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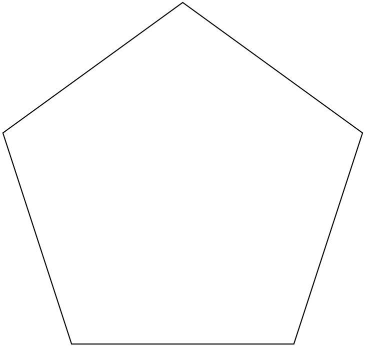

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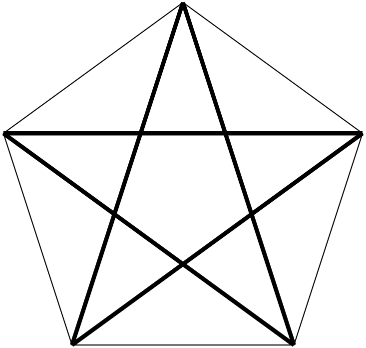

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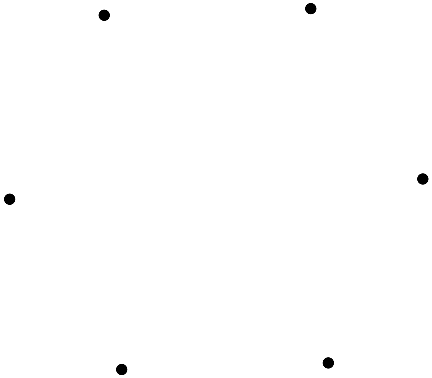

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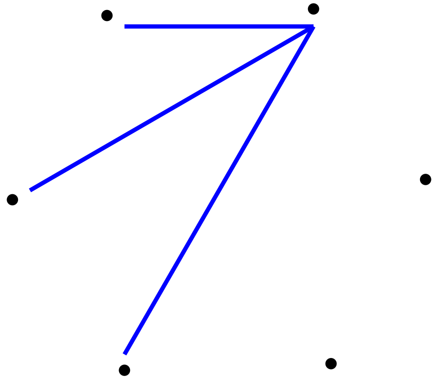

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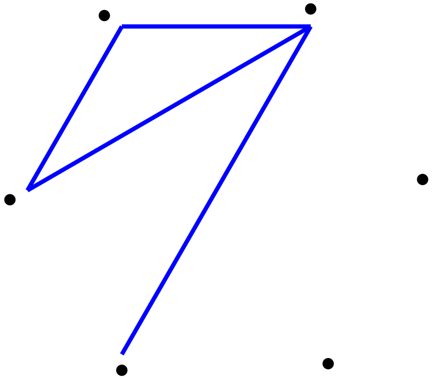

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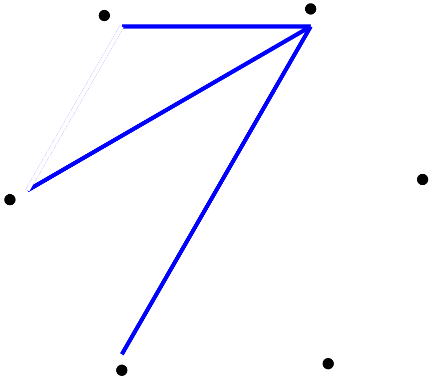

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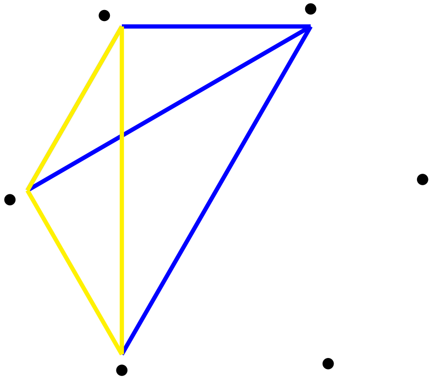

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For every coloring of the edges of K_{10} with blue and yellow there will be a either a yellow triangle (K_3) or a blue K_4

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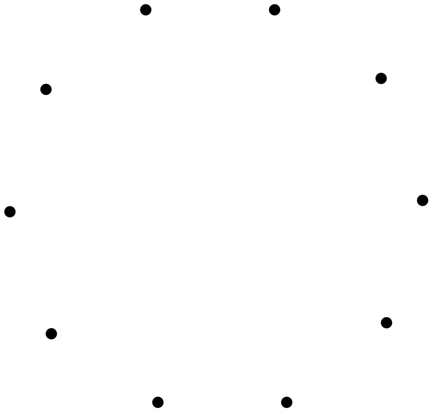

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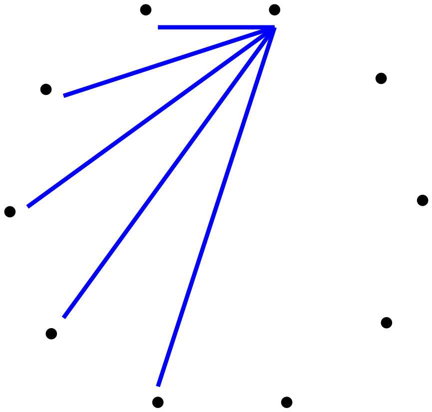

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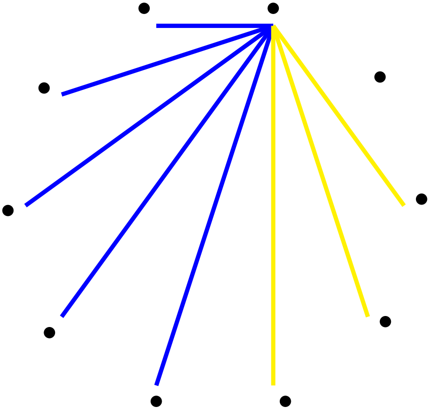

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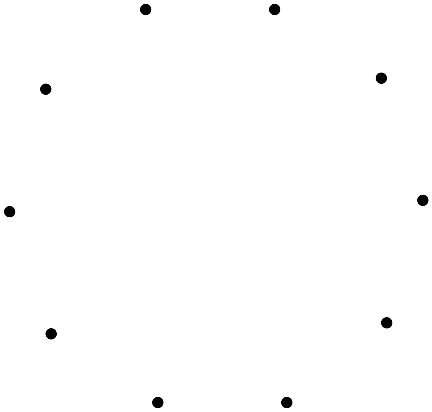

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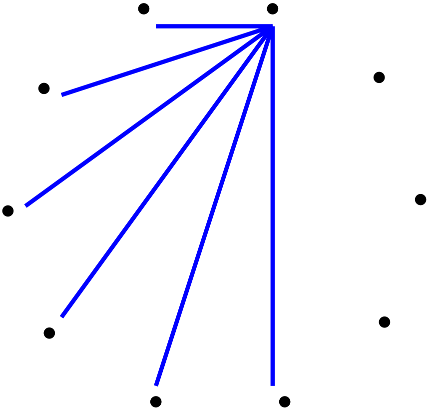

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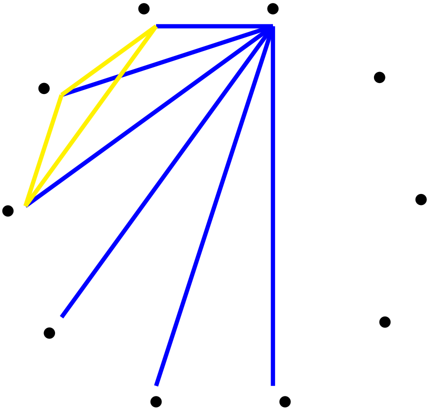

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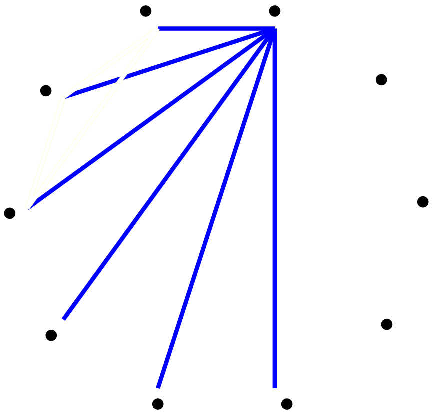

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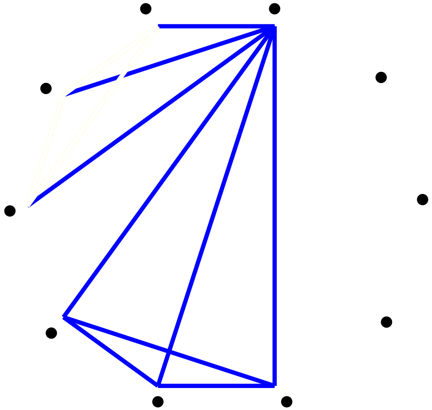

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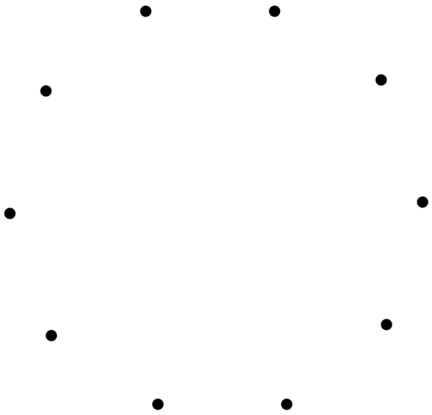

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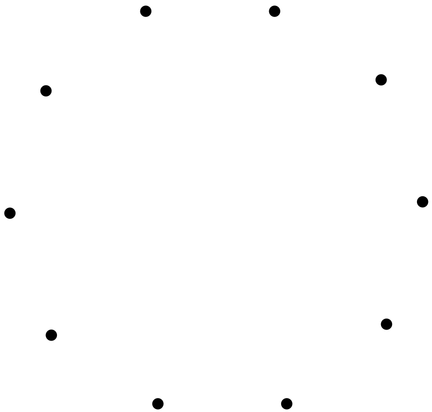

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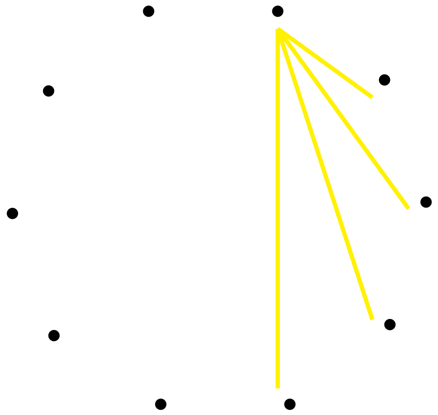

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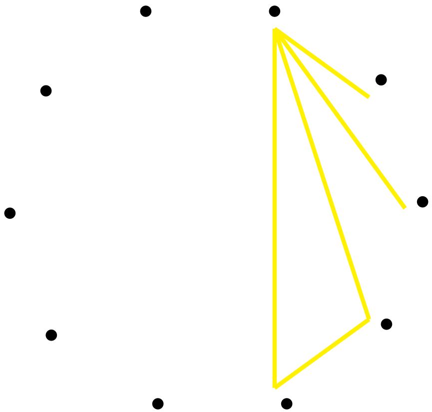

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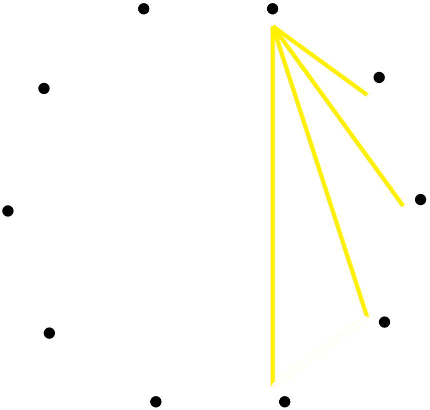

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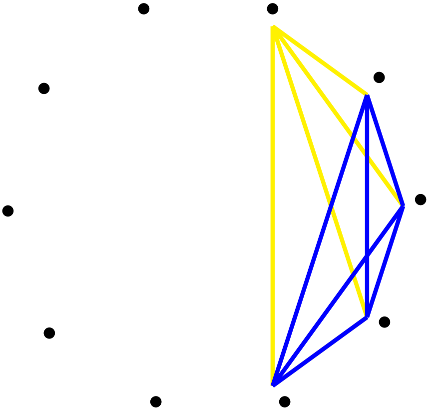

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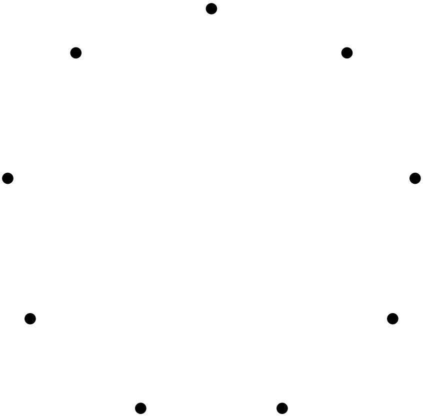

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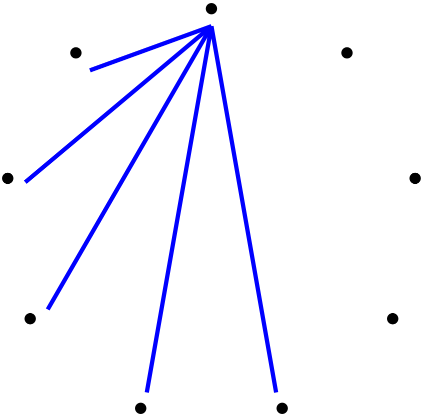

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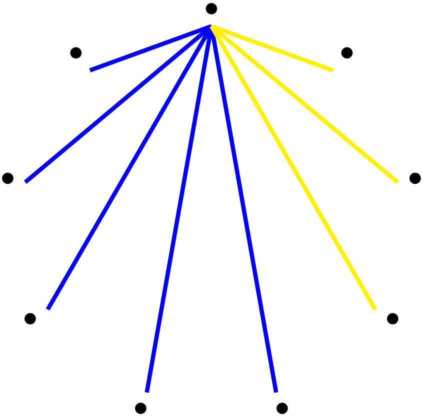

Ramsey number, in general

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Note that $R(k, 2) = R(2, k) = 2$.

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Theorem

For every $k, l \geq 3$ the following inequality holds:
$R(k, l) \leq R(k - 1, l) + R(k, l - 1)$.

Ramsey number, proof of Lemma

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Proof Take a graph G on $n = R(k-1, l) + R(k, l-1)$ vertices and fix any point x of it.

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Ramsey theorem

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With this theorem and the easy observation: $R(2,k) = R(k,2) = k$ we get that $R(k,l) \leq \binom{k+l-2}{k-1} = \binom{k+l-2}{l-1}$, **(Ramsey's theorem)** where the proof is by induction, the induction step being the theorem above and the base cases are $R(2,k) = R(k,2) = k$.

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k,l	2	3	4	5	6
2	2*	3*	4*	5*	6*
3	3*	6*	10	15	21
4	4*	10	20		
5	5*	15			
6	6*	21			

In this table the values denoted by a $*$ are exact values.

A few Ramsey numbers

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3	3*	6*	9^a	14^b	20
4	4*	9^a	18^c		
5	5*	14^b			
6	6*	20			

In this table the values for $R(3,4) = 9$, $R(3,5) = 14$ and $R(4,4) = 18$ are exact too.

Ramsey numbers: a generalization

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Ramsey numbers: a generalization, cont'd, Turan number

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Ramsey numbers: a generalization, cont'd, Turan number

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Ramsey numbers: a generalization, cont'd, Turan number

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Definition

The Turan number $ex(n, H)$ is the maximum number of edges of a graph on n vertices without contaning H as a subgraph.

Ramsey numbers: a generalization, cont'd, Turan number

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$$ex(n, P_4) = \begin{cases} n & \text{for } 3|n \\ n-1 & \text{for } 3 \nmid n \end{cases}$$