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Chromatic number of cycles

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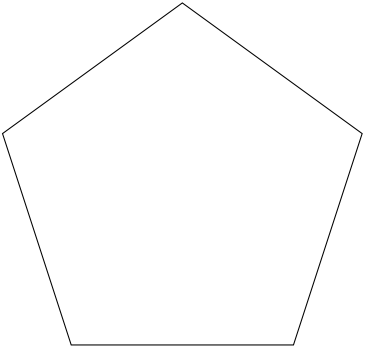

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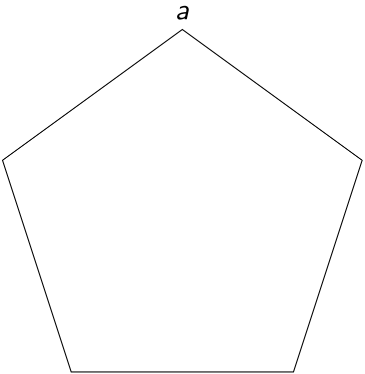

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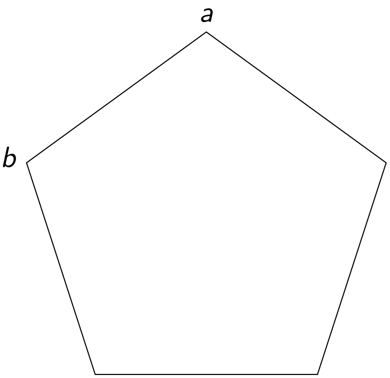

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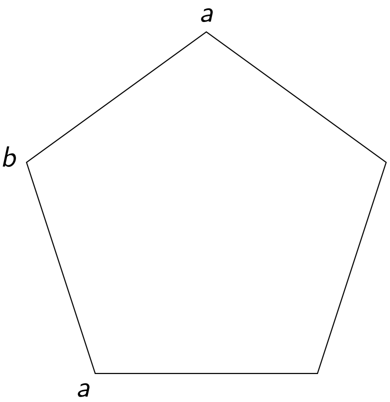

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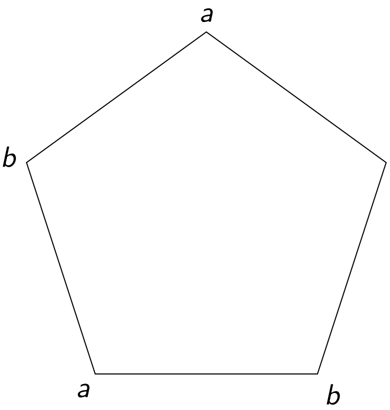

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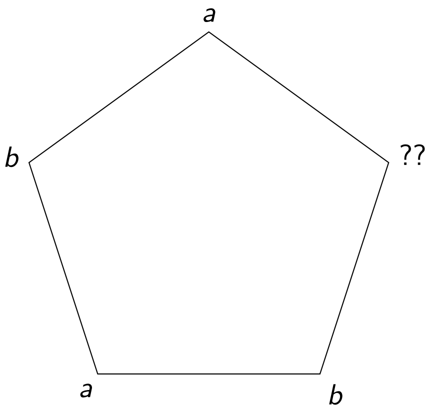

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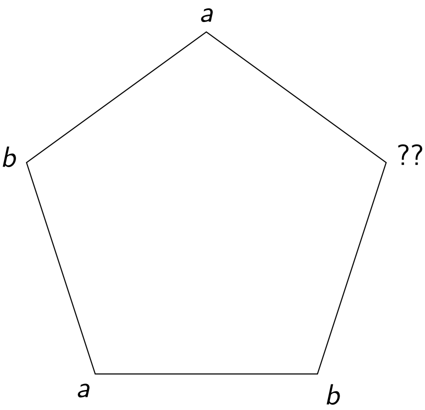

$\chi(C_5) = 3$ but $C_5 \not\supseteq K_3$!

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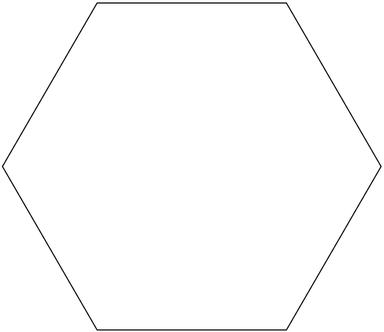

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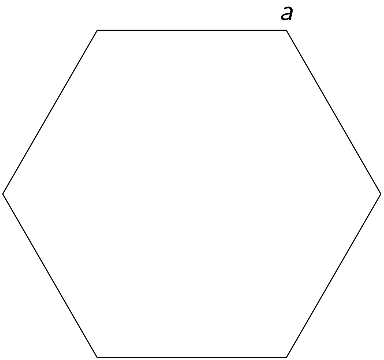

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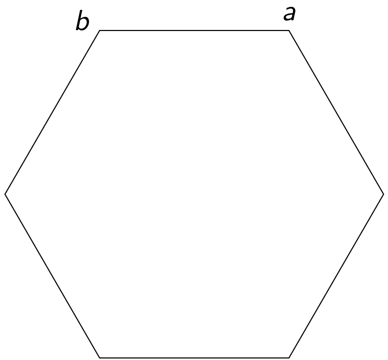

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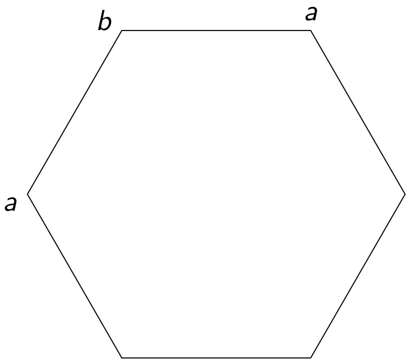

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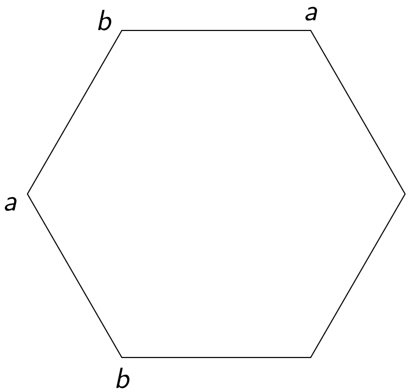

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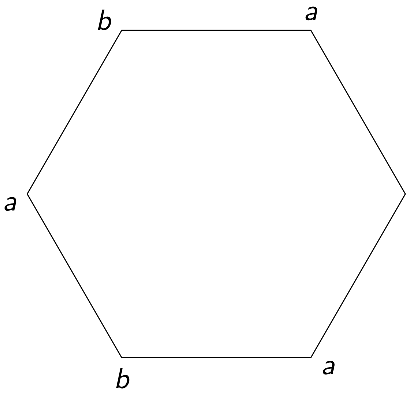

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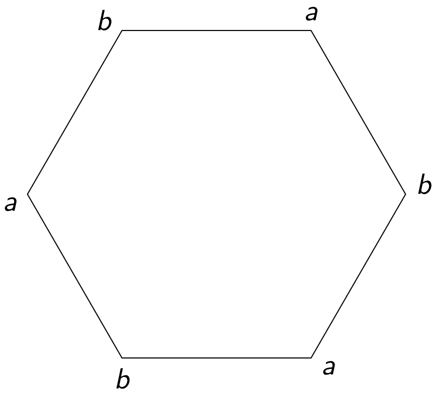

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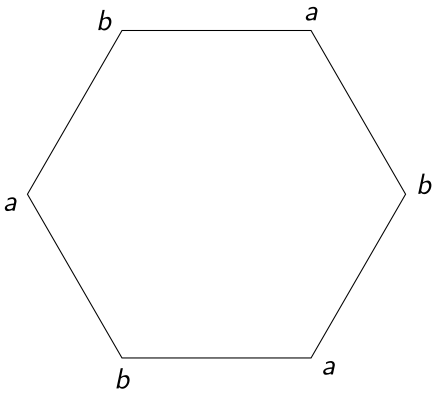

$$\chi(C_n) = \begin{cases} 2 & \text{for } 2 | n \\ 3 & \text{for } 2 \nmid n \end{cases}$$

Greedy algorithm

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The greedy (coloring) algorithm of a graph G w.r.t. the order of the vertices $x_1, x_2, \dots x_n$ using colors $1, 2, \dots$: color x_1 to 1 and for $i \geq 2$ color x_i the smallest color itr allows to have $x_1, x_2, \dots, x_i$ good colored.

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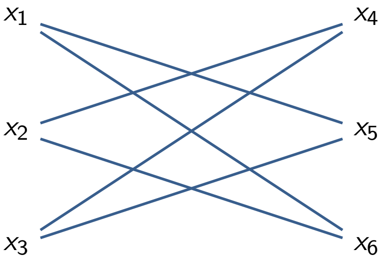

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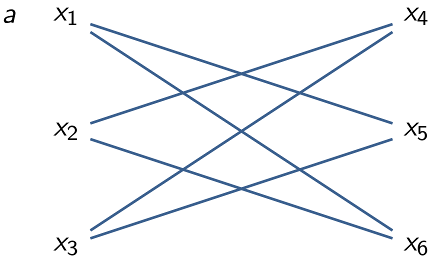

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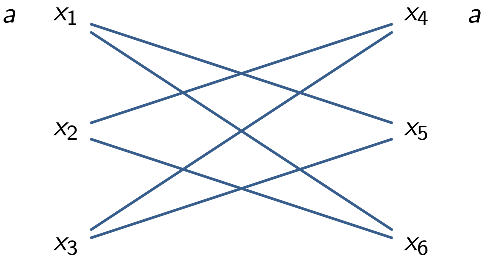

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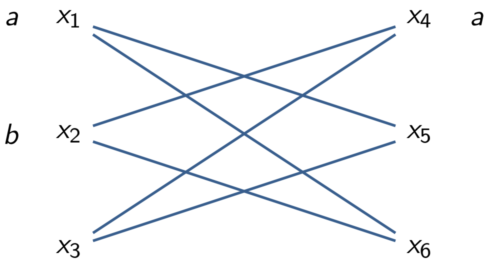

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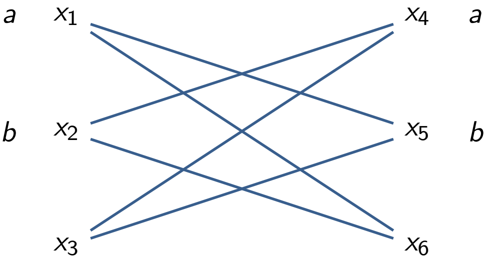

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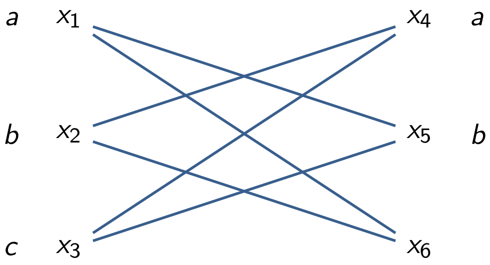

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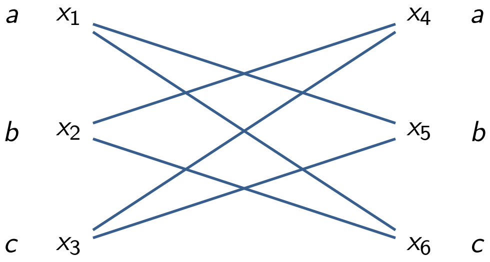

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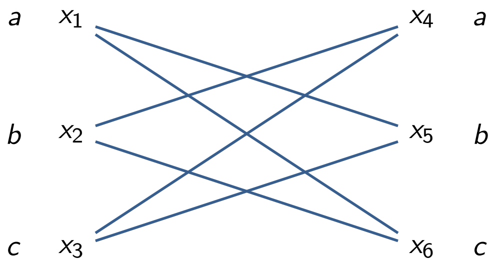

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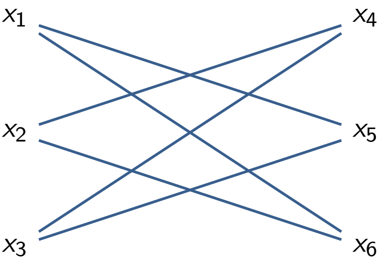

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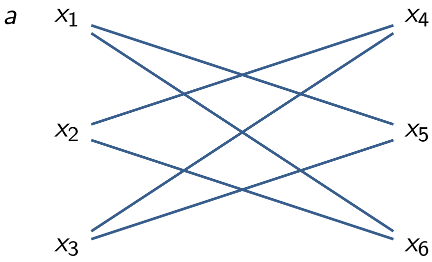

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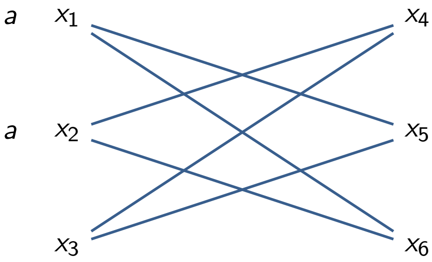

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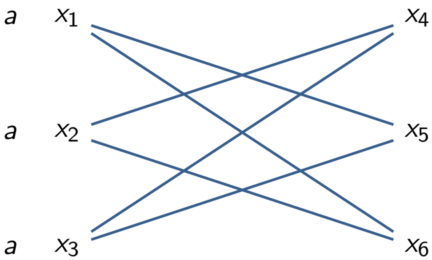

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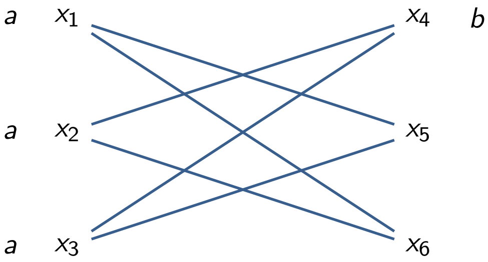

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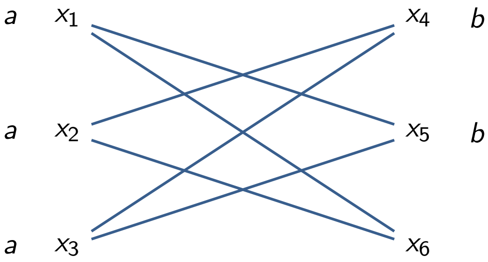

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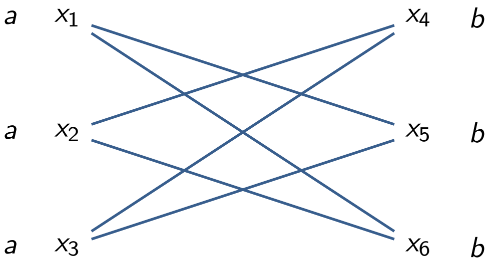

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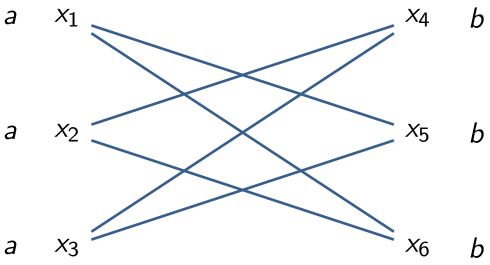

Proposition

For every graph G there exists an ordering of the vertices w.r.t. the greedy algorithm requires only (as low as possible) $\chi(G)$ colors.

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On the other hand, if $C_{2k+1} \nsubseteq G$ we can good color G by two colors in the following way:

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For $k = 4$ and $g = 4$, G is