardas_Zitikis@umanitoba.ca

Abstract. It is well known that, asymptotically, the appropriately normalized uniform Vervaat process, i.e., the integrated uniform Bahadur–Kiefer process properly normalized, behaves like the square of the uniform empirical process. We give a complete description of the strong and weak asymptotic behaviour in sup-norm of this representation of the Vervaat process and, likewise, we study also its pointwise asymptotic behaviour.

Keywords. Empirical process, quantile process, Bahadur–Kiefer process, Vervaat process, Vervaat error process, Kiefer process, Brownian bridge, Wiener process, strong approximation, law of the iterated logarithm, convergence in distribution.

2000 Mathematics Subject Classification. Primary 60F15; Secondary 60F05, 60F17.

[1]Research supported by the Hungarian National Foundation for Scientific Research, Grant No. T 029621.

[2]Research supported by an NSERC Canada Grant at Carleton University, Ottawa.

[3]Research supported by a PSC CUNY grant, No. 6-69413.

[4]Research supported by a grant of the University of Manitoba, Winnipeg, and by an NSERC Canada Grant at the Laboratory for Research in Statistics and Probability, Carleton University and University of Ottawa, Ottawa.

1 Introduction and preliminary results

Let $U_1,\ U_2,\ldots$ be independent copies of a random variable U uniformly distributed over the interval $[0,1]$. Let

$$F_n(t) := \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}\{U_k \le t\}, \quad 0 \le t \le 1,$$

denote the empirical distribution function based on $U_1,\ U_2,\ldots,U_n$, where $\mathbf{1}$ is the indicator function. Let F_n^{-1} be the left-continuous inverse of F_n. We denote the empirical and quantile processes over the interval $[0,1]$ by

$$\begin{aligned}\alpha_n(t) &:= n^{1/2}(F_n(t)-t), \quad 0\le t\le 1,\\ \beta_n(t) &:= n^{1/2}(F_n^{-1}(t)-t), \quad 0\le t\le 1,\end{aligned}$$

respectively. The sum

$$R_n(t) := \alpha_n(t)+\beta_n(t), \quad 0\le t\le 1,$$

of the empirical and quantile processes is known in the literature as the Bahadur–Kiefer process (cf. Bahadur, 1966, Kiefer, 1967, 1970).

The Bahadur–Kiefer process enjoys some remarkable asymptotic properties, which are of interest in statistical quantile data analysis (cf., e.g., Csörgő, 1983, Shorack and Wellner, 1986). We summarize the most relevant results of Kiefer (1967, 1970) in this regard in the following theorem. Throughout the paper, we use the notation $\log x := \log\max(x,e)$ and $\log_2 x := \log\log x$.

Theorem A *For every fixed $t\in(0,1)$, we have*

$$n^{1/4}R_n(t) \to_d (t(1-t))^{1/4}\mathcal{N}(|\widetilde{\mathcal{N}}|)^{1/2}, \quad n\to\infty, \tag{1.1}$$

$$\limsup_{n\to\infty}\frac{n^{1/4}|R_n(t)|}{(\log_2 n)^{3/4}} = (t(1-t))^{1/4}\frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.}, \tag{1.2}$$

where $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are independent standard normal variables and $\to_d$ denotes convergence in distribution. Also,

$$\lim_{n\to\infty} n^{1/4}(\log n)^{-1/2}\frac{\|R_n\|}{(\|\alpha_n\|)^{1/2}} = 1 \quad \text{a.s.}, \tag{1.3}$$

where $\|f\| := \sup_{0\le t\le 1}|f(t)|$ denotes the uniform sup-norm of f.

Theorem A is due to Kiefer (1967, 1970), except for (1.3), which he announced but proved only convergence in probability (cf. Theorem 1A, and the two sentences right after, in Kiefer, 1970). The upper bound for the almost sure convergence in (1.3) was proved by Shorack (1982), and the lower bound by Deheuvels and Mason (1990). For a review of these results and for further developments along these lines we refer to Deheuvels and Mason (1992), Einmahl (1996), Csörgő and Szyszkowicz (1998).

Via using the usual and the other laws of the iterated logarithm for α_n, (1.3) immediately implies

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_2 n)^{-1/4}\|R_n\| = 2^{-1/4} \quad \text{a.s.}, \tag{1.4}$$

$$\liminf_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_2 n)^{1/4}\|R_n\| = \frac{\pi^{1/2}}{8^{1/4}} \quad \text{a.s.}, \tag{1.5}$$

while a direct application of (1.3) together with the weak convergence of α_n to a Brownian bridge B gives

$$n^{1/4}(\log n)^{-1/2}\|R_n\| \to_d (\|B\|)^{1/2}, \quad n \to \infty. \tag{1.6}$$

Nevertheless, the following result, which one can immediately conclude also by combining (1.1) with (1.6), is true, and it was first formulated and proved directly by Vervaat (1972b).

Theorem B (Vervaat, 1972b) *The statement*

$$a_n R_n \to_d Y, \quad n \to \infty,$$

cannot hold true in the space $D[0,1]$ (endowed with the Skorohod topology) for any sequence $\{a_n\}$ of positive real numbers and any non-degenerate random element Y of $D[0,1]$.

In view of Theorems A and B now, it is of interest to see the asymptotic behaviour of the Bahadur–Kiefer process possibly in other norms as well. In this regard we quote

Theorem C (Csörgő and Shi, 1998) *For any $p \in [2,\infty)$, we have*

$$\lim_{n\to\infty} n^{1/4}\frac{\|R_n\|_p}{(\|\alpha_n\|_{p/2})^{1/2}} = c_0(p) \qquad \text{a.s.}, \tag{1.7}$$

where

$$c_0(p) := (\mathbf{E}|\mathcal{N}|^p)^{1/p} = \sqrt{2}\left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}}\right)^{1/p}, \tag{1.8}$$

$\mathcal{N}$ stands, as before, for a standard normal variable, and $\|f\|_p := \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}$, the L_p norm of f.

In particular, (1.7) yields L_p versions of the laws of the iterated logarithm (LIL's) for the Bahadur–Kiefer process R_n. We also note that (1.7) combined with the weak convergence of α_n to a Brownian bridge B implies that, for $p \in [2,\infty)$,

$$n^{1/4}\|R_n\|_p \to_d c_0(p)(\|B\|_q)^{1/2}, \quad n \to \infty.$$

This, in combination with (1.6), yields Theorem B again.

Vervaat's (1972b) proof of Theorem B was based, in a most crucial and elegant way, on the following integrated Bahadur–Kiefer process

$$I_n(t) := \int_0^t R_n(s)ds, \quad 0 \le t \le 1.$$

Concerning the latter process, he established the weak convergence of

$$V_n(t) := 2n^{1/2} I_n(t) \tag{1.9}$$

to B^2, the square of a Brownian bridge, as well as a functional LIL for V_n, via proving the following theorem (for a discussion of related details we refer to Csörgő and Zitikis, 1999a, b).

Theorem D (Vervaat, 1972a, b) *We have*

$$\lim_{n\to\infty} (\log_2 n)^{-1}\|V_n - \alpha_n^2\| = 0 \qquad \text{a.s.} \tag{1.10}$$

$$\lim_{n\to\infty} \|V_n - \alpha_n^2\| = 0 \qquad \textit{in probability.} \tag{1.11}$$

In particular, in the space $C[0,1]$,

$$V_n \to_d B^2, \quad n \to \infty. \tag{1.12}$$

For use of terminology, we call the process V_n of (1.9) the uniform Vervaat process. For further references and elaboration on this terminology we refer to Section 1 of Zitikis (1998).

As a consequence of (1.12), Vervaat (1972b) concluded Theorem B. We also note that (1.10) yields the LIL for V_n in sup-norm (cf. Corollary 1.1 (Vervaat 1972a, b) in Csörgő and Zitikis, 1999a, b).

2 The Vervaat error process, main results

Bahadur (1966) introduced R_n as the remainder term in the representation

$$\beta_n = -\alpha_n + R_n$$

of the quantile process β_n in terms of the empirical process α_n. As we have seen in Theorems A and C, the remainder term R_n, i.e., the Bahadur–Kiefer process, is asymptotically smaller than the main term α_n, i.e., the empirical process, in both the L_p and sup-norm topologies.

In a similar vein, we can think of the process

$$Q_n(t) := V_n(t) - \alpha_n^2(t), \quad 0 \le t \le 1, \tag{2.1}$$

that appears in both statements (1.10) and (1.11) of Theorem D as the remainder term Q_n in the following representation

$$V_n = \alpha_n^2 + Q_n \tag{2.2}$$

of the uniform Vervaat process V_n in terms of the square of the empirical process. It is well-known (cf. Zitikis, 1998, for details and references) that the remainder term Q_n in (2.2) is asymptotically smaller than the main term α_n^2. Thus, just like in the case of R_n, one may like to know how small the remainder term Q_n is.

In view of Theorems A and C, one suspects that there should be substantial differences between the asymptotic pointwise, sup- and L_p-norms behaviour of the process Q_n. Indeed, Csörgő and Zitikis (1999a) established the following strong convergence result for $\|Q_n\|_p$.

Theorem E (Csörgő and Zitikis, 1999a) *For any* $p \in [1,\infty)$, *we have*

$$\lim_{n\to\infty} n^{1/4} \frac{\|Q_n\|_p}{(\|\alpha_n\|_{3p/2})^{3/2}} = \frac{1}{\sqrt{3}} c_0(p) \qquad \text{a.s.}, \tag{2.3}$$

where $c_0(p)$ *is defined in* (1.8).

For a comparison of this result to that of Theorem C, as well as for that of their consequences, we refer to Csörgő and Zitikis (1999a), who have also conjectured that in sup-norm the analogue statement of (2.3) should be of the following form:

$$\lim_{n\to\infty} b_n n^{1/4} \frac{\|Q_n\|}{\|\alpha_n\|^{3/2}} = c \qquad \text{a.s.}, \tag{2.4}$$

where b_n is a slowly varying function converging to 0 and c is a positive constant.

One of the aims of this exposition is to prove that this conjecture is true with $b_n = (\log n)^{-1/2}$. In addition, we also study the pointwise behaviour of the Vervaat error process Q_n. We summarize our results in the following theorem, which parallels Theorem A of Kiefer (1967, 1970) concerning the process R_n.

Theorem 2.1 *For every fixed $t \in (0,1)$, we have*

$$n^{1/4} Q_n(t) \to_d (4/3)^{1/2} (t(1-t))^{3/4} \mathcal{N} (|\widetilde{\mathcal{N}}|)^{3/2}, \quad n \to \infty, \tag{2.5}$$

$$\limsup_{n\to\infty} \frac{n^{1/4}|Q_n(t)|}{(\log_2 n)^{5/4}} = (t(1-t))^{3/4} \frac{2^{11/4} 3^{1/4}}{5^{5/4}} \quad \text{a.s.}, \tag{2.6}$$

where $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are independent standard normal variables. Also,

$$\lim_{n\to\infty} n^{1/4} (\log n)^{-1/2} \frac{\|Q_n\|}{(\|\alpha_n\|)^{3/2}} = (4/3)^{1/2} \quad \text{a.s.} \tag{2.7}$$

As a consequence of this theorem, as well as that of Theorem E combined with (2.7), we have the following corollary, which confirms Conjecture 2.1 of Csörgő and Zitikis (1999a).

Corollary 2.1 *The statement*

$$a_n Q_n \to_d Y, \quad n \to \infty,$$

cannot hold true in the space $D[0,1]$ for any sequence $\{a_n\}$ of positive real numbers and for any non-degenerate random element Y of the space $D[0,1]$.

Another consequence via (2.7) is the following corollary.

Corollary 2.2 *We have*

$$\limsup_{n\to\infty} n^{1/4} (\log n)^{-1/2} (\log_2 n)^{-3/4} \|Q_n\| = \frac{2^{1/4}}{3^{1/2}} \quad \text{a.s.}, \tag{2.8}$$

$$\liminf_{n\to\infty} n^{1/4} (\log n)^{-1/2} (\log_2 n)^{3/4} \|Q_n\| = \frac{\pi^{3/2}}{3^{1/2} 2^{5/4}} \quad \text{a.s.}, \tag{2.9}$$

$$n^{1/4} (\log n)^{-1/2} \|Q_n\| \to_d (4/3)^{1/2} \|B\|^{3/2}, \quad n \to \infty, \tag{2.10}$$

where B is a standard Brownian bridge.

We note that (2.8) and (2.9) follow from (2.7) by means of the usual and the other LIL's for α_n. As to (2.10), it results from a direct application of (2.7) together with the weak convergence of α_n to a Brownian bridge B.

Remark 2.1 In the literature we also find general forms of the Vervaat process that are based on random variables $X_1, X_2, \ldots$ replacing the uniform [0,1] random variables $U_1, U_2, \ldots$ In particular, such a general Vervaat process first appeared and was put to good use in Csörgő and Zitikis (1996). We refer to Zitikis (1998) for a detailed survey on this subject. For related though rather different limit theorems for the general Vervaat process, we refer to Csörgő and Zitikis (1998). It is obvious that the results of this paper can be generalized in such a way that they would cover general forms of the Vervaat process as well. However, a solution of this problem under reasonably optimal assumptions may constitute a rather challenging mathematical task which is definitely not within the scope of the present paper. For a recent review of Vervaat and Vervaat error processes we refer to Csörgő and Zitikis (1999b). □

3 The Vervaat error process in terms of a Kiefer process

We introduce some two-parameter Gaussian processes. Let $\{W(x,y),\ x \geq 0,\ y \geq 0\}$ be a Wiener (Brownian) sheet, i.e., a two-parameter Gaussian process with $\mathbf{E}W(x,y) = 0$ and covariance function

$$\mathbf{E}W(x_1,y_1)\,W(x_2,y_2) = (x_1 \wedge x_2)(y_1 \wedge y_2).$$

Next we define a Kiefer process $\{K(x,y),\ 0 \leq x \leq 1,\ y \geq 0\}$ by

$$K(x,y) := W(x,y) - xW(1,y),$$

where $W(x,y)$ is a Wiener sheet. A Kiefer process $K(x,y)$ can be characterized as a mean zero Gaussian process with covariance function

$$\mathbf{E}K(x_1,y_1)\,K(x_2,y_2) = (x_1 \wedge x_2 - x_1x_2)(y_1 \wedge y_2).$$

Remark 3.1 In this paper we define, without loss of generality, all Kiefer processes $K(x,y)$ and Brownian bridges $B(x)$ to be equal to zero if $x < 0$ or $x > 1$. □

Concerning the uniform empirical process α_n, Komlós *et al.* (1975) established the following fundamental embedding theorem.

Theorem F (Komlós, Major and Tusnády, 1975) *On a suitable probability space, the uniform empirical process* $\{\alpha_k(x),\ 0 \leq x \leq 1,\ k = 1,2,\dots\}$ *and a Kiefer process* $\{K(x,k),\ 0 \leq x \leq 1,\ k = 1,2,\dots\}$ *can be so constructed that, as* $n \to \infty$,

$$\max_{1\leq k\leq n}\sup_{0\leq x\leq 1}|\alpha_k(x) - n^{-1/2}K(x,k)| = \mathcal{O}\left(\frac{(\log n)^2}{n^{1/2}}\right) \qquad \text{a.s.}$$

Combining (1.4) with Theorem F, we arrive at (cf. Csörgő and Révész, 1975):

Proposition A *On the probability space of Theorem* F *for the uniform quantile process* $\{\beta_k(x),\ 0 \leq x \leq 1,\ k = 1,2,\dots\}$ *with the Kiefer process* $\{K(x,k),\ 0 \leq x \leq 1,\ k = 1,2,\dots\}$ *of Theorem* F, *as* $n \to \infty$, *we have*

$$\max_{1\leq k\leq n}\sup_{0\leq x\leq 1}|\beta_k(x) + n^{-1/2}K(x,k)| = \mathcal{O}\left(\frac{(\log n)^{1/2}(\log_2 n)^{1/4}}{n^{1/4}}\right) \qquad \text{a.s.} \tag{3.1}$$

We note in passing (cf. Deheuvels, 1998 and references therein) that the almost sure rate of convergence in (3.1) cannot be improved even when approximating the uniform quantile process by any other Kiefer process.

For further use we quote the following two results of A.H.C. Chan (1977) (cf. Theorem 1.15.2, and Theorems S.1.14.2 and S.1.15.1, respectively, in Csörgő and Révész, 1981).

Theorem G (Chan, 1977) *Let* $K(\cdot,\cdot)$ *be a Kiefer process, and let* $\{h_n\}$ *be a non-increasing sequence of positive numbers such that* $\lim_{n\to\infty}(\log \frac{1}{h_n})/\log_2 n = \infty$. *Then*

$$\lim_{n\to\infty}\sup_{0\leq t\leq 1-h_n}\sup_{0\leq s\leq h_n}\frac{|K(t+s,n) - K(t,n)|}{(2nh_n\log(1/h_n))^{1/2}} = 1 \qquad \text{a.s.}$$

Theorem H (Chan, 1977) *Let* $0 < \varepsilon_T \le \frac{1}{2}$, $0 < a_T \le T$ *be functions of* T *such that* ε_T *and* a_T/T *are non-increasing and* a_T *is non-decreasing. Define* $K((x_1, x_2], t) = K(x_2, t) - K(x_1, t)$ $(0 \le x_1 < x_2 \le 1)$. *Then almost surely,*

$$\begin{aligned} &\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le z\le a_T} \sup_{0\le x\le 1-\varepsilon_T} \sup_{0\le s\le\varepsilon_T} \beta_T |K((x, x+s], t+z) - K((x, x+s], t)| \\ = &\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le x\le 1-\varepsilon_T} \beta_T |K((x, x+\varepsilon_T], t+a_T) - K((x, x+\varepsilon_T], t)| = 1, \end{aligned}$$

where

$$\beta_T = \left(2a_T\varepsilon_T(1-\varepsilon_T)\left(\log\frac{T}{\varepsilon_T a_T} + \log_2 T\right)\right)^{-1/2}.$$

If in addition, $\lim_{T\to\infty}(\log \frac{T}{\varepsilon_T a_T})/\log_2 T = \infty$, *then* $\limsup_{T\to\infty}$ *can be replaced by* $\lim_{T\to\infty}$.

The main result of this section is the following strong approximation of the Vervaat error process $Q_n(t)$ defined in (2.1) via a Kiefer process.

Theorem 3.1 *On the probability space of Theorem* F, *using the there constructed Kiefer process* $K(\cdot,\cdot)$, $Q_n(\cdot)$ *can be approximated as follows. As* $n \to \infty$ *we have*

$$\sup_{0<t<1} |Q_n(t) - Z_n(t)| = \mathcal{O}(n^{-3/8}(\log n)^{3/4}(\log_2 n)^{5/8}) \qquad \text{a.s.}, \tag{3.2}$$

where $\{Z_n(t),\, 0<t<1,\, n=1,2,\dots\}$ *is defined by*

$$Z_n(t) := 2\frac{K(t,n)}{n}\int_0^1 \left(K\left(t - s\frac{K(t,n)}{n}, n\right) - K(t,n)\right) ds. \tag{3.3}$$

The proof of this theorem is based on the next two lemmas, which are of interest on their own.

Lemma 3.1 *Let*

$$A_n(t) := 2n^{1/2}\int_{F_n^{-1}(t)}^{t} (\alpha_n(u) - \alpha_n(t))\, du, \quad 0<t<1,\ n=1,2,\dots \tag{3.4}$$

Then

$$Q_n(t) = A_n(t) - R_n^2(t), \tag{3.5}$$

and, consequently, as $n \to \infty$,

$$\sup_{0<t<1} |Q_n(t) - A_n(t)| = \mathcal{O}(n^{-1/2}(\log n)(\log_2 n)^{1/2}) \qquad \text{a.s.} \tag{3.6}$$

Proof. We have the following easy-to-check representation

$$V_n(t) = -\alpha_n(t)\beta_n(t) + A_n(t) - (\alpha_n(t) + \beta_n(t))\beta_n(t)$$

for all $t \in (0,1)$ and $n = 1,2,\dots$ (cf., Vervaat, 1972a, Shorack and Wellner, 1986, Zitikis, 1998). By (2.1), $Q_n(t) = V_n(t) - \alpha_n^2(t) = A_n(t) - R_n^2(t)$, yielding (3.5). The conclusion (3.6) follows from (3.5) and (1.4). □

Remark 3.2 While in this paper we are yet to study the stochastic process $Q_n(t)$ in terms of $A_n(t)$, it is interesting to note that, as a result of the representation (3.5), we already know the best possible stochastic behaviour of their difference $A_n(t) - Q_n(t) = R_n^2(t)$. Moreover, on account of having (1.1)–(1.8), we can immediately write down exact analogues for the difference process $A_n(t) - Q_n(t)$ via $R_n^2(t)$. □

Lemma 3.2 *On the probability space of Theorem* F, *using the there constructed Kiefer process* $K(\cdot,\cdot)$, *the stochastic process* $A_n(t)$ *of Lemma* 3.1 *can be approximated such that when* $n\to\infty$,

$$\sup_{0<t<1}|A_n(t)-Z_n(t)|=\mathcal{O}(n^{-3/8}(\log n)^{3/4}(\log_2 n)^{3/8})\qquad \text{a.s.},$$

where $Z_n(t)$ *is as in Theorem* 3.1.

Proof. By a change of variables $u=t-s(t-F_n^{-1}(t))=t+sn^{-1/2}\beta_n(t)$ in (3.4), we have

$$A_n(t)=-2\beta_n(t)\int_0^1\left(\alpha_n\left(t+s\frac{\beta_n(t)}{n^{1/2}}\right)-\alpha_n(t)\right)ds.$$

The usual LIL for β_n confirms that $\|\beta_n\|=\mathcal{O}((\log_2 n)^{1/2})$ almost surely (when $n\to\infty$). Therefore, by Theorem F, when $n\to\infty$,

$$\begin{aligned}A_n(t) &= -2n^{-1/2}\beta_n(t)\int_0^1\left(K\left(t+s\frac{\beta_n(t)}{n^{1/2}},n\right)-K(t,n)\right)ds\\ &\quad+\mathcal{O}\left(n^{-1/2}(\log n)^2(\log_2 n)^{1/2}\right),\qquad \text{a.s.,}\end{aligned}\tag{3.7}$$

uniformly in $t\in(0,1)$.

For all $t,s\in(0,1)$, we have

$$\begin{aligned}&\left|K\left(t+s\frac{\beta_n(t)}{n^{1/2}},n\right)-K\left(t-s\frac{K(t,n)}{n},n\right)\right|\\ :=\ &|K(u+v,n)-K(u,n)|\\ \le\ &\sup_{0\le u\le 1}\ \sup_{0\le v\le h_n}|K(u+v,n)-K(u,n)|,\end{aligned}$$

where

$$u=t-s\frac{K(t,n)}{n},\qquad v=s\left(\frac{\beta_n(t)}{n^{1/2}}+\frac{K(t,n)}{n}\right)$$

and, on account of Proposition A, $h_n=\mathcal{O}(n^{-3/4}(\log n)^{1/2}(\log_2 n)^{1/4})$ $(n\to\infty)$, almost surely. Thus, according to Theorem G, almost surely, when $n\to\infty$,

$$K\left(t+s\frac{\beta_n(t)}{n^{1/2}},n\right)-K\left(t-s\frac{K(t,n)}{n},n\right)=\mathcal{O}\left(n^{1/8}(\log n)^{3/4}(\log_2 n)^{1/8}\right),$$

uniformly in $t,s\in(0,1)$. Inserting this into (3.7) and using again the LIL for β_n, we arrive at:

$$\begin{aligned}A_n(t) &= -2n^{-1/2}\beta_n(t)\int_0^1\left(K\left(t-s\frac{K(t,n)}{n},n\right)-K(t,n)\right)ds\\ &\quad+\mathcal{O}(n^{-3/8}(\log n)^{3/4}(\log_2 n)^{5/8})\qquad \text{a.s.}\end{aligned}\tag{3.8}$$

According to Proposition A,

$$-2n^{-1/2}\beta_n(t)=2\frac{K(t,n)}{n}+\mathcal{O}\left(\frac{(\log n)^{1/2}(\log_2 n)^{1/4}}{n^{3/4}}\right)\tag{3.9}$$

almost surely and uniformly in $t\in(0,1)$. On the other hand, applying Theorem G to the integrand in (3.9) with $h_n=\mathcal{O}(n^{-1/2}(\log_2 n)^{1/2})$, we obtain:

$$\int_0^1\left(K\left(t-s\frac{K(t,n)}{n},n\right)-K(t,n)\right)ds=\mathcal{O}\left(n^{1/4}(\log n)^{1/2}(\log_2 n)^{1/4}\right),$$

almost surely and uniformly in $t \in (0,1)$. Plugging this and (3.9) into (3.8) yields that almost surely, when $n \to \infty$,

$$A_n(t) = Z_n(t) + \mathcal{O}\left(\frac{(\log n)(\log_2 n)^{1/2}}{n^{1/2}}\right) + \mathcal{O}(n^{-3/8}(\log n)^{3/4}(\log_2 n)^{5/8}),$$

uniformly in $t \in (0,1)$. This yields Lemma 3.2. □

Proof of Theorem 3.1. Follows from Lemmas 3.1 and 3.2. □

We mention that the process $Z_n(t)$ was put to use in Csörgő and Zitikis (1999a) in their study of $Q_n(t)$ in L_p norm. Moreover, they also remarked that their conjecture as stated in (2.4) is equivalent to stating it in terms of Z_n instead of Q_n. Likewise, in view of Theorem 3.1 now, the proof of Theorem 2.1 can, and will, be based on proving the statements (2.5)–(2.7) with the process $Z_n(t)$ replacing $Q_n(t)$ in all of them. The latter goal in turn will be achieved via using the next simple, though essential, observation that is borrowed from Einmahl (1996).

Proposition B *Let $\{K(x,y),\ 0 \le x \le 1,\ y \ge 0\}$ be a Kiefer process. For any fixed $0 \le u < v \le 1$, the process*

$$\left\{\frac{K(u+(v-u)x,y) - xK(v,y) - (1-x)K(u,y)}{\sqrt{v-u}},\ x \in [0,1],\ y \ge 0\right\}$$

is a Kiefer process that is independent of $\{K(s,y),\ s \in [0,u],\ y \ge 0\}$ and $\{K(s,y),\ s \in [v,1],\ y \ge 0\}$.

Based on Theorem 3.1 and this crucial observation, the respective proofs of the *pointwise* statements of (2.5)–(2.6) and the proof of the *uniform* property as in (2.7) of the process Q_n will take different routes. Hence, our next Section 4 is devoted to proving (2.5)–(2.6), while Section 5 will be on establishing the almost sure ratio statement of (2.7).

4 Pointwise behaviour of the Vervaat error process

This section is devoted to the proof of (2.5) and (2.6) of Theorem 2.1, concerning the pointwise asymptotics of $Q_n(t)$. We first establish a strong approximation of $Z_n(t)$ of (3.3) for any fixed $t \in (0,1)$ by a process in n, which is an integral of a Wiener sheet in its first parameter over a random interval that is independent of this Wiener sheet in hand.

Lemma 4.1 *Given $Z_n(t)$ as in (3.3), then for any fixed $t \in (0,1)$ one can define a Wiener sheet $W^*(\cdot,\cdot)$ such that, as $n \to \infty$, we have*

$$Z_n(t) = 2\int_0^{|K(t,n)|/n} W^*(y,n)\,dy + \mathcal{O}\left(\frac{(\log n)^{1/2}\log_2 n}{n^{1/2}}\right) \qquad \text{a.s.,} \tag{4.1}$$

where $\{W^*(y,n),\ y \ge 0,\ n = 1,2,\dots\}$ is independent of $\{K(t,n),\ n = 1,2,\dots\}$.

Proof. Using Proposition B with $u = 0$, $v = t$, and writing $1-x$ instead of x, we see that

$$K_1^*(x,n) := \frac{K(t(1-x),n) - (1-x)K(t,n)}{t^{1/2}}, \quad 0 \le x \le 1, \tag{4.2}$$

is a Kiefer process, independent of $K(t,\cdot)$. Likewise, letting $u=t$, $v=1$, we see that

$$K_2^*(x,n) := \frac{K(t+(1-t)x,n)-(1-x)K(t,n)}{(1-t)^{1/2}}, \quad 0\le x\le 1,$$

is also a Kiefer process, independent of $K(t,\cdot)$. Moreover, $K_1^*(\cdot,\cdot)$ and $K_2^*(\cdot,\cdot)$ are independent. Consequently, the components of the vector

$$(K(t,\cdot),\ K_1^*(\cdot,\cdot),\ K_2^*(\cdot,\cdot)) \tag{4.3}$$

are independent processes.

Defining x via

$$\begin{aligned}(1-x)t &= t - s\frac{K(t,n)}{n} \quad \text{if} \quad K(t,n)>0,\\ t+(1-t)x &= t - s\frac{K(t,n)}{n} \quad \text{if} \quad K(t,n)<0,\end{aligned}$$

in the integral in the definition of $Z_n(t)$ of (3.3), we arrive at the following representation of the latter process for each fixed t:

$$Z_n(t) = \mathcal{I}_1(n)\mathbf{1}\{K(t,n)>0\} - \mathcal{I}_2(n)\mathbf{1}\{K(t,n)<0\}, \tag{4.4}$$

where

$$\begin{aligned}\mathcal{I}_1(n) &:= 2t\int_0^{K(t,n)/(nt)} (K(t(1-x),n)-K(t,n))\,dx,\\ \mathcal{I}_2(n) &:= 2(1-t)\int_0^{-K(t,n)/(n(1-t))} (K(t+(1-t)x,n)-K(t,n))\,dx.\end{aligned}$$

Considering $\mathcal{I}_1(n)$, and remembering that this is the case when $K(t,n)>0$, using the definition (4.2) of $K_1^*(\cdot,\cdot)$, we obtain: almost surely as $n\to\infty$,

$$\begin{aligned}\mathcal{I}_1(n) &= 2t^{3/2}\int_0^{K(t,n)/(nt)} K_1^*(x,n)\,dx - 2tK(t,n)\int_0^{K(t,n)/(nt)} x\,dx\\ &= 2t^{3/2}\int_0^{K(t,n)/(nt)} K_1^*(x,n)\,dx - \frac{t(K(t,n))^3}{n^2t^2}\\ &= 2t^{3/2}\int_0^{K(t,n)/(nt)} K_1^*(x,n)\,dx + \mathcal{O}\left(\frac{(\log_2 n)^{3/2}}{n^{1/2}}\right),\end{aligned} \tag{4.5}$$

the last line following from the LIL for the Kiefer process with fixed t. Since $K_1^*(\cdot,\cdot)$ is a Kiefer process independent of $K(t,\cdot)$, we can use the representation

$$K_1^*(x,n) = W_1^*(x,n) - xW_1^*(1,n),$$

where $W_1^*(\cdot,\cdot)$ is a Wiener sheet, independent of $K(t,\cdot)$. This independence property will be crucial in our use later on.

Observe that

$$\int_0^{K(t,n)/(nt)} K_1^*(x,n)\,dx = \int_0^{K(t,n)/(nt)} W_1^*(x,n)\,dx - \frac{K^2(t,n)\,W_1^*(1,n)}{2t^2\,n^2}.$$

By the LIL for the Kiefer process K and the Wiener process $W_1^*(1,\cdot)$, we have, when $n\to\infty$, $K^2(t,n)\,W_1^*(1,n)=\mathcal{O}(n^{3/2}(\log_2 n)^{3/2})$ almost surely. Going back to (4.5), we obtain: as $n\to\infty$, almost surely

$$\begin{aligned}\mathcal{I}_1(n) &= 2t^{3/2}\int_0^{K(t,n)/(nt)} W_1^*(x,n)\,dx+\mathcal{O}\left(\frac{(\log_2 n)^{3/2}}{n^{1/2}}\right)\\ &= 2\int_0^{K(t,n)/n} t^{1/2}W_1^*(\frac{y}{t},n)\,dy+\mathcal{O}\left(\frac{(\log_2 n)^{3/2}}{n^{1/2}}\right). \qquad (4.6)\end{aligned}$$

Similarly, in the case $K(t,n)<0$, we can show that as $n\to\infty$, for $\mathcal{I}_2(n)$ of (4.4) we have, almost surely,

$$\mathcal{I}_2(n)=2\int_0^{-K(t,n)/n}(1-t)^{1/2}W_2^*(\frac{y}{1-t},n)\,dy+\mathcal{O}\left(\frac{(\log_2 n)^{3/2}}{n^{1/2}}\right), \qquad (4.7)$$

where, just like $W_1^*(\cdot,\cdot)$ of (4.6), the Wiener sheet $W_2^*(\cdot,\cdot)$ of (4.7) is also independent of $K(t,\cdot)$ (cf. (4.3)).

Combining (4.6) and (4.7) with (4.4) yields that, for each fixed $t\in(0,1)$, as $n\to\infty$,

$$Z_n(t)=2\int_0^{|K(t,n)|/n} W^*(y,n)\,dy+\mathcal{O}\left(\frac{(\log_2 n)(\log n)^{1/2}}{n^{1/2}}\right) \qquad \text{a.s.},$$

where

$$W^*(y,n):=t^{1/2}W_1^*(\frac{y}{t},n)\mathbf{1}\{K(t,n)>0\}-(1-t)^{1/2}W_2^*(\frac{y}{1-t},n)\mathbf{1}\{K(t,n)<0\}.$$

Since $W^*(\cdot,\cdot)$ is a Wiener sheet, independent of $K(t,\cdot)$, this yields Lemma 4.1. □

The rest of this section is devoted to the proof of (2.5) and (2.6) in Theorem 2.1. For the sake of clarity, they are proved separately.

Proof of (2.5). For each fixed n and T, $\frac{3^{1/2}}{n^{1/2}T^{3/2}}\int_0^T W^*(y,n)\,dy$ is a standard normal variable. Thus, by conditioning, if T is a random variable independent of $W^*(\cdot,n)$, then $\frac{3^{1/2}}{n^{1/2}T^{3/2}}\int_0^T W^*(y,n)\,dy$ is a standard normal variable, independent of T. In view of the independence of $K(t,\cdot)$ and $W^*(\cdot,\cdot)$, we have, for each fixed n and t, (taking $T:=|K(t,n)|/n$)

$$\begin{aligned}&2\int_0^{|K(t,n)|/n} W^*(y,n)\,dy\\ =\;&\left(\frac{4}{3}\right)^{1/2}\frac{(t(1-t))^{3/4}}{n^{1/4}}\left(\frac{n^{1/2}|T|}{(t(1-t))^{1/2}}\right)^{3/2}\left(\frac{3^{1/2}}{n^{1/2}T^{3/2}}\int_0^T W^*(y,n)\,dy\right)\\ =_d\;&\left(\frac{4}{3}\right)^{1/2}(t(1-t))^{3/4}n^{-1/4}(|\widetilde{\mathcal{N}}|)^{3/2}\mathcal{N},\end{aligned}$$

where "$=_d$" denotes identity in distribution, and $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are independent standard normal random variables. This, in light of (3.2) and (4.1), yields (2.5). □

Proof of (2.6). Fix again $t\in(0,1)$. Define

$$\begin{aligned}W_t(n) &:= \frac{K(t,n)}{(t(1-t))^{1/2}}, \qquad n\ge 1,\\ W_t^*(x,y) &:= (t(1-t))^{1/4}W^*\left(\frac{x}{(t(1-t))^{1/2}},y\right), \qquad x\ge 0,\ y\ge 0.\end{aligned}$$

Clearly, W_t is a Wiener process (restricted on $\mathbf{N}^*$), and $W_t^*(\cdot,\cdot)$ is a Wiener sheet, independent of $\{K(t,n),\ n=1,2,\dots\}$, thus of W_t as well. We also observe that

$$\begin{aligned}&\int_0^{|K(t,n)|/n} W^*(y,n)\,dy\\ =\quad & (t(1-t))^{3/4}\frac{(2\log_2 n)^{5/4}}{n^{1/4}}\int_0^{|W_t(n)|/(2n\log_2 n)^{1/2}} \frac{W_t^*(u(\frac{2\log_2 n}{n})^{1/2},n)}{n^{1/4}(2\log_2 n)^{3/4}}\,du.\end{aligned}$$

By Theorem 1.1 of Deheuvels and Mason (1992), as $n\to\infty$, the set of limit points of

$$\left\{\left(\frac{W_t(n)}{(2n\log_2 n)^{1/2}},\ \frac{W_t^*(u(\frac{2\log_2 n}{n})^{1/2},n)}{n^{1/4}(2\log_2 n)^{3/4}}\right),\ u\in[0,1]\right\}$$

is almost surely equal to

$$\begin{aligned}\mathcal{D} := \{(c,f) :\ & f \text{ absolutely continuous with respect to the Lebesgue measure},\\ & |c|\in(0,1),\quad f(0)=0,\quad c^2+\textstyle\int_0^{|c|}(f'(u))^2\,du\le 1\}.\end{aligned}\tag{4.8}$$

Consequently, with probability one,

$$\limsup_{n\to\infty}\frac{n^{1/4}}{(\log_2 n)^{5/4}}\int_0^{|K(t,n)|/n} W^*(y,n)\,dy = 2^{5/4}(t(1-t))^{3/4}\sup_{(c,f)\in\mathcal{D}}\int_0^{|c|} f(u)\,du.$$

We now determine the value of the "sup" expression on the right hand side. Integrating by parts, using the Cauchy–Schwarz inequality and (4.8), we get, for each $(c,f)\in\mathcal{D}$,

$$\begin{aligned}&\int_0^{|c|} f(u)\,du = \int_0^{|c|}(|c|-u)f'(u)\,du\\ \le\ & \left(\int_0^{|c|}(|c|-u)^2\,du\right)^{1/2}\left(\int_0^{|c|}(f'(u))^2\,du\right)^{1/2} \le \left(\frac{|c|^3(1-c^2)}{3}\right)^{1/2}.\end{aligned}$$

Since $|c|^3(1-c^2)\le 2\times 3^{3/2}/5^{5/2}$ for any $c\in[-1,1]$, this yields

$$\sup_{(c,f)\in\mathcal{D}}\int_0^{|c|} f(u)\,du \le \frac{2^{1/2}3^{1/4}}{5^{5/4}}.\tag{4.9}$$

Choosing

$$c=(3/5)^{1/2},\qquad f(u)=\frac{2^{1/2}3^{1/4}}{5^{1/4}}u-\frac{5^{1/4}}{2^{1/2}3^{1/4}}u^2,$$

it is seen that in (4.9) we have, in fact, an equality. Accordingly, with probability one,

$$\limsup_{n\to\infty}\frac{n^{1/4}}{(\log_2 n)^{5/4}}\int_0^{|K(t,n)|/n} W^*(y,n)\,dy = (t(1-t))^{3/4}\frac{2^{7/4}3^{1/4}}{5^{5/4}}.$$

This yields (2.6) in view of (4.1) and (3.2). □

5 Uniform behaviour of the Vervaat error process

This section is devoted to establishing the *uniform* property (2.7) in Theorem 2.1. In view of Theorems 3.1 and F it suffices to show that

$$\limsup_{n\to\infty} \frac{n}{(\log n)^{1/2}} \frac{\|Z_n\|}{\|K(\cdot,n)\|^{3/2}} \le \left(\frac{4}{3}\right)^{1/2} \qquad \text{a.s.,} \tag{5.1}$$

$$\liminf_{n\to\infty} \frac{n}{(\log n)^{1/2}} \frac{\|Z_n\|}{\|K(\cdot,n)\|^{3/2}} \ge \left(\frac{4}{3}\right)^{1/2} \qquad \text{a.s.,} \tag{5.2}$$

where $\|Z_n\| := \sup_{0\le t\le 1}|Z_n(t)|$ and $\|K(\cdot,n)\| := \sup_{0\le t\le 1}|K(t,n)|$.

Proof of (5.1). Recall that (cf. (3.3)), by definition,

$$\begin{aligned} Z_n(t) &= 2\frac{K(t,n)}{n}\int_0^1 \left(K\left(t-s\frac{K(t,n)}{n},n\right)-K(t,n)\right)ds \\ &= 2\int_0^{K(t,n)/n}(K(t-z,n)-K(t,n))\,dz. \end{aligned} \tag{5.3}$$

Let

$$N = N(n) := \lfloor n^{1/2}(\log_2 n)^{-4}\rfloor, \tag{5.4}$$

and let $t_i = t_i(n) := i/N$ (for $0 \le i \le N$). Define, for $u\in[0,1]$,

$$B^*_{i,n}(u) := \frac{N^{1/2}}{n^{1/2}}\left(K\left(t_i+\frac{u}{N},n\right) - uK(t_{i+1},n) - (1-u)K(t_i,n)\right).$$

It follows from Proposition B that for each fixed n,

$$\{B^*_{i,n}(\cdot),\ i=0,1,\ldots,N-1\}$$

are independent Brownian bridges, and independent of $\{K(t_i,n),\ i=0,1,\ldots,N-1\}$.

Let $t\in[t_i,t_{i+1}]$. Then

$$Z_n(t) = 2\left(\int_0^{K(t_i,n)/n} + \int_{K(t_i,n)/n}^{K(t,n)/n}\right)(K(t-z,n)-K(t,n))\,dz.$$

By Theorem G, when $n\to\infty$,

$$\max_{0\le i\le N-1}\ \sup_{t_i\le t\le t_{i+1}} \frac{|K(t,n)-K(t_i,n)|}{n} = \mathcal{O}\left(\frac{(\log n)^{1/2}}{N^{1/2}n^{1/2}}\right) \qquad \text{a.s.}$$

On the other hand, by the LIL for the Kiefer process and Theorem G,

$$K(t-z,n)-K(t,n) = \mathcal{O}\left(n^{1/4}(\log n)^{1/2}(\log_2 n)^{1/4}\right) \qquad \text{a.s.,}$$

uniformly in $0\le i\le N-1$, $t\in[t_i,t_{i+1}]$ and $z\in[K(t_i,n)/n, K(t,n)/n]$. Therefore,

$$\int_{K(t_i,n)/n}^{K(t,n)/n}(K(t-z,n)-K(t,n))\,dz = \mathcal{O}\left(\frac{(\log n)(\log_2 n)^{1/4}}{N^{1/2}n^{1/4}}\right) \qquad \text{a.s.}$$

As a consequence, for $t \in [t_i, t_{i+1}]$, we have, almost surely when $n \to \infty$,

$$Z_n(t) = 2\int_0^{K(t_i,n)/n} (K(t-z,n) - K(t,n))\,dz + \mathcal{O}\left(\frac{(\log n)(\log_2 n)^{1/4}}{N^{1/2} n^{1/4}}\right), \tag{5.5}$$

where $\mathcal{O}$ is uniform in $i = 0, 1, \ldots, N-1$ and $t \in [t_i, t_{i+1}]$.

Let i be such that $K(t_i, n) \le 0$. If $t \in [t_i, t_{i+1}]$, then $t = t_i + v/N$ for some $v \in [0,1]$. Accordingly, by writing $A_i := N|K(t_i,n)|/n$,

$$\begin{aligned} &\int_0^{K(t_i,n)/n} (K(t-z,n) - K(t,n))\,dz \\ = \;& -\frac{n^{1/2}}{N^{3/2}} \int_0^{A_i} (B^*_{i,n}(v+y) - B^*_{i,n}(v))\,dy - \frac{K(t_{i+1},n) - K(t_i,n)}{N} \int_0^{A_i} y\,dy. \end{aligned}$$

Since $\int_0^{A_i} y\,dy = N^2 K^2(t_i,n)/2n^2$, an application of the LIL for the Kiefer process and Theorem G yields that with probability one,

$$\begin{aligned} &\int_0^{K(t_i,n)/n} (K(t-z,n) - K(t,n))\,dz \\ = \;& -\frac{n^{1/2}}{N^{3/2}} \int_0^{A_i} (B^*_{i,n}(v+y) - B^*_{i,n}(v))\,dy + \mathcal{O}\left(\frac{N^{1/2}(\log n)^{1/2}\log_2 n}{n^{1/2}}\right), \end{aligned} \tag{5.6}$$

uniformly in $i = 0, 1, \ldots, N-1$.

Note that $B^*_{i,n}$ can be represented as

$$B^*_{i,n}(u) := -W^*_{i,n}(u) + uW^*_{i,n}(1),$$

where (for each fixed n) $\{W^*_{i,n},\, i = 0, 1, \ldots, N-1\}$ are independent Wiener processes which are also independent of $\{K(t_i,n),\, i = 0, 1, \ldots, N-1\}$. Hence, almost surely,

$$\begin{aligned} &\int_0^{A_i} (B^*_{i,n}(v+y) - B^*_{i,n}(v))\,dy \\ = \;& -\int_0^{A_i} (W^*_{i,n}(v+y) - W^*_{i,n}(v))\,dy + W^*_{i,n}(1)\int_0^{A_i} y\,dy \\ = \;& -\int_0^{A_i} (W^*_{i,n}(v+y) - W^*_{i,n}(v))\,dy + \mathcal{O}\left(\frac{N^2(\log n)^{1/2}\log_2 n}{n}\right), \end{aligned} \tag{5.7}$$

the last equality following from the LIL for the Kiefer process and the fact that almost surely, $\sup_{0\le i\le N-1} |W^*_{i,n}(1)| = \mathcal{O}((\log n)^{1/2})$. This last fact can be easily checked using the usual estimate for Gaussian tails and the Borel–Cantelli lemma, regardless of the dependency structure of the standard normal variables $\{W^*_{i,n}(1),\, i = 0, 1, \ldots, N-1;\, n = 1, 2, \ldots\}$.

Putting (5.5), (5.6) and (5.7) together, and taking into account the definition of N in (5.4), we obtain:

$$Z_n(t) = \frac{2n^{1/2}}{N^{3/2}} \int_0^{A_i} (W^*_{i,n}(v+y) - W^*_{i,n}(v))\,dy + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right), \tag{5.8}$$

for $t = t_i + v/N$, $0 \le v \le 1$, $K(t_i,n) \le 0$.

If $K(t_i,n) > 0$, then by introducing the Kiefer process $K^{\leftarrow}(t,n) := K(1-t,n)$, we can write

$$\begin{aligned} &2\int_0^{K(t_i,n)/n} (K(t-z,n) - K(t,n))\,dz \\ = \;& 2\int_0^{K(t_i,n)/n} (K^{\leftarrow}(1-t+z,n) - K^{\leftarrow}(1-t,n))\,dz, \end{aligned}$$

and, similarly to the argument leading to (5.8), we can find independent Wiener processes $\{W^{\leftarrow}_{i,n},\ i = 0,1,\ldots,N-1\}$ which are also independent of $\{K(t_i,n),\ i = 0,1,\ldots,N-1\}$, such that almost surely,

$$Z_n(t) = \frac{2n^{1/2}}{N^{3/2}} \int_0^{A_i} (W^{\leftarrow}_{i,n}(v+y) - W^{\leftarrow}_{i,n}(v))\, dy + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right), \tag{5.9}$$

for $t = t_i + v/N$, $0 \le v \le 1$, $K(t_i,n) > 0$.

Combining (5.8) and (5.9), we see that with probability one, for $t = t_i + v/N$, $v \in [0,1]$,

$$Z_n(t) = \frac{2n^{1/2}}{N^{3/2}} \int_0^{A_i} (W_{i,n}(v+y) - W_{i,n}(v))\, dy + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right), \tag{5.10}$$

where

$$W_{i,n}(\cdot) := W^{*}_{i,n}(\cdot)\mathbf{1}\{K(t_i,n) \le 0\} + W^{\leftarrow}_{i,n}(\cdot)\mathbf{1}\{K(t_i,n) > 0\},$$

$i = 0,1,\ldots,N-1$, are Wiener processes, independent of $\{K(t_i,n),\ i = 0,1,\ldots,N-1\}$. Note that we do *not* claim that the Wiener processes $\{W_{i,n},\ i = 0,1,\ldots,N-1\}$ are independent between themselves.

For each n, we split $\{0,1,\ldots,N-1\}$ into two (random) parts:

$$\begin{aligned}
\mathcal{J}_1 = \mathcal{J}_1(n) &:= \{i : |K(t_i,n)| \le n^{1/2}(\log_2 n)^{-1},\ 0 \le i \le N-1\},\\
\mathcal{J}_2 = \mathcal{J}_2(n) &:= \{i : |K(t_i,n)| > n^{1/2}(\log_2 n)^{-1},\ 0 \le i \le N-1\}.
\end{aligned}$$

If $i \in \mathcal{J}_1$, then on applying the LIL for the Kiefer process and Theorem G we conclude

$$\int_0^{K(t_i,n)/n} (K(t-z,n) - K(t,n))\, dz = \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}(\log_2 n)^{3/2}}\right) \qquad \text{a.s.}$$

uniformly in $i \in \mathcal{J}_1$. In view of (5.5), we obtain: when $\to \infty$,

$$\max_{i\in\mathcal{J}_1} \sup_{t\in[t_i,t_{i+1}]} |Z_n(t)| = \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}(\log_2 n)^{3/2}}\right) \qquad \text{a.s.} \tag{5.11}$$

For $i \in \mathcal{J}_2$, we consider the variables

$$X_{i,n} := \frac{3^{1/2}}{A_i^{3/2}} \sup_{0\le v\le 1} \left| \int_0^{A_i} (W_{i,n}(v+y) - W_{i,n}(v))\, dy \right|.$$

According to (5.10), almost surely when $n \to \infty$,

$$\max_{i\in\mathcal{J}_2} \sup_{t\in[t_i,t_{i+1}]} |Z_n(t)| \le \frac{2\|K(\cdot,n)\|^{3/2}}{3^{1/2}\, n} \max_{i\in\mathcal{J}_2} X_{i,n} + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right),$$

which, combined with (5.11), yields

$$\|Z_n\| \le \frac{2\|K(\cdot,n)\|^{3/2}}{3^{1/2}\, n} \max_{i\in\mathcal{J}_2} X_{i,n} + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right) \qquad \text{a.s.} \tag{5.12}$$

We now show that the variables $X_{i,n}$, $i \in \mathcal{J}_2$ have Gaussian-like tails. To this end, we fix $A > 0$ and introduce the mean zero Gaussian process

$$X(v) := \frac{3^{1/2}}{A^{3/2}} \int_0^{A} (W(v+y) - W(v))\, dy, \qquad v \in [0,1].$$

It is straightforward to compute its covariance:

$$\mathbf{E}X(u)X(v) = \begin{cases} 1-\frac{3|v-u|}{2A}+\frac{|v-u|^3}{2A^3} & \text{if } |v-u|\le A, \\ 0 & \text{if } |v-u|>A, \end{cases} \tag{5.13}$$

and therefore

$$\mathbf{E}\left(X(v)-X(u)\right)^2 = \begin{cases} \frac{3|v-u|}{A}-\frac{|v-u|^3}{A^3} & \text{if } |v-u|\le A, \\ 2 & \text{if } |v-u|>A. \end{cases} \tag{5.14}$$

According to a well-known inequality of Fernique (1964), for any $x>0$,

$$\mathbf{P}\left(\sup_{0\le v\le 1}|X(v)|\ge x\left(\sigma^2+4\int_1^\infty \varphi(p^{-s^2})\,ds\right)\right)\le 4p^2\int_x^\infty e^{-s^2/2}\,ds,$$

where $p\ge 2$ is arbitrary, and σ and φ are such that

$$\mathbf{E}(X(v))^2\le\sigma^2, \qquad \mathbf{E}\left(X(v)-X(u)\right)^2\le\varphi^2(v-u).$$

In view of (5.13) and (5.14), we can choose $\sigma=1$ and $\varphi(h)=(3h/A)^{1/2}$. Since

$$\int_1^\infty \varphi(p^{-s^2})\,ds\le\frac{\sqrt{3}}{\sqrt{A}}\int_1^\infty e^{-(s\log p)/2}\,ds=\frac{2\sqrt{3}}{\sqrt{pA}\,\log p},$$

this leads to:

$$\mathbf{P}\left(\sup_{0\le v\le 1}|X(v)|\ge x\left(1+\frac{8\sqrt{3}}{\sqrt{pA}\,\log p}\right)\right)\le 4p^2\int_x^\infty e^{-s^2/2}\,ds. \tag{5.15}$$

Recall that $W_{i,n}$ is independent of $K(t_i,n)$. Therefore, applying (5.15) to $W=W_{i,n}$ and $A=A_i$ (for $i\in\mathcal{J}_2$) we obtain:

$$\mathbf{P}\left(X_{i,n}\ge x\left(1+\frac{8\sqrt{3}\,(\log_2 n)^3}{\sqrt{p}\,\log p}\right)\right)\le 4p^2\int_x^\infty e^{-s^2/2}\,ds,$$

where, with N as in (5.4), we used the fact that $A_i\ge(\log_2 n)^{-6}$ for all $i\in\mathcal{J}_2$.

Let $\varepsilon\in(0,1)$. We choose $p=\varepsilon^{-2}(\log_2 n)^6$ and $n_0=n_0(\varepsilon)$ such that for all $n\ge n_0$, $8\sqrt{3}\,(\log_2 n)^3/\sqrt{p}\,\log p\le\varepsilon$. Thus for any $x>0$ and $n\ge n_0$,

$$\mathbf{P}(X_{i,n}\ge(1+\varepsilon)x)\le\frac{4(\log_2 n)^{12}}{\varepsilon^4}\int_x^\infty e^{-s^2/2}\,ds\le\frac{4(\log_2 n)^{12}}{\varepsilon^4}\frac{e^{-x^2/2}}{x},$$

which yields

$$\mathbf{P}\left(\max_{i\in\mathcal{J}_2}X_{i,n}\ge(1+\varepsilon)x\right)\le\frac{4N(\log_2 n)^{12}e^{-x^2/2}}{\varepsilon^4 x}\le\frac{4n^{1/2}(\log_2 n)^8e^{-x^2/2}}{\varepsilon^4 x}.$$

Taking $x:=(1+\varepsilon)(\log n)^{1/2}$, we obtain:

$$\mathbf{P}\left(\max_{i\in\mathcal{J}_2}X_{i,n}\ge(1+\varepsilon)^2(\log n)^{1/2}\right)\le\frac{4(\log_2 n)^8}{\varepsilon^4(1+\varepsilon)(\log n)^{1/2}n^{\varepsilon}}.$$

Let $n_k = \lfloor k^{2/\varepsilon} \rfloor$. By the Borel–Cantelli lemma and (5.12), almost surely when $k \to \infty$,

$$\|Z_{n_k}\| \leq \frac{2(1+\varepsilon)^2 (\log n_k)^{1/2} \, \|K(\cdot, n_k)\|^{3/2}}{3^{1/2}\, n_k} + \mathcal{O}\left(\frac{(\log n_k)^{1/2}}{n_k^{1/4} \log_2 n_k} \right). \tag{5.16}$$

Let $n_k \leq n < n_{k+1}$. Note that $n_{k+1} - n_k = \mathcal{O}(n_k^{1-\varepsilon/2})$, $k \to \infty$. By (5.3),

$$\begin{aligned} |Z_n(t) - Z_{n_k}(t)| \;\; &\leq \;\; 2 \left| \int_0^{K(t,n)/n} \Delta_{k,n,t}(z)\, dz \right| \\ &\quad + 2 \left| \int_{K(t,n_k)/n_k}^{K(t,n)/n} (K(t-z, n_k) - K(t, n_k))\, dz \right|, \end{aligned}$$

where $\Delta_{k,n,t}(z) := K(t-z, n) - K(t-z, n_k) - K(t,n) + K(t, n_k)$. According to Theorem H and the LIL for the Kiefer process, when $k \to \infty$,

$$\Delta_{k,n,t}(z) = \mathcal{O}\left(n_k^{(1-\varepsilon)/4} (\log n_k)^{1/2} (\log_2 n_k)^{1/4} \right) \qquad \text{a.s.},$$

uniformly in $t \in (0,1)$, $z \in [0, K(t,n)/n]$ and $n_k \leq n < n_{k+1}$. Thus, by the LIL for the Kiefer process,

$$\int_0^{K(t,n)/n} \Delta_{k,n,t}(z)\, dz = \mathcal{O}\left(n_k^{-(1+\varepsilon)/4} (\log n_k)^{1/2} (\log_2 n_k)^{3/4} \right) \qquad \text{a.s.},$$

uniformly in $t \in (0,1)$ and $n_k \leq n < n_{k+1}$.

On the other hand, by the LIL for the Kiefer process and Theorem G,

$$K(t-z, n_k) - K(t, n_k) = \mathcal{O}\left(n_k^{1/4} (\log n_k)^{1/2} (\log_2 n_k)^{1/4} \right) \qquad \text{a.s.},$$

uniformly in $t \in (0,1)$, $z \in [K(t,n_k)/n_k, K(t,n)/n]$ and $n_k \leq n < n_{k+1}$. Since Corollary 1.12.4 of Csörgő and Révész (1981) implies that

$$\sup_{0 \leq t \leq 1} |K(t,n) - K(t,n_k)| = \mathcal{O}\left(n_k^{1/2-\varepsilon/4} (\log n_k)^{1/2} \right) \qquad \text{a.s.}, \tag{5.17}$$

uniformly in $n_k \leq n < n_{k+1}$, it follows that

$$\max_{n_k \leq n < n_{k+1}} \sup_{t \in [0,1]} \left| \frac{K(t,n)}{n} - \frac{K(t,n_k)}{n_k} \right| = \mathcal{O}\left(n_k^{-1/2-\varepsilon/4} (\log n_k)^{1/2} \right) \qquad \text{a.s.}$$

Therefore, almost surely when $k \to \infty$,

$$\int_{K(t,n_k)/n_k}^{K(t,n)/n} (K(t-z, n_k) - K(t, n_k))\, dz = \mathcal{O}\left(n_k^{-(1+\varepsilon)/4} (\log n_k) (\log_2 n_k)^{1/4} \right),$$

uniformly in $t \in [0,1]$ and $n_k \leq n < n_{k+1}$. We have therefore proved that

$$\max_{n_k \leq n < n_{k+1}} \|Z_n - Z_{n_k}\| = \mathcal{O}\left(n_k^{-(1+\varepsilon)/4} (\log n_k) (\log_2 n_k)^{1/4} \right), \qquad \text{a.s.}$$

Going back to (5.16) and taking (5.17) into account, we obtain, when $n \to \infty$,

$$\begin{aligned} \|Z_n\| \;\; &\leq \;\; \frac{2(1+\varepsilon)^2\, (\log n)^{1/2}}{3^{1/2}\, n} \|K(\cdot, n)\|^{3/2} + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4} \log_2 n} \right) \\ &= \;\; (1 + o(1)) \frac{2(1+\varepsilon)^2\, (\log n)^{1/2}}{3^{1/2}\, n} \|K(\cdot, n)\|^{3/2} \qquad \text{a.s.}, \end{aligned}$$

where in the last line we used the other LIL for the Kiefer process (Kuelbs, 1979, Mogulskii, 1979):

$$(\|K(\cdot,n)\|)^{-1} = \mathcal{O}\left(\frac{(\log_2 n)^{1/2}}{n^{1/2}}\right) \qquad \text{a.s.} \tag{5.18}$$

This also completes the proof of (5.1). □

Proof of (5.2). Let N and t_i, $i = 0, 1, \ldots, N$ be as before (cf. (5.4)). Combining Theorem G with

$$\left(\sup_{0\le t\le 1} K(t,n)\right)^{-1} = \mathcal{O}\left(\frac{\log n}{n^{1/2}}\right) \qquad \text{a.s.} \tag{5.19}$$

(cf. Csáki and Shi, 1998), we conclude

$$\lim_{n\to\infty} \frac{\max_{0\le i\le N-1} K(t_i,n)}{\sup_{0\le t\le 1} K(t,n)} = 1 \qquad \text{a.s.} \tag{5.20}$$

Let $t_0^+ = t_0^+(\omega, n)$ be such that $K(t_0^+, n) = \max_{0\le i\le N-1} K(t_i, n)$ and consider the set of indices $\mathcal{I}_n := \{i : t_0^+ - h_n \le t_i \le t_0^+ + h_n\}$, where $h_n = (\log n)^{-4}$. Fix $\varepsilon \in (0,1)$. For $i \in \mathcal{I}_n$, we obtain from Theorem G, (5.19) and (5.20) that when n is sufficiently large,

$$\begin{aligned} K(t_i,n) &\ge K(t_0^+,n) - \mathcal{O}\left((nh_n \log(1/h_n))^{1/2}\right) \\ &\ge (1-\varepsilon)K(t_0^+,n) > 0. \end{aligned} \tag{5.21}$$

Hence, for $i \in \mathcal{I}_n$, we can apply (5.9) to $t = t_i$ and $v = 0$ to see that when $n \to \infty$,

$$\begin{aligned} Z_n(t_i) &= \frac{2n^{1/2}}{N^{3/2}} \int_0^{NK(t_i,n)/n} W_{i,n}^{\leftarrow}(y)\,dy + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right) \\ &= \frac{2}{3^{1/2}\,n}(K(t_i,n))^{3/2} Y_{i,n} + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right), \end{aligned} \tag{5.22}$$

where, for each n, $\{W_{i,n}^{\leftarrow}, i = 0, 1, \ldots, N-1\}$ are independent Wiener processes which are also independent of $\{K(t_i, n), i = 0, 1, \ldots, N-1\}$, and

$$Y_{i,n} := \frac{3^{1/2} n^{3/2}}{N^{3/2}} \frac{1}{(K(t_i,n))^{3/2}} \int_0^{NK(t_i,n)/n} W_{i,n}^{\leftarrow}(y)\,dy, \qquad i \in \mathcal{I}_n.$$

For each n, $\{Y_{i,n}, i \in \mathcal{I}_n\}$ are independent standard normal variables. Since $\#(\mathcal{I}_n) \ge N/(\log n)^4$, we have, by the usual estimate for Gaussian tails,

$$\begin{aligned} &\mathbf{P}\left(\max_{i\in\mathcal{I}_n} |Y_{i,n}| < (1-\varepsilon)^{1/2}(\log n)^{1/2}\right) \\ \le\ & \left(1 - \frac{2^{1/2}+o(1)}{(1-\varepsilon)^{1/2}(\pi\log n)^{1/2}} \frac{1}{n^{(1-\varepsilon)/2}}\right)^{N/(\log n)^4} \\ \le\ & \exp\left(-\frac{2^{1/2}+o(1)}{(1-\varepsilon)^{1/2}(\pi\log n)^{1/2}} \frac{N/(\log n)^4}{n^{(1-\varepsilon)/2}}\right) \\ =\ & \exp\left(-\frac{2^{1/2}+o(1)}{(1-\varepsilon)^{1/2}\pi^{1/2}} \frac{n^{\varepsilon/2}}{(\log n)^{9/2}(\log_2 n)^4}\right), \end{aligned}$$

which is summable. By the Borel–Cantelli lemma, almost surely for all large n,

$$\max_{i\in\mathcal{I}_n}|Y_{i,n}|\geq(1-\varepsilon)^{1/2}(\log n)^{1/2}.$$

Inserting this into (5.22) and (5.21) yields that, almost surely,

$$\begin{aligned}\|Z_n\| &\geq \max_{i\in\mathcal{I}_n} Z_n(t_i)\\ &\geq \frac{2(1-\varepsilon)^{3/2}(\log n)^{1/2}}{3^{1/2}\,n}(K(t_0^+,n))^{3/2}+\mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right)\\ &\geq \frac{2(1-\varepsilon)^{2}(\log n)^{1/2}}{3^{1/2}\,n}\left(\sup_{0\leq t\leq 1}K(t,n)\right)^{3/2}+\mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right),\end{aligned}$$

where in the last inequality we used (5.20). This also holds true for $-\inf_{0\leq t\leq 1}K(t,n)$ in place of $\sup_{0\leq t\leq 1}K(t,n)$, by means of a similar argument with $K(t_0^+,n)$ replaced by $-K(t_0^-,n) = -\min_{0\leq i\leq N-1}K(t_i,n)$. Consequently,

$$\|Z_n\|\geq\frac{2(1-\varepsilon)^{2}(\log n)^{1/2}}{3^{1/2}\,n}\|K(\cdot,n)\|^{3/2}+\mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right)\qquad\text{a.s.}$$

In light of (5.18), this yields (5.2). □

References

[1] Bahadur, R.R. (1966). A note on quantiles in large samples. *Ann. Math. Statist.* **37** 577–580.

[2] Chan, A.H.C. (1977). *On the Increments of Multiparameter Gaussian Processes.* Ph.D. Thesis, Carleton University.

[3] Csáki, E. and Shi, Z. (1998). Some liminf results for two-parameter processes. *Stoch. Process. Appl.* **78** 27–46.

[4] Csörgő, M. (1983). *Quantile Processes with Statistical Applications.* SIAM, Philadelphia.

[5] Csörgő, M. and Révész, P. (1975). Some notes on the empirical distribution function and the quantile process. In: *Coll. Math. Soc. J. Bolyai* **11** *Limit Theorems of Probability Theory*, Keszthely, Hungary, pp. 59–71. North Holland, Amsterdam.

[6] Csörgő, M. and Révész, P. (1981). *Strong Approximations in Probability and Statistics.* Academic Press, New York.

[7] Csörgő, M. and Shi, Z. (1998). *An L^p-view of the Bahadur–Kiefer theorem.* Tech. Report No. **324** of the Laboratory for Research in Statistics and Probability, Carleton University and University of Ottawa, Ottawa.

[8] Csörgő, M. and Szyszkowicz, B. (1998). Sequential quantile and Bahadur–Kiefer processes. In: *Order Statistics: Theory & Methods* (eds. N. Balakrishnan and C.R. Rao) pp. 631–688. Elsevier Science B.V., Amsterdam.

[9] Csörgő, M. and Zitikis, R. (1996). Strassen's LIL for the Lorenz curve. *J. Multivariate Analysis* **59** 1–12.

[10] Csörgő, M. and Zitikis, R. (1998). *On the general Bahadur–Kiefer, quantile, and Vervaat processes: old and new.* Tech. Report No. **323** of the Laboratory for Research in Statistics and Probability, Carleton University and University of Ottawa, Ottawa. To appear in: *Bolyai Society Mathematical Studies: A Volume in Honour of Pál Révész.*

[11] Csörgő, M. and Zitikis, R. (1999a). *The Vervaat process in L_p-spaces.* Tech. Report No. **328** of the Laboratory for Research in Statistics and Probability, Carleton University and University of Ottawa, Ottawa.

[12] Csörgő, M. and Zitikis, R. (1999b). On the Vervaat and Vervaat-error processes. *Acta Applicandae Mathematicae* **58** 91–105.

[13] Deheuvels, P. (1988). On the local oscillations of empirical and quantile processes. In: *Asymptotic Methods in Probability and Statistics – a Volume in Honour of Miklós Csörgő* (ed. B. Szyszkowicz) pp. 127–134. Elsevier Science B.V., Amsterdam.

[14] Deheuvels, P. and Mason, D.M. (1990). Bahadur–Kiefer-type processes. *Ann. Probab.* **18** 669-697.

[15] Deheuvels, P. and Mason, D.M. (1992). A functional LIL approach to pointwise Bahadur–Kiefer theorems. In: *Probability in Banach Spaces* (R.M. Dudley *et al.,* eds.) pp. 255-266. Birkhäuser, Boston.

[16] Einmahl, J.H.J. (1996). A short and elementary proof of the main Bahadur–Kiefer theorem. *Ann. Probab.* **24** 526–531.

[17] Fernique, X. (1964). Continuité des processus gaussiens. *C. R. Acad. Sci. Paris* **258** 6058–6060.

[18] Kiefer, J. (1967). On Bahadur's representation of sample quantiles, *Ann. Math. Statist.* **38** 1323–1342.

[19] Kiefer, J. (1970). Deviations between the sample quantile process and the sample df. In: *Nonparametric Techniques in Statistical Inference* (ed. M.L. Puri) pp. 299–319. Cambridge University Press, Cambridge.

[20] Komlós, J., Major, P. and Tusnády, G. (1975). An approximation of partial sums of independent R.V.'s and the sample DF. I. *Z. Wahrsch. verw. Gebiete* **32** 111–131.

[21] Kuelbs, J. (1979). Rates of growth for Banach space valued independent increment processes. *Lecture Notes in Mathematics* **709** pp. 151–169. Springer, New York.

[22] Mogulskii, A.A. (1979). On the law of the iterated logarithm in Chung's form for functional spaces. *Th. Probab. Appl.* **24** 405–413.

[23] Shorack, G.R. (1982). Kiefer's theorem via the Hungarian construction. *Z. Wahrsch. Verw. Gebiete* **61** 369–373.

[24] Shorack, G.R. and Wellner, J.A. (1986). *Empirical Processes with Applications to Statistics.* Wiley, New York.

[25] Vervaat, W. (1972a). Functional central limit theorems for processes with positive drift and their inverses. *Z. Wahrsch. Verw. Gebiete* **23** 245–253.

[26] Vervaat, W. (1972b). *Success Epochs in Bernoulli Trials: with Applications to Number Theory.* Mathematical Centre Tracts **42** (second edition in 1977). Matematisch Centrum, Amsterdam.

[27] Zitikis, R. (1998). The Vervaat process. In: *Asymptotic Methods in Probability and Statistics – a Volume in Honour of Miklós Csörgő* (ed. B. Szyszkowicz) pp. 667–694. Elsevier Science B.V., Amsterdam.